Topological recursion, Painlevé equations and cluster algebras

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Contents

Topological recursion

- **Topological recursion** ([Eynard-Orantin 07]) is a remarkable algorithm which associate a hierarchy of
  - meromorphic differential forms $\omega_n^{(g)}(z_1, \ldots, z_n)$ ("EO differentials")
  - numbers $F^{(g)}$ ("free energy" or "symplectic invariants")

  for $g \geq 0$, $n \geq 1$ to a given algebraic curve Σ (called spectral curve).
  TR originates in the "loop equation" in the context of matrix model.

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Example: $\Sigma = \begin{cases} x(z) = \theta^{1/2}(z + z^{-1}) \\ y(z) = \frac{\theta^{1/2}}{2}(z - z^{-1}) \end{cases} \Leftrightarrow \ y^2 = \dfrac{x^2}{4} - \theta$ (Weber curve)

$$\rightsquigarrow F^{(g)} = \chi(\mathcal{M}_g)\theta^{2-2g} = \frac{B_{2g}}{2g(2g-2)}\theta^{2-2g}$$: Harer-Zagier-Penner

Gromov-Witten invariants, Hurwitz numbers, Weil-Peterson volume, knot invariants, etc. appears in this context.

TR and integrable systems of Painlevé type

TR is closely related to τ-functions of integrable systems:

- $$\begin{cases} x(z) = z^2 + t_1 \\ y(z) = z - \frac{1}{2}\sum_{k=0}^{\infty} t_{k+2} z^{k+2} \end{cases}$$

  $\rightsquigarrow \quad \log \tau(\mathbf{t}, \hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} F^{(g)}(\mathbf{t})$: Witten-Kontsevich KdV τ-function

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- Painlevé τ-functions [Bergère, Borot, Eynard, I-Marchal-Saenz] :

$$\begin{cases} x(z) = z^2 - 2\sqrt{t/6} \\ y(z) = 2z(z^2 - 3\sqrt{t/6}) \end{cases}$$

$$\rightsquigarrow \quad \log \tau(t, \hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} F^{(g)}(t)$$: τ-function for the Painlevé I.

$$\hbar^2 \frac{d^2}{dt^2} \log \tau(t, \hbar) = -q(t, \hbar) \ \text{ where } q \text{ satisfies } \ (P_{\mathrm{I}}) : \hbar^2 \frac{d^2 q}{dt^2} = 6q^2 - t.$$

TR and WKB analysis

- **Quantum curve** [Gukov-Sułkowski, Mulase-Dumitrescu, Bouchard-Eynard,...]

$$\begin{cases} x(z) = \theta^{1/2}(z + z^{-1}) \\ y(z) = \frac{\theta^{1/2}}{2}(z - z^{-1}) \end{cases} \Leftrightarrow \; y^2 = \frac{x^2}{4} - \theta \quad \text{(Weber curve)}$$

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$\rightsquigarrow \omega_n^{(g)}(z_1, \ldots, z_n)$: EO differentials.

$$\omega_1^{(0)}(z_1) = y(z)dx(z), \quad \omega_2^{(0)}(z_1, z_2) = B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2},$$

$$\omega_3^{(0)}(z_1, z_2, z_3) = \frac{1}{2\theta}\left(\frac{1}{(z_1 - 1)^2(z_2 - 1)^2(z_3 - 1)^2} + \frac{1}{(z_1 + 1)^2(z_2 + 1)^2(z_3 + 1)^2}\right)dz_1 dz_2 dz_3.$$

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The formal series

$$\psi(x, \hbar) = \exp\left(\sum_{g \geq 0, n \geq 1} \frac{\hbar^{2g-2+n}}{n!} \frac{1}{2^n} \int_{1/z_1}^{z_1} \cdots \int_{1/z_n}^{z_n} \omega_n^{(g)}(z_1, \ldots, z_n)\right)\Bigg|_{z_1 = \cdots = z_n = z(x)}$$

is a WKB (formal) solution of $\left\{\hbar^2 \frac{d^2}{dx^2} - \left(\frac{x^2}{4} - \theta\right)\right\} \psi(x, \hbar) = 0.$

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Topological recursion [Eynard-Orantin 07]

To a given genus 0 algebraic curve with a parametrization (spectral curve)

$$\Sigma : \begin{cases} x = x(z) \\ y = y(z) \end{cases} \qquad (x(z), y(z)\text{: rational function of } z \in \mathbb{P}^1),$$

and for $g \geq 0$, $n \geq 1$, define $\omega_n^{(g)}(z_1, \ldots, z_n) = W_n^{(g)}(z_1, \ldots, z_n) dz_1 \cdots dz_n$ by

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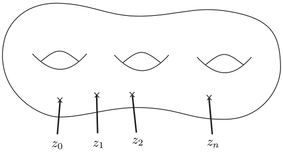
$$\omega_1^{(0)}(z) := y(z)dx(z), \quad \omega_2^{(0)}(z_1, z_2) := \frac{dz_1 dz_2}{(z_1 - z_2)^2} \quad \text{(Bergman kernel)}$$

$$\omega_{n+1}^{(g)}(z_0, z_1, \ldots, z_n) := \sum_{a\,:\,\text{branch point}} \operatorname*{Res}_{z=a} K(z, z_0) \Big(\omega_{n+2}^{(g-1)}(z, \bar{z}, z_1, \ldots, z_n) + \sum_{\substack{g_1 + g_2 = g,\ I_1 \sqcup I_2 = \{1, \ldots, n\}, \\ \text{except for } (g_i = 0\ \&\ I_i = \emptyset)}} \omega_{1+|I_1|}^{(g_1)}(z, z_{I_1})\, \omega_{1+|I_2|}^{(g_2)}(\bar{z}, z_{I_2}) \Big).$$

- **Branch point** are zeros of dx (assume that all branch points are generic).
- $\bar{z}$ is a **local conjugate point** of z near a branch point.
- $K(z, z_0) := \dfrac{1}{2(y(z) - y(\bar{z}))dx(z)} \displaystyle\int_{w=z}^{w=\bar{z}} \omega_2^{(0)}(z_0, w)$ is the **recursion kernel**.

Diagrammatic expression of topological recursion

"$\omega_n^{(g)}(z_1, \ldots, z_n) \longleftrightarrow$ genus g Riemann surface with n marked points$''$



$$\omega_{n+1}^{(g)}(z_0, z_1, \ldots, z_n)$$

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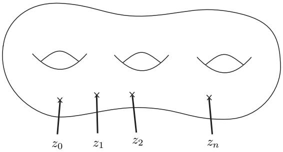
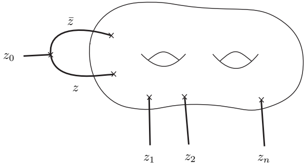
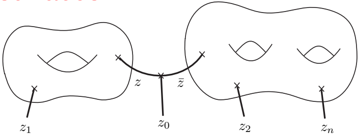


$\omega_{n+1}^{(g)}(z_0, z_1, \ldots, z_n)$

"Degeneration" of
Riemann surfaces



$K(z, z_0)\omega_{n+2}^{(g-1)}(z, \bar{z}, z_1, \ldots, z_n)$



$K(z, z_0)\omega_{1+|I_1|}^{(g_1)}(z, z_{I_1})\omega_{1+|I_2|}^{(g_2)}(\bar{z}, z_{I_2})$

Symplectic invariant (Free energy)

For $g \geq 2$, define **symplectic invariant** of the spectral curve C by

$$F^{(g)} := \frac{1}{2-2g} \sum_{a\,:\,\text{branch points}} \operatorname*{Res}_{z=a} \Phi(z)\omega_1^{(g)}(z) \qquad \left(\Phi(z) := \int^z y(z)dx(z)\right)$$

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$F^{(g)}$ for $g = 0, 1$ are also defined (in a different mannar).

- $F^{(g)}$ is **invariant** under **symplectic transformation** of spectral curves (which preserves $dx \wedge dy$).
- A formula for **variation** of spectral curve: For a family of spectral curves C_t,

$$\frac{dF^{(g)}(t)}{dt} = \int_\Omega \left(\Lambda(z,t)\omega_1^{(g)}(z,t)\right) dz \quad (\text{for } g \geq 1).$$

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Painlevé II and isomonodromy systems

- **Painlevé II** (with a small parameter $\hbar$):

$$\boxed{(P_{\mathrm{II}}) \ : \ \hbar^2 \frac{d^2 q}{dt^2} = 2q^3 + tq - \theta + \frac{\hbar}{2}} \qquad (\theta \neq 0)$$

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- Isomonodromy system:

$$\boxed{\hbar \frac{\partial \Psi}{\partial x} = A(x,t,\hbar)\Psi, \quad \hbar \frac{\partial \Psi}{\partial t} = B(x,t,\hbar)\Psi} \qquad \left(\hbar\Big(\frac{\partial A}{\partial t} - \frac{\partial B}{\partial x}\Big) + [A,B] = 0 \right)$$

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  - Jimbo-Miwa (JM) system:

$$A_{\mathrm{JM}} = \begin{pmatrix} x^2 + p + t/2 & u(x-q) \\ -\frac{2}{u}(xp + qp + \theta) & -(x^2 + p + t/2) \end{pmatrix}, \quad B_{\mathrm{JM}} = \begin{pmatrix} \frac{x}{2} & \frac{u}{2} \\ -\frac{p}{u} & -\frac{x}{2} \end{pmatrix}.$$

  - Harnad-Tracy-Widom (HTW) system:

$$A_{\mathrm{HTW}} = \begin{pmatrix} -q + \frac{\theta}{2x} & x - p - 2q^2 - t \\ \frac{1}{2} + \frac{p}{2x} & q - \frac{\theta}{2x} \end{pmatrix}, \quad B_{\mathrm{HTW}} = \begin{pmatrix} \frac{x}{2} & \frac{u}{2} \\ -\frac{p}{u} & -\frac{x}{2} \end{pmatrix}.$$

τ-function

- Compatibility condition implies

$$\hbar\frac{dq}{dt} = p + q^2 + \frac{t}{2}, \quad \hbar\frac{dp}{dt} = -2pq - \theta, \quad \hbar\frac{du}{dt} = -qu.$$

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- The (Jimbo-Miwa-Ueno) **τ-function** for (P_{II}) is defined by

$$\boxed{\hbar^2 \frac{d}{dt} \log \tau := H} \qquad \text{(up to constant)}$$

where $H = H(q, p, t) := \dfrac{p^2}{2} + \left(q^2 + \dfrac{t}{2}\right) p + \theta q$ is the Hamiltonian of (P_{II}).

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- We restrict our discussion to a (formal) solution of Painlevé II of the form:

$$\begin{aligned} q &= q(t,\hbar) &= q_0(t) + \hbar q_1(t) + \hbar^2 q_2(t) + \cdots \\ p &= p(t,\hbar) &= p_0(t) + \hbar p_1(t) + \hbar^2 p_2(t) + \cdots \end{aligned}$$

  - The leading term satisfies $p_0 + q_0^2 + t/2 = 0$ and $2q_0p_0 + \theta = 0$ ($\rightsquigarrow 2q_0^3 + tq_0 - \theta = 0$).
  - Subleading terms $q_n(t), p_n(t)$ are recursively determined.

TR and Painlevé τ-function

(Semi-classical) spectral curve is given by $\det\left(y - A(x,t,\hbar)\right)\Big|_{\hbar=0} = 0$.

- For JM case (recall that $p_0 + q_0^2 + t/2 = 0$ and $2q_0p_0 + \theta = 0$):

$$y^2 = (x-q_0)^2\left(x+q_0+\sqrt{-\theta/q_0}\right)\left(x+q_0-\sqrt{-\theta/q_0}\right)$$
: JM curve (genus = 0)

- For HTW case: $y^2 = \frac{1}{2x^2}\left(x - \frac{\theta}{2q_0}\right)^2(x+2q_0^2)$: HTW curve (genus = 0)

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- For HTW case: $y^2 = \dfrac{1}{2x^2}\left(x - \dfrac{\theta}{2q_0}\right)^2 (x+2q_0^2)$: HTW curve (genus = 0)

Theorem [I-Marchal 14]

(i) Let $F^{(g)}_{\mathrm{JM}}(t)$ be the symplectic invariants of JM curve. Then,

$$\log\tau_{\mathrm{JM}}(t,\hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} F^{(g)}_{\mathrm{JM}}(t) \quad \text{satisfies} \quad \hbar^2 \frac{d}{dt}\log\tau_{\mathrm{JM}}(t,\hbar) = H(t,\hbar).$$

(ii) Let $F^{(g)}_{\mathrm{HTW}}(t)$ be the symplectic invariants of HTW curve. Then,

$$\log\tau_{\mathrm{HTW}}(t,\hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} F^{(g)}_{\mathrm{HTW}}(t) \quad \text{satisfies} \quad \hbar^2 \frac{d}{dt}\log\tau_{\mathrm{HTW}}(t,\hbar) = H(t,\hbar) + \frac{t^2}{8}.$$

Relationship between $F^{(g)}_{\mathrm{JM}}$ and $F^{(g)}_{\mathrm{HTW}}$

For $g \geq 1$, the difference is a constant:

$$\begin{aligned}
F^{(1)}_{\mathrm{JM}}(t) - F^{(1)}_{\mathrm{HTW}}(t) &= \frac{1}{24}\log\left(\frac{\theta^3}{4}\right), \\
F^{(2)}_{\mathrm{JM}}(t) - F^{(2)}_{\mathrm{HTW}}(t) &= \frac{1}{240}\theta^{-2}, \\
F^{(3)}_{\mathrm{JM}}(t) - F^{(3)}_{\mathrm{HTW}}(t) &= -\frac{1}{1008}\theta^{-4}, \\
&\vdots
\end{aligned}$$

Theorem [I-Marchal 14]

$$F^{(g)}_{\mathrm{JM}}(t) - F^{(g)}_{\mathrm{HTW}}(t) = \frac{B_{2g}}{2g(2g-2)}\theta^{2-2g} \quad \left(= \chi(\mathcal{M}_g)\theta^{2-2g}\right) \quad \text{(for } g \geq 2\text{)}.$$

Isomonodromy system as quantum curve

Theorem ([I 16]. C.f., [I-Saenz 15])

$\omega_n^{(g)}(z_1, \ldots, z_n; t)$: the EO differential for JM curve.

$$\psi(x,t,\hbar) := \exp\left(\sum_{g \geq 0, n \geq 1} \frac{\hbar^{2g-2+n}}{n!} \frac{1}{2^n} \int_{\bar{z}_1}^{z_1} \cdots \int_{\bar{z}_n}^{z_n} \omega_n^{(g)}(z_1, \ldots, z_n; t) \right) \Bigg|_{z_1 = \cdots = z_n = z(x)}$$

gives a WKB solution of the (scalar version of) Jimbo-Miwa isomonodromy system for (P_{II}):

$$\left\{ \left(\hbar \frac{\partial}{\partial x} \right)^2 + f(x,t,\hbar) \left(\hbar \frac{\partial}{\partial x} \right) + g(x,t,\hbar) \right\} \psi(x,t,\hbar) = 0$$

$$f = O(\hbar), \quad g = (x - q_0)^2 \left(x + q_0 + \sqrt{-\theta/q_0} \right) \left(x + q_0 - \sqrt{-\theta/q_0} \right) + O(\hbar)$$

Contents

Exact WKB analysis and cluster algebras [I-Nakanishi], [Gaiotto-Moore-Neitzke]

$$\left(\hbar^2 \frac{d^2}{dx^2} - Q(x)\right)\psi(x,\hbar) = 0$$

$$\rightsquigarrow \psi(x,\hbar) = \exp\left(\hbar^{-1}\int^x \lambda(x,\hbar)dx\right) \quad : \quad \text{WKB (formal) solution}$$

$$\lambda(x,\hbar) = \sum_{k\geq 0} \hbar^k \lambda_k(x) = \sqrt{Q(x)} + O(\hbar)$$

Exact WKB analysis	Cluster algebras
(saddle-free) Stokes graph	skew-symmetric matirx B
mutation of Stokes graphs	mutation of B
(Borel sum of) Voros symbols $e^{W_{\beta_i}} = \exp\left(\hbar^{-1}\int_{\beta_i}(\lambda(x,\hbar) - \lambda_0(x))dx\right)$	cluster x-variables x_i
(Borel sum of) Voros symbols $e^{V_{\gamma_i}} = \exp\left(\hbar^{-1}\oint_{\gamma_i}\lambda(x,\hbar)dx\right)$	cluster y-variable y_i
$\exp\left(\hbar^{-1}\oint_{\gamma_i}\lambda_0(x)dx\right)$	coefficients
Stokes phenomenon for Voros symbols	mutation of cluster variables

Weber equation ($\leftrightarrow A_1$-cluster algebra)

$$\left(\hbar^2 \frac{d^2}{dx^2} - Q(x)\right)\psi(x,\hbar) = 0, \qquad Q(x) = \frac{x^2}{4} - \theta.$$

Theorem [Voros 83, Takei 07]

$$V_\gamma = 2\pi i \theta \cdot \hbar^{-1}, \quad W_\beta = \sum_{g\geq 1} \frac{(1-2^{1-2g})\cdot B_{2g}}{2g(2g-1)} \cdot \left(\frac{\hbar}{\theta}\right)^{2g-1}$$

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- The Borel transform (i.e., $\hbar^k \mapsto y^{k-1}/(k-1)!$) of W_β is

$$\sum_{g\geq 1} \frac{(1-2^{1-2g})\cdot B_{2g}}{2g(2g-1)\theta^{2g-1}} \cdot \frac{y^{2g-2}}{(2g-2)!} = \frac{1}{2y}\cdot\left(\frac{1}{e^{y/2\theta}-1} + \frac{1}{e^{y/2\theta}+1} - \frac{2\theta}{y}\right) =: f(y)$$

- When $\theta \notin i\mathbb{R}$, the Borel sum of W_β is computable explicitly:

$$\mathcal{S}[W_\beta] = \int_0^\infty e^{-y/\hbar} f(y)dy = \log\left(\frac{\sqrt{2\pi}}{\Gamma(\theta/\hbar + 1/2)}\right) + \frac{\theta}{\hbar}\left(\log\left(\frac{\theta}{\hbar}\right) - 1\right). \quad (\mathrm{Re}\,\theta > 0)$$

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- The Borel transform (i.e., $\hbar^k \mapsto y^{k-1}/(k-1)!$) of W_β is

$$\sum_{g \geq 1} \frac{(1 - 2^{1-2g}) \cdot B_{2g}}{2g(2g-1)\theta^{2g-1}} \cdot \frac{y^{2g-2}}{(2g-2)!} = \frac{1}{2y} \cdot \left(\frac{1}{e^{y/2\theta} - 1} + \frac{1}{e^{y/2\theta} + 1} - \frac{2\theta}{y}\right) =: f(y)$$

- When $\theta \notin i\mathbb{R}$, the Borel sum of W_β is computable explicitly:

$$\mathcal{S}[W_\beta] = \int_0^\infty e^{-y/\hbar} f(y) dy = \log\left(\frac{\sqrt{2\pi}}{\Gamma(\theta/\hbar + 1/2)}\right) + \frac{\theta}{\hbar}\left(\log\left(\frac{\theta}{\hbar}\right) - 1\right). \quad (\mathrm{Re}\, \theta > 0)$$

- Poles $y = 2m\pi i \theta$ ($m \in \mathbb{Z}_{\neq 0}$) of $f(y)$ cause ambiguity (Stokes phenomenon).

Stokes phenomenon = cluster mutation

For $\theta \in i\mathbb{R}_{>0}$:

$$\begin{aligned}
\mathcal{S}_-[W_\beta] - \mathcal{S}_+[W_\beta] &= \int_0^{\infty e^{-i\epsilon}} e^{-y/\hbar} f(y)dy - \int_0^{\infty e^{+i\epsilon}} e^{-y/\hbar} f(y)dy \\
&= \sum_{m=1}^{\infty} 2\pi i \operatorname*{Res}_{y=-2m\pi i\theta} e^{-y/\hbar} f(y)dy \\
&= \sum_{m=1}^{\infty} \frac{(-1)^m}{m} e^{2m\pi i\theta/\hbar} = \log(1 + e^{2\pi i\theta/\hbar})
\end{aligned}$$

Thus we have

$\mathcal{S}_-[e^{W_\beta}] = \mathcal{S}_+[e^{W_\beta}] \cdot (1 + e^{V_\gamma})$: (the simplest) cluster transformation.

Stokes phenomenon = cluster mutation

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Thank you for your attention !