Equivariant Verlinde algebra for Higgs bundles

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Verlinde formulae

- C smooth complex projective curve of genus g
- fix rank $n \in \mathbb{Z}_{>0}$, degree $d \in \mathbb{Z}$ and level $k \in \mathbb{Z}_{>0}$
- $\mathcal{N}_n^d$ moduli space of semi-stable rank n fixed degree d vector bundles on C; projective and smooth when $(d, n) = 1$
- $L \in \mathrm{Pic}(\mathcal{N}_n^d) \cong \mathbb{Z}$ ample generator of Picard group
- Verlinde formula (1988) for $\dim H^0(\mathcal{N}_n^d; L^k) = \chi(\mathcal{N}_n^d, L^k)$
- e.g. for $n = 2$ $d = 1$
  $\dim H^0(\mathcal{N}_2^1, L^k) = \sum_{j=1}^{2k+1} (-1)^{j+1} \left(\frac{k+1}{sin^2(\frac{j\pi}{2k+2})} \right)^{g-1} =$
  $\frac{1}{2} \mathrm{Res}_{z=1} \frac{(4k+4)^g}{\left(z^{k+1} - z^{-(k+1)}\right)((1-1/z)(1-z))^{g-1}} \frac{dz}{z}$
- proved for
  - $k = 1$ by (Beauville–Narasimhan–Ramanan 1988)
  - $n = 2$ by (Szenes, Bertram–Szenes 1993)
  - $\vdots$
  - in all generality by (Teleman–Woodward, 2009)

Verlinde algebra

- Verlinde formula = partition function of a $1+1D$ TQFT
- $1+1D$ TQFT determined by a Frobenius algebra
  i.e. finite dimensional comm. $\mathbb{C}$-algebra + symmetric pairing
- $R := R(\mathrm{SU}_n)) \cong$ character ring of SU_n
- $R \cong R(T_n)^{S_n} \cong (\mathbb{Z}[z_1, \dots, z_n]/(z_1 \cdots z_n - 1))^{S^n}$
  irrep $\chi_\lambda \in Irr(\mathrm{SU}_n) \leftrightsquigarrow s_\lambda \in R$ Schur function
  $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0) \in \mathbb{Z}^{n-1}$
- $Ver_n^k := \mathbb{C} \otimes_{\mathbb{Z}} R/(s_{(k+1)}, s_{(k+2)}, \dots, s_{(k+n-1)})$ has basis $\{s_\lambda\}_{\lambda_1 \leq k}$
- declaring $\langle s_\lambda, s_{\eta^\dagger} \rangle = \delta_{\lambda\eta} \rightsquigarrow$ non-degenarate pairing

Theorem (Goodman-Wenzl 1990; Gepner 1991; Witten 1993)

$(Ver_n^k, \langle , \rangle)$ is a Frobenius algebra (i.e. $\langle a, bc \rangle = \langle ab, c \rangle$).
$\cong$ Verlinde algebra, with partition function giving Verlinde formulae

- geometrization: Borel-Weil-Bott theory $\rightsquigarrow$
  $\chi_{T_n}(\mathcal{F}; L_\lambda) = s_\lambda \in R(T_n)^{S_n}$ on flag variety $\mathcal{F} := \mathrm{SU}_n/T_n$

Equivariant Verlinde formulae

- $\mathcal{M}_n^d \supset T^*\mathcal{N}_n^d$ moduli ss rank n fixed degree d Higgs bundles
- i.e. Higgs bundles of the form (E,ϕ) where $rank(E)=n$, $\det(E)=\Lambda$, $deg(\Lambda)=d$ and $\phi\in End_0(E)\otimes K$
- $\mathbb{T}:=\mathbb{C}^\times$ acts on $\mathcal{M}_n^d$ by scaling Higgs field
- $L\in \mathrm{Pic}(\mathcal{M}_n^d)$ ample generator with $\mathbb{T}$ action trivial on $L^k|_{\mathcal{N}_n^d}$
- $\mathbb{T}$ acts on $H^0(\mathcal{M}_n^d,L^k)$ with weights ≥ 0
- $\mathrm{grdim}(H^0(\mathcal{M}_n^d,L^k))=\sum_{i=0}^\infty \dim(H^0(\mathcal{M}_n^d,L^k)^i)t^i\in\mathbb{Z}[[t]]$
- (Paradan 2011) $\rightsquigarrow$ $\chi_\mathbb{T}(\mathcal{M}_n^d,L^k)=\sum_{F_i}\int_{F_i} ch_\mathbb{T}(L^k|_{F_i}\otimes \mathrm{Sym}_t N^*F_i)Todd(TF_i)$
- $F_i\subset(\mathcal{M}_n^d)^\mathbb{T}$ fixed point components
- (Hausel–Szenes, 2003) direct computation $\rightsquigarrow$ $\chi_\mathbb{T}(\mathcal{M}_2^1,L^k)=$
$$\sum_{a=1,t,1/t}\operatorname*{Res}_{z=a}\frac{\frac{2^{2g-1}}{(1-t)^{g-1}}\left[k+1+\frac{zt}{1-zt}+\frac{t/z}{1-t/z}\right]^g}{\left[z^{k+1}\frac{1-t/z}{1-tz}-z^{-(k+1)}\frac{1-tz}{1-t/z}\right][(1-1/z)(1-z)(1-t/z)(1-tz)]^{g-1}}\frac{dz}{z},$$
- (Hausel–Szenes, 2003) conjecture for higher n
- recently (Halpern-Leistner 2016) and (Andersen–Gukov–Pei 2016) gave formulas for $\chi_\mathbb{T}(\mathcal{M}_G,L^k)$ for general G building on the work of (Teleman–Woodward 2009)

Equivariant Verlinde algebra for Higgs bundles

- by physics arguments (Gukov, Pei 2015)
  $\chi_{\mathbb{T}}(\mathcal{M}_n^d, L^k)$ arises from a $1+1$D TQFT, i.e. a Frobenius algebra, dubbed *equivariant Verlinde algebra*
- we construct this algebra explicitly:
  t-deforming Goodman-Wenzl presentation of Ver_n^k
- step 1: find t-deformation $R_t(\mathrm{SU}_n)$ of $R(\mathrm{SU}_n)$ and of basis s_λ
- step 2: find deformed ideal $\mathrm{I_t}$ in the deformation
  $\rightsquigarrow QVer_n^k := \overline{\mathbb{C}(t)} \otimes_{\mathbb{Z}(t)} R_t(\mathrm{SU}_n)/\mathrm{I_t}$
- step 3: define pairing $\langle , \rangle_t$ and show it yields symmetric algebra $(QVer_n^k, \langle , \rangle_t)$
- step 4: check partition function giving $\chi_{\mathbb{T}}(\mathcal{M}_n^d, L^k)$
  and compare with properties described in
  (Gukov–Pei, 2015), (Gukov–Pei–Yan–Ye 2016),
  (Andersen–Gukov–Pei, 2016)

step 1. Definition of $R_t(\mathrm{SU}_n)$ and χ_λ^t

- recall geometrisation of $R(\mathrm{SU}_n)$:
  $\chi_{T_n}(\mathcal{F}; L_\lambda) = \chi_\lambda \in R(T_n)^{S_n}$ on flag variety $\mathcal{F} := \mathrm{SU}_n/T_n$
- consider $T^*\mathcal{F}$ and $L_\lambda := \pi^* L_\lambda \in \mathrm{Pic}_{\mathbb{T}}(T^*\mathcal{F})$
- $\chi_\lambda^t := \chi_{\mathrm{T}_n \times \mathbb{T}}(T^*\mathcal{F}, L_\lambda) \in R(\mathrm{T}_n \times \widehat{\mathbb{T}})^{S_n} \cong R(\mathrm{SU}_n)[[t]] =: R_t(\mathrm{SU}_n)$
- χ_λ^t computed by (Gupta/Brylinski 1987):
  $\chi_\lambda^t = E_\lambda \in R(T_n)^{S_n}[[t]]$ can be obtained
  $E_\lambda = t_\lambda(t) P_\lambda / \psi_t$
  $P_\lambda = \frac{\sum_{w \in S_n} (-1)^{\sigma(w)} z^{w(\lambda)} w(\Delta_t)}{\Delta_1 t_\lambda(t)} \in R(\mathrm{SU}_n)[t]$ Hall-Littlewood
  $\Delta_t = z^\rho \prod_{\alpha \in \Phi^-} (1 - tz^\alpha)$; $\psi_t = \prod_{\alpha \in \Phi} (1 - tz^\alpha)$
  $t_\lambda(t) = \sum_{w \in St_{S_n}(\lambda)} t^{l(w)}$

Theorem (Gupta 1987)

$$\langle P_\lambda, E_{\eta^\dagger} \rangle = \frac{1}{n!} \mathrm{Res}_{z=0} \frac{E_\lambda \psi_t}{t_\lambda(t)} E_{\eta^\dagger} \psi_1 \frac{dz}{z} = \delta_{\lambda\eta}$$

step 2. Definition of $QVer_n^k$

- for $\alpha \in \Phi$ define $b_\alpha := z^{(k+n)\alpha} \prod_{\beta\in\Phi}(1 - tz^\beta)^{-\langle\alpha,\beta\rangle} \in R(T_n)(t)$ $\rightsquigarrow$ non-symmetric Bethe-Ansatz equation $b_\alpha = 1$
- we have $b_{\alpha+\beta} = b_\alpha b_\beta$ and $b_{w\alpha} = w(b_\alpha)$
- thus $\mathrm{I}'_t := (1 - b_{\alpha_1}, \dots, 1 - b_{\alpha_{n-1}}) = (1 - b_\alpha)_{\alpha\in\Phi} \lhd R(T_n)(t)$
- then $\mathrm{Spec}(\mathbb{F}R(T_n)/\mathrm{I}'_t) \subset T_n(\mathbb{F})$ with $\mathbb{F} := \overline{\mathbb{C}(t)}$ i.e. the solutions of the Bethe-Ansatz equations $b_{\alpha_1} = 1, \dots, b_{\alpha_{n-1}} = 1$ in $T_n(\mathbb{F})$ are invariant under S_n
- $\mathrm{Spec}_n^k := \mathrm{Spec}\,(\mathbb{F}R(T_n)/\mathrm{I}'_t) \setminus \mathrm{Spec}\,(\mathbb{F}R(T_n)/(1 - z^\alpha)_{\alpha\in\Phi})$
- for $\lambda = (k + 1 = \lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0)$ form $B_\lambda = \frac{\sum_{w\in S_n}(-1)^{\sigma(w)} z^{w(\lambda)}(1-tz^{w(-\theta)})w(\Delta_t)}{\Delta_1 t_\lambda(t)} \in R(T_n)^{S_n}[t]$ symmetric Bethe-Ansatz polynomial then $B_\lambda\Delta_1 \in \mathrm{I}'_t$
- $\lambda_m := (k + 1, 1, \dots, 1, 0, \dots, 0) = k\omega_1 + \omega_m$ $\mathrm{I}_t := (B_{\lambda_1}, \dots, B_{\lambda_{n-1}}) \lhd R(T_n)^{S_n}[t]$
- $QVer_n^k := \mathbb{F}R(T_n)^{S_n}[t]/\mathrm{I}_t$
- then $\mathrm{Spec}_n^k/S_n \subset \mathrm{Spec}(QVer_n^k) \subset T_n(\mathbb{F})/S_n$

step 3. $(QVer_n^k, \langle , \rangle_t)$ is a Frobenius algebra

- $\langle E_\lambda, E_{\eta^\dagger}\rangle_t := \delta_{\lambda\eta}\tilde{t}_\lambda(t)(1-t)^{n-1}$, $\tilde{t}_\lambda(t) = \sum_{w\in St_{\tilde{S}_n^k}(\lambda)} t^{l(w)}$

Theorem (Hausel–Szenes 2016)

$(E_\lambda)_{\lambda_1\le k}$ is a basis of $QVer_n^k$, and $\langle a, bc\rangle_t = \langle ab, c\rangle_t$

- proof: assume $l_1(\lambda), l_1(\eta) \le k$ and denote
  $r = \tau_{k\omega_1}\circ w_{(1,2,\dots,n)} : \mathbb{Z}^{n-1}\to\mathbb{Z}^{n-1}$ then prove:
  $\langle E_\lambda, E_{\eta^\dagger}\rangle_t/(1-t)^{n-1}$
  $=\frac{1}{n!}\sum_{a\in \mathrm{Spec}_n^k}\frac{E_\lambda(a)E_{\eta^\dagger}(a)\psi_t(a)\psi_1(a)}{Jac(\log(b_{\alpha_1}),\dots,\log(b_{\alpha_{n-1}}))(a)}$
  $=\frac{(-1)^{n-1}}{n!}\sum_{a\in \mathrm{Spec}_n^k}\operatorname{Res}_{z=a}\frac{E_\lambda E_{\eta^\dagger}\psi_t\psi_1}{(1-b_{\alpha_1})\cdots(1-b_{\alpha_{n-1}})}\frac{dz}{z}$
  $=\frac{(-1)^{n-1}}{n!}\sum_{a\in \mathrm{Spec}_n^k}\operatorname{Res}_{z=a}\frac{E_{r(\lambda)}E_{r(\eta)^\dagger}\psi_t\psi_1}{(1-b_{\alpha_1})\cdots(1-b_{\alpha_{n-1}})}\frac{dz}{z}$
  observe for some r^m no residue at infinity $\Rightarrow$
  $=\frac{1}{n!}\operatorname{Res}_{z=0}\frac{E_{r^m(\lambda)}E_{r^m(\eta)^\dagger}\psi_t\psi_1}{(1-b_{\alpha_1})\cdots(1-b_{\alpha_{n-1}})}\frac{dz}{z}\overset{Gupta}{=}\delta_{r^m(\lambda)r^m(\eta)}t_{r^m(\lambda)}(t)=\delta_{\lambda\eta}\tilde{t}_\lambda(t)$
- $\Rightarrow |(\lambda)_{\lambda_1\le k}| = \binom{n+k-1}{n-1} \le \dim R_t/I_t \overset{Bezout}{\le} \binom{n+k-1}{n-1}$ ■

step 4. Partition function and other checks

- $\Rightarrow$ rotation on $QVer_n^k$ corresponds to multiplying by $\tilde{P}_{k\omega_1} = P_{k\omega_1} t_{k\omega_1}(t)/\tilde{t}_{k\omega_1}(t)$, and $\tilde{P}^d_{k\omega_1} = \tilde{P}_{k(\omega_1+\cdots+\omega_d)} = \tilde{P}_{k\omega_{1d}}$
- $Tr := QVer_n^k \to \mathbb{F}$ by
  $Tr(E) := \frac{1}{n!}\sum_{a\in\mathrm{Spec}_n^k} \frac{E(a)\psi_t(a)\psi_1(a)(1-t)^{n-1}}{Jac\left(\log(b_{\alpha_1}),\ldots,\log(b_{\alpha_{n-1}})\right)}$
  then $\Rightarrow \langle E_1, E_2\rangle_t = Tr(E_1 E_2)$
- we have $Z_n^k(C_g^d) =$

$$= \frac{1}{n!}\sum_{a\in\mathrm{Spec}_n^k} \tilde{P}_{k\omega_{1d}}(a)\left(\frac{Jac\left(\log(b_{\alpha_1}),\ldots,\log(b_{\alpha_{n-1}})\right)(a)}{\psi_t(a)\psi_1(a)(1-t)^{n-1}}\right)^{g-1}$$

$$= \frac{(-1)^{n-1}}{n!}\sum_{a\in\mathrm{Spec}_n^k} \operatorname*{Res}_{z=a} \frac{\tilde{P}_{k\omega_{1d}} Jac\left(\log(b_{\alpha_1}),\ldots,\log(b_{\alpha_{n-1}})\right)^g}{(1-b_{\alpha_1})\cdots(1-b_{\alpha_{n-1}})(\psi_t\psi_1(1-t)^{n-1})^{g-1}}\frac{dz}{z}$$

$$"=" \frac{1}{n!}\sum_{a\in V(\psi_t\psi_1)} \operatorname*{Res}_{z=a} \frac{\tilde{P}_{k\omega_{1d}} Jac\left(\log(b_{\alpha_1}),\ldots,\log(b_{\alpha_{n-1}})\right)^g}{(1-b_{\alpha_1})\cdots(1-b_{\alpha_{n-1}})(\psi_t\psi_1(1-t)^{n-1})^{g-1}}\frac{dz}{z}$$

- when $n=2$, $d=1$ this agrees with (Hausel–Szenes 2003)

$$\sum_{a=1,t,1/t} \operatorname*{Res}_{z=a} \frac{\frac{2^{2g-1}}{(1-t)^{g-1}}\left[k+1+\frac{t/z}{1-t/z}+\frac{tz}{1-tz}\right]^g}{\left[z^{k+1}\frac{1-t/z}{1-tz} - z^{-(k+1)}\frac{1-tz}{1-t/z}\right][(1-1/z)(1-z)(1-t/z)(1-tz)]^{g-1}}\frac{dz}{z}$$

- $QVer_2^k$ and $QVer_3^k$ matches (Gukov–etal, 2015,2016)

Problems

- We can analogously define $(QVer_n^k, \langle , \rangle_t)$ for any compact simply-connected semisimple Lie group G. Is it a Frobenius algebra?
- (Korff 2011) constructs a deformation of Ver_n^k in another presentation. Is it isomorphic with our $QVer_n^k$?
- We understand how to read off $\chi(\mathcal{M}_\mu, L^k)$ for parabolic Higgs bundles $\mathcal{M}_\mu$ from $QVer_n^k$. How to include irregular singularities?
- Hall–Littlewood polynomials deform to Macdonald polynomials. Is there a corresponding further deformation of $QVer_n^k$? What does it compute?
- can we enhance our $1+1$D TQFT to a $2+1$D TQFT deforming the Jones–Witten TQFT?
- Is there a representation theory of deformations of affine Kac-Moody algebras or Hecke algebras/quantum groups at root of unity behind $QVer_n^k$?
- Can we relate $\chi_{\mathbb{T}}(\mathcal{M}_n^d, L^k)$ along the Hitchin map to the abelianization program of (Atiyah–Hitchin 1987)?