MODEL COMPLETENESS OF GENERIC GRAPHS IN RATIONAL CASES

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ABSTRACT. Let $\mathbf{K}_f$ be an ab initio amalgamation class with an unbounded increasing concave function f. We show that if the predimension function has a rational coefficient and f satisfies a certain assumption then the generic structure of $\mathbf{K}_f$ has a model complete theory.

Hrushovski's amalgamation construction, model completeness 03C10, 03C13, 03C25, 03C30

1. INTRODUCTION

Generic structures constructed by the Hrushovski's amalgamation construction are known to have theories which are nearly model complete. If an amalgamation class has the full amalgamation property then its generic structure has a theory which is not model complete [2]. On the other hand, Hrushovski's strongly minimal structure constructed by the amalgamation construction, refuting a conjecture of Zilber has a model complete theory [5].

We have shown that the generic structure of $\mathbf{K}_f$ for 3-hypergraphs with a coefficient 1 for the predimension function has a model complete theory under some assumption on f [8].

In this paper, we show a similar result for binary graphs with a rational coefficient less than 1 for the predimension function. We have already shown this result for the predimension function with coefficient 1/2 [9]. We treat the general case here.

We essentially use notation and terminology from Baldwin-Shi [3] and Wagner [11]. We also use some terminology from graph theory [4].

For a set X, $[X]^n$ denotes the set of all subsets of X of size n, and $|X|$ the cardinality of X.

We recall some of the basic notions in graph theory we use in this paper. These appear in [4]. Let G be a graph. $V(G)$ denotes the set of vertices of G and $E(G)$ the set of edges of G. $E(G)$ is a subset of $[V(G)]^2$. For $a,b \in V(G)$, ab denotes $\{a,b\}$. $|G|$ denotes $|V(G)|$. The *degree* of a vertex v is the number of edges at v. A vertex of degree 0 is *isolated*. A vertex of degree 1 is a *leaf*. G is a *path* $x_0x_1\dots x_k$ if $V(G) = \{x_0,x_1,\dots,x_k\}$ and $E(G) = \{x_0x_1, x_1x_2,\dots,x_{k-1}x_k\}$ where the x_i are all distinct. x_0 and x_k are *ends* of G. The number of edges of a path is its *length*. A path of length 0 is a single vertex. G is a *cycle* $x_0x_1\dots x_{k-1}x_0$ if $k \geq 3$, $V(G) = \{x_0,x_1,\dots,x_{k-1}\}$ and $E(G) = \{x_0x_1, x_1x_2,\dots,x_{k-2}x_{k-1}, x_{k-1}x_0\}$ where the x_i are all distinct. The number of edges of a cycle is its *length*. A non-empty graph G is *connected* if any two of its vertices are linked by a path in G. A *connected component* of a graph G is a maximal connected subgraph of G. A *forest* is a graph not containing any cycles. A *tree* is a connected forest.

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To see a graph G as a structure in the model theoretic sense, it is a structure in language $\{E\}$ where E is a binary relation symbol. $V(G)$ will be the universe, and $E(G)$ will be the interpretation of E. The language $\{E\}$ will be called *the graph language*.

Suppose A is a graph. If $X \subseteq V(A)$, $A|X$ denotes the substructure B of A such that $V(B) = X$. If there is no ambiguity, X denotes $A|X$. We usually follow this convention. $B \subseteq A$ means that B is a substructure of A. A substructure of a graph is an induced subgraph in graph theory. $A|X$ is the same as $A[X]$ in Diestel's book [4].

We say that X is *connected* in A if X is a connected graph in the graph theoretical sense [4]. A maximal connected substructure of A is a *connected component* of A.

Let A, B, C be graphs such that $A \subseteq C$ and $B \subseteq C$. AB denotes $C|(V(A) \cup V(B))$, $A \cap B$ denotes $C|(V(A) \cap V(B))$, and $A - B$ denotes $C|(V(A) - V(B))$. If $A \cap B = \emptyset$, $E(A,B)$ denotes the set of edges xy such that $x \in A$ and $y \in B$. We put $e(A,B) = |E(A,B)|$. $E(A,B)$ and $e(A,B)$ depend on the graph in which we are working. When we are working in a graph G, we sometimes write $E_G(A,B)$ and $e_G(A,B)$ respectively.

Let D be a graph and A, B, and C substructures of D. We write $D = B \otimes_A C$ if $D = BC$, $B \cap C = A$, and $E(D) = E(B) \cup E(C)$. $E(D) = E(B) \cup E(C)$ means that there are no edges between $B - A$ and $C - A$. D is called a *free amalgam of B and C over A*. If A is empty, we write $D = B \otimes C$, and D is also called a *free amalgam of B and C*.

Definition 1. Let α be a real number such that $0 < \alpha < 1$.

(1) For a finite graph A, we define a predimension function δ by $\delta(A) = |A| - \alpha|E(A)|$.
(2) Let A and B be substructures of a common graph. Put $\delta(A/B) = \delta(AB) - \delta(B)$.

Definition 2. Let A and B be graphs with $A \subseteq B$, and suppose A is finite.

$A \leq B$ if whenever $A \subseteq X \subseteq B$ with X finite then $\delta(A) \leq \delta(X)$.
$A < B$ if whenever $A \subsetneq X \subseteq B$ with X finite then $\delta(A) < \delta(X)$.
We say that A is *closed* in B if $A < B$.

If α is irrational then $\leq$ and $<$ are the same relations, but they are different if α is a rational number. Our relation $<$ is often denoted by $\leqslant$ in the literature and some people use $\leq^*$ for our $<$. Since we want to use the relation $\leq$ as well, we use the symbol $<$ for the closed substructure relation.

Let $\mathbf{K}_\alpha$ be the class of all finite graphs A such that $\emptyset < A$.

The following facts appear in [3, 11, 12].

Fact 1. *Let A, B, C be finite substructures in a common graph.*

(1) *If $A \cap C$ is empty then $\delta(A/C) = \delta(A) - \alpha e(A,C)$.*
(2) *If $A \cap C$ is empty and $B \subseteq C$ then $\delta(A/B) \geq \delta(A/C)$.*
(3) *$A \leq B$ if and only if $\delta(X/A) \geq 0$ for any $X \subseteq B$.*
(4) *$A < B$ if and only if $\delta(X/A) > 0$ for any $X \subseteq B$ with $X - A$ non-empty.*
(5) *$A \leq A$.*
(6) *If $A \leq B$ then $A \cap C \leq B \cap C$.*
(7) *If $A \leq B$ and $B \leq C$ then $A \leq C$.*
(8) *If $A \leq C$ and $B \leq C$ then $A \cap B \leq C$.*
(9) *$A < A$.*
(10) *If $A < B$ then $A \cap C < B \cap C$.*
(11) *If $A < B$ and $B < C$ then $A < C$.*
(12) *If $A < C$ and $B < C$ then $A \cap B < C$.*

Proof. (1), (3), (4), (5) and (9) are immediate from the definitions.

(2) Suppose $B \subseteq C$ and $A \cap C$ is empty. It is clear that $E(A,B) \subseteq E(A,C)$. Therefore, the statement follows from (1).

Proofs of (6) and (10) are similar. We show (10). Suppose $A < B$. If $A \cap C = B \cap C$ then $A \cap C < B \cap C$ by (9). Suppose $A \cap C \subsetneq B \cap C$. Let X be a graph with $A \cap C \subsetneq X \leq B \cap C$. Put $X_1 = X - A$. Then $\delta(X/A \cap C) = \delta(X_1/A \cap C)$ by Definition 1 (2). We have $\delta(X_1/A \cap C) \geq \delta(X_1/A)$ by (2). Since X_1 is non-empty, we also have $\delta(X_1/A) > 0$ by the assumption $A < B$ and (4). Therefore, $\delta(X/A \cap C) > 0$.

Proofs of (7) and (11) are similar. We show (11). Suppose $A < B$ and $B < C$. Let X be a graph with $A \subsetneq X \subseteq C$. We have $A < X \cap B < X$ by (10). Since $A \subsetneq X$, we have $A \subsetneq X \cap B$ or $X \cap B \subsetneq X$. Hence $\delta(A) < \delta(X \cap B)$ or $\delta(X \cap B) < \delta(X)$. Therefore, $\delta(A) < \delta(X)$ anyway.

(8) follows from (6) and (7). (12) follows from (10) and (11). □

Fact 2. *Let $D = B \otimes_A C$.*

1. $\delta(D/A) = \delta(B/A) + \delta(C/A)$.
2. *If $A \leq C$ then $B \leq D$.*
3. *If $A \leq B$ and $A \leq C$ then $A \leq D$.*
4. *If $A < C$ then $B < D$.*
5. *If $A < B$ and $A < C$ then $A < D$.*

Proof. (1) By Definition 1 (2), $\delta(D/A) = \delta(D/C) + \delta(C/A) = \delta(B/C) + \delta(C/A)$. Let $B' = B - C = B - A$. Then $E(B',C) = E(B',A)$ since $D = B \otimes_A C$. By Fact 1 (1), we have $\delta(B/C) = \delta(B') - \alpha e(B',C) = \delta(B') - \alpha e(B',A) = \delta(B'/A) = \delta(B/A)$.

(4) Suppose $A < C$. Let U be a graph with $B \subsetneq U \subseteq D$. Then $U = B \otimes_A (U \cap C)$. Put $U' = U - B = U \cap (C - A)$. U' is a substructure of $C - A$ and non-empty. We have $\delta(U'/A) > 0$ by $A < C$. Also, $E(U',B) = E(U',A)$ by $BC = B \otimes_A C$. We have $\delta(U'/B) = \delta(U') - \alpha e(U',B) = \delta(U') - \alpha e(U',A) = \delta(U'/A) > 0$.

(5) follows from (4) and the transitivity of $<$. (2) and (3) can be shown similarly. □

Lemma 1. (1) *Let A, B, C and D be graphs with $D = B \otimes C$ and $A \subseteq D$. Then $\delta(D/A) = \delta(B/A \cap B) + \delta(C/A \cap C)$.*

(2) *Let D be a graph and A a substructure of D. Let $\{D_1, D_2, \ldots, D_k\}$ be the set of all connected components of D where the D_i are all distinct. Then*

$$\delta(D/A) = \sum_{i=1}^{k} \delta(D_i/A \cap D_i).$$

Proof. (1) Put $B' = B - A$, and $C' = C - A$. By Fact 1 (1), $\delta(D/A) = \delta(B'C'/A) = \delta(B'C') - \alpha e(B'C',A)$. Since $B'C' = B' \otimes C'$, we have

$$\delta(B'C') = \delta(B') + \delta(C') \text{ and } e(B'C',A) = e(B',A) + e(C',A).$$

Since there are no edges between B and C,

$$e(B',A) = e(B',A \cap B) \text{ and } e(C',A) = e(C',A \cap C).$$

Hence,

$$\begin{aligned} \delta(D/A) &= \delta(B') + \delta(C') - \alpha e(B',A \cap B) - \alpha e(C',A \cap C) \\ &= \delta(B/A \cap B) + \delta(C/A \cap C). \end{aligned}$$

(2) D is a free amalgam of the all connected components of D. The statement follows from (1). □

Let B, C be graphs and $g : B \to C$ a graph embedding. g is a *closed embedding* of B into C if $g(B) < C$. Let A be a graph with $A \subseteq B$ and $A \subseteq C$. g is a *closed embedding over* A if g is a closed embedding and $g(x) = x$ for any $x \in A$.

In the rest of the paper, $\mathbf{K}$ denotes a class of finite graphs closed under isomorphisms.

Definition 3. Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$. $(\mathbf{K}, <)$ has the *amalgamation property* if for any finite graphs $A, B, C \in \mathbf{K}$, whenever $g_1 : A \to B$ and $g_2 : A \to C$ are closed embeddings then there is a graph $D \in \mathbf{K}$ and closed embeddings $h_1 : B \to D$ and $g_2 : C \to D$ such that $h_1 \circ g_1 = h_2 \circ g_2$.

$\mathbf{K}$ has the *hereditary property* if for any finite graphs A, B, whenever $A \subseteq B \in \mathbf{K}$ then $A \in \mathbf{K}$.

$\mathbf{K}$ is an *amalgamation class* if $\emptyset \in \mathbf{K}$ and $\mathbf{K}$ has the hereditary property and the amalgamation property.

A countable graph M is a *generic structure* of $(\mathbf{K}, <)$ if the following conditions are satisfied:

(1) If $A \subseteq M$ and A is finite then there exists a finite graph $B \subseteq M$ such that $A \subseteq B < M$.
(2) If $A \subseteq M$ then $A \in \mathbf{K}$.
(3) For any A, $B \in \mathbf{K}$, if $A < M$ and $A < B$ then there is a closed embedding of B into M over A.

Let A be a finite structure of M. By Fact 1 (12), there is a smallest B satisfying $A \subseteq B < M$, written $\mathrm{cl}(A)$. The set $\mathrm{cl}(A)$ is called a *closure* of A in M.

Fact 3. [3, 11, 12] *Let $(\mathbf{K}, <)$ be an amalgamation class. Then there is a generic structure of $(\mathbf{K}, <)$. Let M be a generic structure of $(\mathbf{K}, <)$. Then any isomorphism between finite closed substructures of M can be extended to an automorphism of M.*

Definition 4. Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$. A graph $A \in \mathbf{K}$ is *absolutely closed* in $\mathbf{K}$ if whenever $A \subseteq B \in \mathbf{K}$ then $A < B$.

Note that the notion of being absolutely closed in $\mathbf{K}$ is invariant under isomorphisms.

Theorem 5. *Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$ and M a generic structure of $(\mathbf{K}, <)$. Assume that M is countably saturated. Suppose for any $A \in \mathbf{K}$ there is $C \in \mathbf{K}$ such that $A < C$ and C is absolutely closed in $\mathbf{K}$. Then the theory of M is model complete.*

Proof. Let T be the theory of M in the graph language. Since M is countably saturated, every finite type without parameters is realised in M. Our aim is to show that T is model compete.

Claim 1. *Every finite type realised in M is generated by a single existential formula of the graph language.*

Let A be a finite substructure of M. We show that $\mathrm{tp}(A)$ is generated by an existential formula. Consider the closure $\mathrm{cl}(A)$ of A in M. $\mathrm{cl}(A)$ is finite by the definition. By the assumption of the theorem, there is $B \in \mathbf{K}$ such that $\mathrm{cl}(A) < B$ and B is absolutely closed in $\mathbf{K}$. Since $\mathrm{cl}(A) < B$ and $\mathrm{cl}(A) < M$, we can embed B in M over $\mathrm{cl}(A)$ as a closed substructure of M. So, We can assume that $B \subseteq M$ and $\mathrm{cl}(A) < B < M$.

Suppose $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_m\}$. Let

$$\psi(x_1, \dots, x_n, y_1, \dots, y_m) = \mathrm{qftp}(a_1, \dots, a_n, b_1, \dots, b_m)$$

be a formula representing the quantifier-free type of (A, B). Then $(a_1, \dots, a_n)$ realises an existential formula $\exists y_1 \dots y_m \psi(x_1, \dots, x_n, y_1, \dots, y_m)$. Let $\varphi(x_1, \dots, x_n)$ denote this formula. We show that $\varphi(x_1, \dots, x_n)$ determines $\mathrm{tp}(a_1, \dots, a_n)$.

Let $c_1, \dots, c_n \in M$ be arbitrary. Assume that $(c_1, \dots, c_n)$ satisfies $\varphi(x_1, \dots, x_n)$. We show that $(c_1, \dots, c_n)$ realises $\mathrm{tp}(a_1, \dots, a_n)$.

There is $d_1, \dots, d_m \in M$ such that $M \models \psi(c_1, \dots, c_n, d_1, \dots, d_m)$. Then

$$\mathrm{qftp}(c_1, \dots, c_n, d_1, \dots, d_m) = \mathrm{qftp}(a_1, \dots, a_n, b_1, \dots, b_m).$$

Hence, there is a graph isomorphism σ_0 such that $\sigma_0(d_i) = b_i$ for $i = 1, \dots, m$ and $\sigma_0(c_i) = a_i$ for $i = 1, \dots, n$. Put

$$C = M|\{c_1, \dots, c_n\} \text{ and } D = M|\{d_1, \dots, d_m\}.$$

Then $\sigma_0 : D \to B$ is a graph isomorphism such that $\sigma_0|C$ is a graph isomorphism from C to A.

D is also absolutely closed in $\mathbf{K}$. Hence D is closed in M. Therefore, σ_0 can be extended to an graph automorphism σ of M by Fact 3. Hence, $\mathrm{tp}(c_1, \dots, c_n) = \mathrm{tp}(a_1, \dots, a_n)$. The claim is proved.

By the claim, every formula is equivalent to an existential formula modulo T. Therefore, T is model complete. □

Definition 6. Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$. $(\mathbf{K}, <)$ has the *free amalgamation property* if whenever $D = B \otimes_A C$ with $B, C \in \mathbf{K}$, $A < B$ and $A < C$ then $D \in \mathbf{K}$.

By Fact 2 (4), we have the following.

Fact 4. *Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$. If $(\mathbf{K}, <)$ has the free amalgamation property then it has the amalgamation property.*

Definition 7. Let $\mathbb{R}^+$ be the set of non-negative real numbers. Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing concave (convex upward) unbounded function. Assume that $f(0) = 0$, and $f(1) \le 1$. Define $\mathbf{K}_f$ as follows:

$$\mathbf{K}_f = \{A \in \mathbf{K}_\alpha \mid B \subseteq A \Rightarrow \delta(B) \ge f(|B|)\}.$$

Note that if $\mathbf{K}_f$ is an amalgamation class then the generic structure of $(\mathbf{K}_f, <)$ has a countably categorical theory [12].

The following is the main theorem.

Theorem 8. *Let $\alpha = m/d < 1$ with relatively prime positive integers m and d. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing concave unbounded function. Assume that $f(0) = 0$, $f(1) \le 1$, and $f(x) + 1/d \ge f(2x)$ for any positive integer x.*

Then $(\mathbf{K}_f, <)$ has the free amalgamation property and the theory of the generic structure of $(\mathbf{K}_f, <)$ is model complete.

In the rest of the paper, we assume that the assumption of Theorem 8 holds:

Assumption 9. (1) $\alpha = m/d < 1$ where m and d are relatively prime positive integers.
(2) $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing concave unbounded function.
(3) $f(0) = 0$, $f(1) \le 1$.
(4) $f(x) + 1/d \ge f(2x)$ for any positive integer x.

In order to discuss if a given graph is in $\mathbf{K}_f$ or not, the following definition will be convenient.

Definition 10. Let B be a graph and $c \ge 0$ an integer. B is *normal* to f if $\delta(B) \ge f(|B|)$. B is *c-normal* to f if $\delta(B) \ge f(|B| + c)$. B is *c-critical to* f if B is c-normal to f and c is maximal with this property.

The following three lemmas are immediate from the definitions and Assumption 9 above.

Lemma 2. *Let A be a finite graph.*

(1) *Suppose A is normal to f and non-empty. Then $\delta(A) > 0$.*
(2) *$A \in \mathbf{K}_f$ if and only if every substructure of A is normal to f.*
(3) *Let c and c' be integers such that $0 \le c \le c'$. If A is c'-normal to f then A is c-normal to f, and in particular, A is normal to f.*
(4) *Let A be normal to f. Let n be an integer such that $\delta(A) \ge f(n)$ but $\delta(A) < f(n+1)$. Such an n uniquely exists. Let $c = n - |A|$. Then A is c-critical to f. c is a unique integer u such that A is u-critical to f.*
(5) *Let B be another graph such that $\delta(A) = \delta(B)$, $|A| \le |B|$ and A and B are normal to f. Then B is c-critical to f if and only if A is $(|B| - |A| + c)$-critical to f.*

Proof. (1) Since A is non-empty, we have $0 < |A|$. By Assumption 9, $f(0) = 0$, and f is strictly increasing. Hence $\delta(A) \ge f(|A|) > 0$.

(2) By the definitions.

(3) Suppose A is c'-normal to f. Then $\delta(A) \ge f(|A| + c')$. Since $|A| + c' \ge |A| + c$ and f is strictly increasing, we have $f(|A| + c') \ge f(|A| + c)$. Hence, $\delta(A) \ge f(|A| + c)$. So, A is c-normal to f. Since $c' \ge 0$, A is 0-normal to f. This means that A is normal to f.

(4) Since A is normal to f, we have $\delta(A) \ge f(|A|)$. Since the function f is unbounded and increasing, there is an integer x such that $\delta(A) < f(x)$. Hence, we can choose an integer $n \ge |A|$ such that $\delta(A) \ge f(n)$ but $\delta(A) < f(n+1)$. Since f is strictly increasing, such an n is unique. Let $c = n - |A|$. Then $n = |A| + c$. Then $\delta(A) \ge f(n) = f(|A| + c)$ but $\delta(A) < f(x)$ for any $x \ge n + 1 = |A| + c + 1$. Therefore, A is c-critical to f. Such a c is also unique by (3).

(5) Since B is normal to f, we have $\delta(B) \ge f(|B|)$. Since $\delta(A) = \delta(B)$, $|A| \le |B|$, and f is strictly increasing, we have $\delta(A) \ge f(|B|) \ge f(|A|)$. Hence, A is also normal to f. Let n be the unique integer such that $\delta(A) \ge f(n)$ but $\delta(A) < f(n+1)$. Then B is $(n - |B|)$-critical to f and A is $(n - |A|)$-critical to f by (4). (5) holds because $n - |B| = c$ if and only if $n - |A| = |B| - |A| + c$. □

Lemma 3. *Recall that $\alpha = m/d < 1$ with relatively prime positive integers m and d. Let $B \in \mathbf{K}_f$. Suppose $|B| \ge m$, and B is c-critical to f with $0 \le c < m$. Then B is absolutely closed in $\mathbf{K}_f$.*

Proof. Suppose B is not absolutely closed in $\mathbf{K}_f$. Then there is a proper extension $B' \in \mathbf{K}_f$ of B with $\delta(B) \ge \delta(B')$.

If $\delta(B) > \delta(B')$ then $\delta(B) \ge \delta(B') + 1/d \ge f(|B'|) + 1/d \ge f(2|B'|)$. Since $m \le |B| \le |B'|$, B must be m-normal. But this contradicts the assumption that B is c-critical with $c < m$.

Otherwise, we have $\delta(B) = \delta(B')$. Let $k = |B' - B|$. Then $0 < k \le c < m$. We have $0 = \delta(B'/B) = \delta(B' - B) - \alpha e(B' - B, B) = k - l\alpha = k - l(m/d)$ for some integer $l \ge 0$. Hence, $m/d = k/l$ with $k < m$. But this is impossible because m and d are relatively prime. □

Lemma 4. *Let A, U be graphs such that $A \subseteq U$, $\delta(A) \le \delta(U)$, and A is $|U - A|$-normal to f. Then U is normal to f.*

Proof. $\delta(U) \ge \delta(A) \ge f(|A| + |U - A|) = f(|U|)$. □

Lemma 5. *Recall that $\alpha = m/d < 1$ with relatively prime positive integers m and d. Let $A = A' \otimes P$ where A' is non-empty and P consists of isolated points of A. Assume A' is normal to f.*

(1) *If $|P| \geq 2$ then A is $3m|A|$-normal to f.*
(2) *If $|P| = 1$ then A is $m|A|$-normal to f.*

Proof. Put $n = |P|$. We have $|A| = |A'| + n$ and $\delta(A) = \delta(A') + n \geq f(|A'|) + nd/d \geq f(2^{nd}|A'|)$.

(1) We have $n \geq 2$. We show that A is $5m|A|$-normal to f if $\alpha \neq 1/2$ and A is $3m|A|$-normal to f if $\alpha = 1/2$.

Assume $\alpha \neq 1/2$. Then $d \geq 3$. Hence $nd \geq 6$ and thus $2^{nd} > 10nd$. Therefore, $2^{nd}|A'| > 10nd|A'| \geq 10n(m+1)|A'| > 5(m+1)(|A'|+n) = 5(m+1)|A|$. Hence, $\delta(A) \geq f((5m+1)|A|)$. This means that A is $5m|A|$-normal to f.

Now, assume $\alpha = 1/2$. Then $m = 1$ and $d = 2$. $nd = 2n \geq 4$ since $n \geq 2$. Hence, $2^{nd}|A'| \geq 4nd|A'| = 8n|A'| > 4(|A'|+n) = 4|A| = (3m+1)|A|$. Hence, $\delta(A) > f((3m+1)|A|)$. This means that A is $3m|A|$-normal to f.

(2) Suppose $n = 1$. Since $d \geq 2$ and $2|A'| \geq |A|$, we have $2^d|A'| \geq 2d|A'| \geq (m+1)|A|$. Therefore, $\delta(A) \geq f(2^d|A'|) \geq f((m+1)|A|)$. This means that A is $m|A|$-normal to f. □

Lemma 6. (1) *Let $D = B \otimes_A C$ with $\delta(A) < \delta(B)$ and $\delta(A) < \delta(C)$. If B and C are normal to f then D is normal to f.*
(2) *Let $D = B \otimes C$. If B and C are normal to f then D is normal to f.*

Proof. (1) By symmetry, we can assume that $|C| \leq |B|$. Then $|D| \leq 2|B|$. Also, $\delta(D) = \delta(B) + \delta(C) - \delta(A) > \delta(B)$ since $\delta(C) - \delta(A) > 0$. Hence,

$$\delta(D) \geq \delta(B) + 1/d \geq f(|B|) + 1/d \geq f(2|B|) \geq f(|D|).$$

Therefore, D is normal to f.

(2) By Lemma 2, we have $\delta(B) > 0$ and $\delta(C) > 0$. We can apply (1) with $A = \emptyset$. □

Proposition 1. $(\mathbf{K}_f, <)$ *has the free amalgamation property. In particular, If $D = B \otimes C$ with $B, C \in \mathbf{K}_f$, then $D \in \mathbf{K}_f$.*

Proof. Let $D = B \otimes_A C$ with $B, C \in \mathbf{K}_f$, $A < B$ and $A < C$. Suppose $U \subseteq D$. If $U \subseteq B$ or $U \subseteq C$ then $U \in \mathbf{K}_f$ since $B, C \in \mathbf{K}_f$. Now, suppose that $U \not\subseteq B$ and $U \not\subseteq C$. Then $U = (U \cap B) \otimes_{U \cap A} (U \cap C)$, $\delta(U \cap B) > \delta(U \cap A)$, and $\delta(U \cap C) > \delta(U \cap A)$ by Fact 1 (10). $U \cap B$ and $U \cap C$ are normal to f since B and C are in $\mathbf{K}_f$. U is normal to f by Lemma 6. Therefore, $D \in \mathbf{K}_f$. □

If we assume that $f(1) = 1$ for our bounding function f, then any single vertex is absolutely closed. In this case, any two structures in $\mathbf{K}_f$ always have a free amalgam over single vertex. With Assumption 1, we will see that any forest belongs to $\mathbf{K}_f$, and any structure in $\mathbf{K}_f$ and any forest have free amalgam over single vertex.

Definition 11. Let B be a graph with $A \subseteq B$. B is an *extension of A by a path of length 1* if $B = A \otimes_a ab$, or $B = A \otimes ab$ with a path ab of length 1. A graph B is an *extension of A by paths* if there is a finite sequence $A_0, A_1, \ldots, A_n$ of graphs such that $A_0 = A$, $A_n = B$, and A_i is an extension of A_{i-1} by a path of length 1 for each $i = 1, \ldots, n$.

Lemma 7. (1) *Let A be a non-empty graph which is normal to f, and B an extension of A by paths. Then B is normal to f.*
(2) *Any finite forest belongs to $\mathbf{K}_f$.*

Proof. (1) Recall that $\alpha = m/d < 1$ with relatively prime integers m and d. Suppose $B = A \otimes_a ab$ with a path ab. Then $|B| = |A| + 1$ and $\delta(B) = \delta(A) + (1-\alpha) \geq \delta(A) + 1/d \geq f(|A|) + 1/d \geq f(2|A|) \geq f(|B|)$. Hence, B is normal to f. Similarly, the path ab is normal to f. If $B = A \otimes ab$ then B is also normal to f by Lemma 6 (2). Iterating this argument, we have the statement of (1).

(2) A single vertex is normal to f by $f(1) \leq 1$. Any forest is an extension by paths of a single vertex. Hence, any forest is normal to f. Since any substructure of a forest is a forest, any forest belongs to $\mathbf{K}_f$. □

Proposition 2. *Let B be a forest and v a vertex of B. Then $B \in \mathbf{K}_f$ and $v < B$.*

Proof. $B \in \mathbf{K}_f$ by Lemma 7 (2). Suppose $v \subsetneq U \subseteq B$. Then U is a forest with $|U| \geq 2$. Let U_0 be a connected component of U with $v \in U_0$. We can write $U = U_0 \otimes U'$.

Case $U_0 = v$. U' is non-empty and thus $\delta(U') > 0$. Hence, $\delta(U) > \delta(U_0) = 1$.

Case $U_0 \neq v$. Then $|U_0| \geq 2$. Since U_0 is a tree, U_0 has $|U_0| - 1$ edges. Hence $\delta(U) \geq \delta(U_0) = |U_0| - (|U_0| - 1)\alpha = 1 + (|U_0| - 1)(1-\alpha) > 1$. □

Proposition 3. *Let C be a cycle. If the length of C is sufficiently large then C belongs to $\mathbf{K}_\alpha$ and any single vertex in C is closed in C.*

Proof. Let k be an integer satisfying $(1-\alpha)k > 1$, and l an integer satisfying $l \geq 2k$. We can write $l = k + k'$ with $k' \geq k$. Let C be a cycle of length l. Then we can write $C = P \otimes_{\{a,b\}} P'$ where P and P' are paths of length k and k' respectively, and a and b are ends of both paths P and P'. Since $\delta(P) = 1 + (1-\alpha)k > 2$, it is easy to see that $\{a,b\}$ is closed in P. With the same argument, $\{a,b\}$ is closed in P' as well. P and P' belong to $\mathbf{K}_f$ by Proposition 2. Hence C belongs to $\mathbf{K}_f$ by the free amalgamation property of $\mathbf{K}_f$.

We have $\delta(C) > \delta(\{a,b\}) > 1$ and any proper substructure of $\delta(C)$ is a free amalgam of paths. Therefore, any single vertex in C is closed in C. □

Definition 12. Let R, S be sets and $\mu : R \to S$ a map. For $Z \subseteq [R]^m$, put

$$\mu(Z) = \{\{\mu(x_1), \ldots, \mu(x_m)\} \mid \{x_1, \ldots, x_m\} \in Z\}.$$

Let B, C, and D be graphs and X a set of vertices. We write $D = B \rtimes_X C$ if $C|X$ has no edges and the following hold:

(1) $V(D) = V(B) \cup V(C)$.
(2) $X = V(B) \cap V(C)$.
(3) $E(D) = E(B) \cup E(C)$.

Since we are assuming that C has no edges on X, B is a usual substructure of D but C may not be a substructure of D in general. If B has no edges on X, then D is the free amalgam of B and C over X.

Lemma 8. *Let D be a graph with $D = B \rtimes_X C$.*

(1) $\delta(D/B) = \delta(C/X)$.
(2) $\delta(D) = \delta(B) + \delta(C/X)$.

Proof. (1) We have $D - B = C - X$, and $E_D(C - X, B) = E_C(C - X, X)$ by the definition of $\rtimes$. The statement follows from Fact 1 (1).

(2) follows from (1). □

Lemma 9. *Let D be a graph with $D = B \rtimes_X C$.*

(1) *If $C|X < C$ then $B < D$.*
(2) *If $C|X \leq C$ then $B \leq D$.*

Proof. (1) Assume $C|X < C$. Suppose $B \subsetneq U \subseteq D$. Then $U = B \rtimes_X U_C$ for some substructure U_C of C with $X \subsetneq U_C$. By Lemma 8 (1), we have $\delta(U/B) = \delta(U_C/X)$ and $\delta(U_C/X) > 0$ by $C|X < C$.

(2) Similar to (1). □

2. Balanced Zero-Sum Sequences

We will use some sequences of numbers to construct structures called twigs or wreaths in a later section. We state and prove some properties of finite zero-sum sequences. Most of them are easy facts but it seems difficult to find them in the literature. We define what we mean by a finite sequence first.

Definition 13. Let $\mathbb{Z}$ be the set of integers, and n a positive integer. $[n]$ denotes the set $\{i \in \mathbb{Z} \mid 0 \leq i < n\}$. Let Y be a set. A *Y-sequence of length n* is a map from $[n]$ to Y. If s is a Y-sequence of length m and t a Y-sequence of length n then a *concatenation* of s and t is a Y-sequence u of length $m+n$ such that $u(i) = s(i)$ for $0 \leq i < m$ and $u(m+j) = t(j)$ for $0 \leq j < n$. st denotes the concatenation of s and t. s^n with a positive integer n denotes the finite sequence obtained by concatenating n copies of s.

Definition 14. Let $\mathbb{R}$ be the set of real numbers and s a $\mathbb{R}$-sequence of length l. $\sum s$ is the value $\sum_{i=0}^{l-1} s(i)$. If $s = uv$ then vu is called a *rotation* of s.

If $s = uvw$, u is called a *prefix* of s, w a *suffix* of s and v a *consecutive subsequence* of s.

Let c be a real number. $c \cdot s$ is a sequence obtained by multiplying c to each entry of s.

$\langle y \rangle$ is a sequence s of length 1 such that $s(0) = y$.

Definition 15. Let s be a finite $\mathbb{R}$-sequence. s is a *zero-sum* sequence if $\sum s = 0$.

Let $c > 0$ be a real number. s is *c-balanced* if whenever u is a consecutive subsequence of s then $|\sum u| < c$.

s has the *positively c-balanced prefix property* if whenever u is a non-empty prefix of s with $u \neq s$ then $0 < \sum u < c$.

Let l be a positive integer and n the length of s. s is a *periodic sequence with period l* if $s(i) = s(i+l)$ for any i with $0 \leq i < i+l < n$.

We state some easy facts first.

Lemma 10. *Let s be a zero-sum $\mathbb{R}$-sequence of length l, c and c' positive real numbers, and n a positive integer.*

1. *If s is c-balanced and $s = uwv$ then $|\sum u + \sum v| < c$.*
2. *s^n is a periodic sequence with period l. It is a zero-sum sequence.*
3. *Any consecutive subsequence of s^n of length l is a zero-sum sequence.*
4. *If s is c-balanced then s^n is also c-balanced.*
5. *If s is c-balanced, then any rotation of s is c-balanced.*
6. *If s has the positively c-balanced prefix property then s is c-balanced.*
7. *If s is c-balanced and c' is a non-zero real number then $c' \cdot s$ is $|cc'|$-balanced.*
8. *Suppose $c' > 0$. s has the positively c-balanced prefix property if and only if $c' \cdot s$ has the positively cc'-balanced prefix property.*

Proof. (2), (7), and (8) are clear.

(1) Suppose s is c-balanced and $s = uwv$. We have $|\sum w| < c$ because s is c-balanced. Since s is a zero-sum sequence, we have $\sum u + \sum w + \sum v = 0$. Hence, $\sum u + \sum v = -\sum w$. Therefore, $|\sum u + \sum v| = |-\sum w| = |\sum w| < c$.

(5) follows from (1).

(3) Let s' be a consecutive subsequence of s^n of length l. Since the length of s' is equal to the length of s, s' is a consecutive subsequence of s^2. Hence $ss = us'v$ for some sequences u, v. Since the length of s' is l, the length of uv is also l. Because u is a prefix of s, v is a suffix of s, we have $uv = s$. So, we have $\sum u + \sum v = \sum s = 0$. Hence, $0 = \sum s + \sum s = \sum ss = \sum us'v = \sum u + \sum s' + \sum v = \sum s'$.

(4) Let s' be a consecutive subsequence of s^n. Since any subsequence of s' of length l has zero-sum by (2), we can assume that the length of s' is less than l. Hence, s' is a subsequence of s^2, and thus we can write $s' = vu$ where v is a suffix of s and u a prefix of s. Since the length of s' is less than l, we can write $s = uwv$. By (1), we have $|\sum s'| = |\sum u + \sum v| < c$.

(6) Let v be a consecutive subsequence of s. Then uv is a prefix of s for some prefix u of s. Since s has the positively c-balanced prefix property, $0 < \sum u < c$ and $0 < \sum uv < c$. We have $\sum v = \sum uv - \sum u$. Hence, $|\sum v| < c$. □

Proposition 4. (1) *Let a and b be positive real numbers such that a/b is a rational number. Let p, q be relatively prime positive integers such that $a/b = p/q$. Then there exists uniquely a zero-sum $\{a, -b\}$-sequence which has the positively $(a+b)$-balanced prefix property. The length of such a sequence is $p+q$.*

(2) *Let b be a non-zero real number. Then $\langle 0 \rangle$ is the unique zero-sum $\{0, b\}$-sequence which has the positively $|b|$-balanced prefix property.*

Proof. (1) By Lemma 10 (7), it is enough to show the statement in the case that $a = p$ and $b = q$.

Let s be a $\{p, -q\}$-sequence with positively $(p+q)$-balanced prefix property. We show that such a sequence s uniquely exists.

Since $s(0)$ must be positive, we have $s(0) = p$.

Suppose $s(i)$ is defined for $i < n$.

If $\sum_{i=0}^{n-1} s(i) \geq q$ then $s(n)$ cannot be p because $\sum_{i=0}^{n} s(i)$ will be $p+q$ or more. Therefore, $s(n)$ must be $-q$.

If $\sum_{i=0}^{n-1} s(i) < q$, then $s(n)$ cannot be $-q$ because $\sum_{i=0}^{n} s(i)$ will be negative. Therefore, $s(n)$ must be p.

Hence, s must satisfy the following two conditions.

(i) $s(0) = p$.
(ii) If $\sum_{i=0}^{n-1} s(i) \geq q$ then $s(n) = -q$. Otherwise, $s(n) = p$.

By induction, we see that such a sequence exists and is unique.

By induction, we can see that $0 \leq \sum_{i=0}^{k} s(i) < p+q$ for any k. Also, we can see that p appears q times in s eventually. Let j be the index such that $s(j)$ is the q'th p in s. If $k < j$, then $\sum_{i=0}^{k} s(i) = lp - l'q$ with $l < q$. Since p and q are relatively prime, $lp - l'q$ cannot be zero. Hence, $\sum_{i=0}^{k} s(i) > 0$ for $k < j$. We also have $\sum_{i=0}^{j} s(i) > 0$ because $s(j) = p > 0$. $0 < \sum_{i=0}^{j} s(i) = qp - l''q = (p - l'')q$ for some integer l''. By the inductive definition of s, $\langle -q \rangle^{p-l''}$ follows. Therefore, $s|[p+q]$ is a zero-sum $\{p, -q\}$-sequence with the positively $(p+q)$-balanced prefix property. It cannot be shorter or longer.

(2) $\langle 0 \rangle$ is a zero-sum $\{0, b\}$-sequence which has the positively $|b|$-balanced prefix property by the definition. It is easy to check that no other sequences can be a zero-sum $\{0, b\}$-sequence. □

Let s be a zero-sum $\{a,-b\}$-sequence with the positively $(a+b)$-balanced prefix property. Since s is $(a+b)$-balanced, any rotation of s^k with a positive integer k is $(a+b)$-balanced. It turns out that any $(a+b)$-balanced zero-sum $\{a,-b\}$-sequence is a rotation of s^k for some positive integer k [10].

3. Zero-Extensions

To prove Theorem 8, given a graph $A \in \mathbf{K}_f$, we would like to construct an extension B of A such that $A < B$ and B is absolutely closed.

Definition 16. Let A and B be graphs. B is a *zero-extension of* A if $A \leq B$ and $\delta(B/A) = 0$. B is a *minimal zero-extension of* A if B is a proper zero-extension of A and minimal with this property. In this case, $A \subsetneq U \subsetneq B$ implies $A < U$.

B is a *biminimal zero-extension of* A if B is a minimal zero-extension of A and whenever $A' \subseteq A$ and $\delta(B-A/A') = 0$ then $A' = A$.

We will use the following facts many times.

Fact 5. *Let A be a substructure of a graph B. The following are equivalent:*

(1) *B is a biminimal zero-extension of A.*
(2) *$\delta(B/A) = 0$ and whenever $D \subsetneq B$ then $A \cap D < D$.*

Proof. We first show that (1) implies (2). Assume (1). We have $\delta(B/A) = 0$ because B is a zero-extension of A.

Suppose D is a proper substructure of B. We show that $A \cap D < D$.

Case $A \cap D = D$. We have $A \cap D < D$ by the definition of $<$.

Case $A \cap D \neq D$. In this case, $D - A$ is non-empty. Suppose $A \cap D \subsetneq U \subseteq D$. We are going to show that $\delta(U/A \cap D) > 0$.

Subcase $U - A = B - A$. We have $D - A = B - A$ because $U \subseteq D \subseteq B$. Hence, $A \cap D \neq A$ since D is a proper substructure of B. Thus, $\delta(U/A \cap D) = \delta(B - A/A \cap D) \geq \delta(B - A/A)$ by Fact 1 (2). Since $\delta(B - A/A) = \delta(B/A) = 0$, we have$\delta(B - A/A \cap D) \geq 0$. $\delta(B - A/A \cap D) \neq 0$ since B is a biminimal extension of A and $A \cap D \neq A$. Hence, $\delta(U/A \cap D) = \delta(B - A/A \cap D) > 0$.

Subcase $U - A \neq B - A$. We have $\delta(U/A \cap D) = \delta(U - A/A \cap D) \geq \delta(U - A/A)$ by Fact 1 (2). Also, $\delta(U - A/A) > 0$ since B is a minimal zero-extension of A and $U - A$ is non-empty because $A \cap D \subsetneq U \subseteq D$. Hence, $\delta(U/A \cap D) > 0$.

(2) is proved.

It is straightforward to see that (2) implies (1). □

Fact 6. *Let $D = B \otimes_A C$ where B and C are zero-extensions of A. Then D is a zero-extension of A.*

Proof. We have $A \leq D$ by Fact 2 (3). We have $\delta(D/A) = 0$ by Fact 2 (1). □

Definition 17. (Twig) Recall that $\alpha = m/d < 1$ with relatively prime positive integers m and d. Let l be the largest integer x such that $x\alpha \leq 1$. Put $r = d \bmod m$.

We have $1 - l\alpha = r/d \geq 0$, $1 - (l+1)\alpha = (r-m)/d < 0$, and

$$|1 - l\alpha| + |1 - (l+1)\alpha| = (1 - l\alpha) - (1 - (l+1)\alpha) = \alpha.$$

Let s be a zero-sum $\{1 - l\alpha, 1 - (l+1)\alpha\}$-sequence of length m with the positively α-balanced prefix property. Such a sequence s exists uniquely by Proposition 4. We call s a *special sequence for* α.

A graph W is called a *general twig associated to* s^k if W can be written as $W = BF$ with substructures B and F having the following properties:

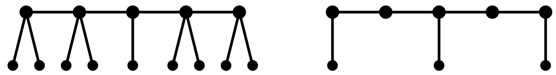

FIGURE 1. A twig for $5/13$ (left) and a twig for $5/7$ (right)

(1) B is a path $b_0 b_1 \cdots b_{km-1}$ of length $km-1$.
(2) F is the set of all leaves of W.
(3) b_0 is adjacent to exactly l leaves in F.
(4) For $i > 0$, if $s^k(i) = 1 - l\alpha$ then b_i is adjacent to exactly $l-1$ leaves in F.
(5) For $i > 0$, if $s^k(i) = 1-(l+1)\alpha$ then b_i is adjacent to exactly l leaves.

Let D be a substructure of W. $B(D)$ denotes $B \cap D$, and $F(D)$ denotes $F \cap D$. If $\alpha = 1/d$, W is a star with d leaves. By the construction, $1 - e(b_0, F(W))\alpha = s^k(0)$, and $1 - (e(b_i, F(W)) + 1)\alpha = s^k(i)$ for $i > 0$.

Let D be a connected substructure of W such that $B(D)$ is non-empty. Since any vertex in $F(D)$ is a leaf of W, $B(D)$ must be a connected substructure of $B(W)$. Then we can see that $B(D)$ is a path $b_j b_{j+1} \cdots b_k$ for some j and k with $j \leq k$. We call D a *non-prefix of* $B(W)$ if $j > 0$ and a *proper prefix of* $B(W)$ if $i = 0$ and $B(D) \neq B(W)$.

In the case that $k = 1$, we call W a *twig associated to* s. In this case, we also call W a *twig for* α without referring to s.

Note that the sequence s^k corresponds to a calculation of $\delta(W/F(W))$ where W is a general twig associated to s^k. See Figure 4.

Example 1. *Let* $\alpha = 5/13$. *Then* $1-2\alpha = 3/13$ *and* $1-3\alpha = -2/13$.

$$s_{5/13} = \langle 1-2\alpha, 1-3\alpha, 1-2\alpha, 1-3\alpha, 1-3\alpha \rangle$$

is the special sequence for $5/13$. *A twig* W *associated to* $s_{5/13}$ *is shown in Figure 1 (left). The upper path is* $B(W)$ *and the set of lower leaves is* $F(W)$.

Example 2. *Let* $\alpha = 5/7$. *Then* $1-\alpha = 2/7$ *and* $1-2\alpha = -3/7$.

$$s_{5/7} = \langle 1-\alpha, 1-\alpha, 1-2\alpha, 1-\alpha, 1-2\alpha \rangle$$

is the special sequence for $5/7$. *A twig associated to* $s_{5/7}$ *is shown in Figure 1 (right).*

Let W be a twig. If $\alpha \leq 1/2$ then $l \geq 2$ in the definition of a twig. Hence, if $\alpha \leq 1/2$ then each vertex in $B(W)$ is adjacent to some leaf in $F(W)$. If $\alpha > 1/2$ then $l = 1$ in the definition of a twig.

Definition 18. (Wreath) Recall that $\alpha = m/d < 1$ with relatively prime positive integers m and d. Let s be the special sequence for α. Let l be an integer such that $1 - l\alpha \geq 0$ and $1-(l+1)\alpha < 0$. Let k be an integer such that $km \geq 3$.

A graph W is called a *wreath associated to* s^k if W can be written as $W = BF$ with the following properties:

(1) B is a cycle $b_0 b_1 \cdots b_{km-1} b_0$ of length km.
(2) F is the set of all leaves of W.
(3) For i with $0 \leq i < km$, if $s^k(i) = 1 - l\alpha$ then b_i is adjacent to exactly $l-1$ leaves in F.
(4) For i with $0 \leq i < km$, if $s^k(i) = 1-(l+1)\alpha$ then b_i is adjacent to exactly l leaves in F.

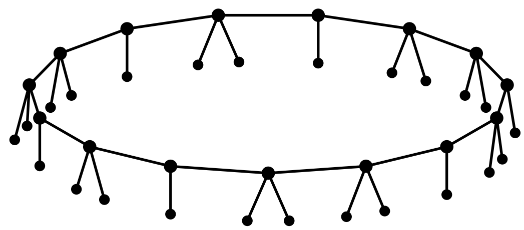

FIGURE 2. A wreath for $5/13$ associated to $s^3_{5/13}$

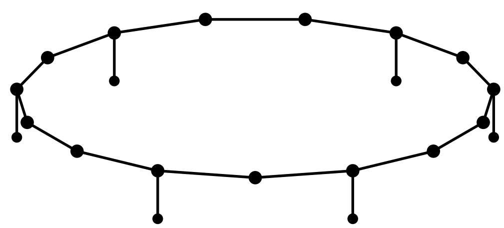

FIGURE 3. A wreath for $5/7$ associated to $s^3_{5/7}$

(a twig) (a part of a wreath)

$1-2\alpha,\ 1-3\alpha,\ 1-2\alpha,\ 1-3\alpha,\ 1-3\alpha \quad 1-2\alpha,\ 1-3\alpha,\ 1-2\alpha,\ 1-3\alpha,\ 1-3\alpha$

FIGURE 4. A special sequence corresponding to a calculation of $\delta(W/F(W))$ ($\alpha = 5/13$).

We also say that W is a wreath for α without referring to s^k.

Let D be a substructure of W. $B(D)$ denotes $B \cap D$, and $F(D)$ denotes $F \cap D$. By the construction, $1-(e(b_i, F(W))+1)\alpha = s^k(i)$ for any i with $0 \le i < km$.

Note that given a twig or a wreath W for α, we have $|F(W)| \ge 2$ by definition. We will use this fact later.

Example 3. *Recall a special sequence $s_{5/13}$ for $5/13$ from Example 1. A twig associated to $s^3_{5/13}$ is shown in Figure 2.*

Example 4. *Recall a special sequence $s_{5/7}$ for $5/7$ from Example 2. A twig associated to $s^3_{5/7}$ is shown in Figure 3.*

Lemma 11. *Any twig for α belongs to $\mathbf{K}_f$. Let W be a wreath for α. If $B(W)$ belongs to $\mathbf{K}_f$ then W belongs to $\mathbf{K}_f$. If $|B(W)| = km$ then $|F(W)| \ge k$.*

Proof. A twig for α is a tree. Therefore, it belongs to $\mathbf{K}_f$ by Proposition 2. W is a wreath. So, it is an extension of cycle $B(W)$ by paths. By Proposition 3 and Lemma 7, W belongs to $\mathbf{K}_f$. Let s be the special sequence for α. If $|B(W)| = km$, then W is a wreath associated to s^k. Hence, any connected substructure of $B(W)$ with m vertices has a vertex adjacent to a leaf in $F(W)$. Therefore, $|F(W)| \geq k$. □

We can prove that any wreath with sufficiently large girth belongs to any amalgamation class with the free amalgamation property by Propositions 2 and 3.

Definition 19. Let W be a twig or a wreath for α and D a substructure of W. A *defect* of D in W is an edge bf of W such that $b \in B(D)$ and $f \in F(W)$, but bf is not an edge of D. An edge bf of W is a defect of D if and only if $b \in B(D)$ but $f \notin F(D)$.

Definition 20. Let W be a twig or a wreath for α and D a connected substructure of W. D is *smooth* if one of the following conditions holds:

(1) $B(D)$ is a cycle.
(2) $B(D)$ is a path $v_0v_1\cdots v_j$ with $0 \leq j$ where v_0 is adjacent to a vertex in $F(D)$.

Twigs and wreaths are designed to make the following lemmas hold.

Lemma 12. *Let s be the special sequence for α, W a general twig associated to s^k with $k \geq 1$, and D a connected substructure of W.*

(1) *If $D = W$ then $\delta(D/F(D)) = 0$.*
(2) *If $B(D) = B(W)$ and $F(D) \neq F(W)$ then $\delta(D/F(D)) > 0$.*
(3) *If $B(D)$ is a non-empty non-prefix of $B(W)$ then $\delta(D/F(D)) > 0$.*
(4) *If $k = 1$ and $B(D)$ is a non-empty proper prefix of $B(W)$ then $\delta(D/F(D)) > 0$.*

Proof. (1) We have $W = B(W)F(W)$, $B(W) \cap F(W) = \emptyset$, $B(W)$ is a path $b_0b_1\ldots b_{km-1}$, $1 - e(b_0, F(W))\alpha = s(0)$, and $1 - (e(b_i, F(W)) + 1)\alpha = s(i)$ for each i with $0 < i < km$.

We show that $\delta(W/F(W)) = 0$. By Fact 1 (1), $\delta(W/F(W)) = \delta(B(W)/F(W)) = \delta(B(W)) - e(B(W), F(W))\alpha$. Since $B(W)$ is a path of length $km-1$, $B(W)$ has km vertices and $km-1$ edges. We have $\delta(B(W)) = km - (km-1)\alpha$. Since $B(W) = b_0b_1\ldots b_{km-1}$, we have $e(B,F) = \sum_{i=0}^{km-1} e(b_i, F(W))$.

Hence,

$$\begin{aligned}
\delta(W/F(W)) &= \delta(B(W)) - e(B(W), F(W))\alpha \\
&= km - (km-1)\alpha - \sum_{i=0}^{km-1} e(b_i, F(W))\alpha \\
&= 1 - e(b_0, F(W))\alpha + \sum_{i=1}^{km-1} (1 - (e(b_i, F(W)) + 1)\alpha) \\
&= s(0) + \sum_{i=1}^{km-1} s^k(i) = \sum s^k = 0.
\end{aligned}$$

(2) Suppose $B(D) = B(W)$, and $F(D)$ is a proper subset of $F(W)$. There must be a defect of D. Hence, $e(B(W), F(D)) < e(B(W), F(W))$. So, we have

$$\begin{aligned}
\delta(D/F(D)) &= \delta(B(D)/F(D)) \\
&= \delta(B(W)/F(D)) \\
&= \delta(B(W)) - e(B(W), F(D))\alpha \\
&> \delta(B(W)) - e(B(W), F(W))\alpha.
\end{aligned}$$

We also have $\delta(B(W)) - e(B(W), F(W))\alpha = 0$ by (1). Therefore, $\delta(D/F(D)) > 0$.

(3), (4) Suppose $B(D) \neq B(W)$ and $B(D)$ is non-empty. Then $B(D)$ is a path $b_p b_{p+1} \cdots b_q$ for some integers p, q with $0 \leq p \leq q \leq km-1$. The length of the path $B(D)$ is $q-p$ and it is less than $km-1$. Note that $e(b,F(D)) \leq e(b,F(W))$ for each $b \in B(D)$. We have

$$\begin{aligned}
\delta(D/F(D)) &= \delta(B(D)) - e(B(D),F(D))\alpha \\
&= (q-p+1)-(q-p)\alpha - \sum_{i=p}^{q} e(b_i,F(D))\alpha \\
&\geq (q-p+1)-(q-p)\alpha - \sum_{i=p}^{q} e(b_i,F(W))\alpha \\
&= 1-e(b_p,F(W))\alpha + \sum_{i=p+1}^{q}(1-(e(b_i,F(W))+1)\alpha) \\
&= 1-e(b_p,F(W))\alpha + \sum_{i=p+1}^{q} s^k(i).
\end{aligned}$$

If $p \geq 1$, then $1-(e(b_p,F(W))+1)\alpha = s^k(p)$. Hence, $1-e(b_p,F(W))\alpha = \alpha + s^k(p)$. Therefore,

$$1-e(b_p,F(W))\alpha + \sum_{i=p+1}^{q} s^k(i) = \alpha + \sum_{i=p}^{q} s^k(i).$$

So, $\delta(D/F(D)) \geq \alpha + \sum u$ for some consecutive subsequence u of s^k. Since s^k is α-balanced, $|\sum u| < \alpha$. Hence, $\delta(D/F(D)) \geq \alpha + \sum u > 0$. So, we have (3).

Suppose $k=1$ and $B(D)$ is a proper prefix of W. Then $p=0$. We have $1-e(b_p,F(W))\alpha = s(0)$. Hence, $\delta(D/F(D)) = \sum u$ with u a proper prefix of s. Since s has the positively α-balanced prefix property, $\sum u > 0$. So, we have (4). □

Lemma 13. *Let s be the special sequence for α, W a wreath associated to s^k with $k \geq 1$, and D a connected substructure of W.*

1. *If $D = W$ then $\delta(D/F(D)) = 0$.*
2. *If $B(D) = B(W)$ and $F(D) \neq F(W)$ then $\delta(D/F(D)) > 0$.*
3. *If $B(D)$ is non-empty and $B(D) \neq B(W)$ then $\delta(D/F(D)) > 0$.*

Proof. The proofs for (1) and (2) go parallel to that for Lemma 12 (1) and (2).

(3) Suppose $B(D)$ is non-empty and $B(D) \neq B(W)$. Then $B(D)$ is a path. So, we can consider D as a substructure of some general twig W' associated to s^{2k} such that $B(D)$ is a non-prefix of $B(W')$. Therefore, $\delta(D/F(D)) > 0$ by Lemma 12 (3). □

Lemma 14. *Let W be a twig or a wreath for α. Then W is a biminimal zero-extension of $F(W)$. In particular, if D is a proper substructure of W then $F(D) < D$ by Fact 5.*

Proof. We have $\delta(W/F(W)) = 0$ by Lemma 12 (1). We show first that W is a minimal zero-extension of $F(W)$. Let U be a substructure of W and suppose that $F(W) \subsetneq U \subsetneq W$. Then $B(U) \subsetneq B(W)$. We want to show that $\delta(U/F(U)) > 0$. Let $\{D_1, D_2, \dots, D_k\}$ be the set of all connected components of U where the D_i are all distinct. By Lemma 1, we have $\delta(U/F(U)) = \sum_{i=1}^{k} \delta(D_i/F(D_i))$. Note that $B(D_i) \subsetneq B(W)$ for each i since $B(U) \subsetneq B(W)$.

If $B(D_i)$ is empty, then $D_i = F(D_i)$. Hence, $\delta(D_i/F(D_i)) = 0$.

Suppose $B(D_i)$ is non-empty. Since $B(D_i) \subsetneq B(W)$, we have $\delta(D_i/F(D_i)) > 0$ by Lemma 12 (3) and (4).

Hence, $\delta(U) > 0$ because there must be i such that $B(D_i)$ is non-empty since $B(U)$ is non-empty.

Now, we show that W is a biminimal zero-extension of $F(W)$. Let U be a substructure of W with $B(U) = B(W)$ and $F(U) \neq F(W)$. Then U is connected. We have $\delta(B(W)/F(U)) = \delta(U/F(U)) > 0$ by Lemma 12 (2).

In the case that W is a wreath for α, we can show the statement similarly by using Lemma 13. $\square$

Lemma 15. *Let $G = A \rtimes_{F(W)} W$ where $A \in \mathbf{K}_f$ and W is a wreath for α with $W \in \mathbf{K}_f$. Let U be a substructure of G where $U = (U \cap A) \rtimes_{F(D)} D$ with D a substructure of W.*

1. *If $(U \cap A) \rtimes_{F(D')} D'$ is normal to f for any connected component D' of D then U is normal to f.*
2. *If $F(D)$ is empty then U belongs to $\mathbf{K}_f$.*
3. *If D is connected and $F(D)$ is non-empty then there is a smooth connected substructure D' of D such that $F(D') = F(D)$ and U is an extension of $(U \cap A) \rtimes_{F(D)} D'$ by paths.*
4. *If $(U \cap A) \rtimes_{F(D')} D'$ is normal to f for any smooth connected substructure D' of D then U is normal to f.*

Proof. (1) If D is connected then the statement is obvious.

Suppose D is not connected. Then $D \neq W$. Let $\{D_1, D_2, \dots, D_k\}$ be the set of all connected components of D where the D_i are all distinct. We have $D_i \neq W$ for each i because $D \neq W$. We can represent

$$U = U_1 \otimes_{U \cap A} U_2 \otimes_{U \cap A} \cdots \otimes_{U \cap A} U_k$$

with $U_i = (U \cap A) \rtimes_{F(D_i)} D_i$ for each i. By Lemmas 14 and 8 (1), we have $\delta(U \cap A) < \delta(U_i)$ if $B(D_i)$ is non-empty. Also, $U_i = U \cap A$ if $B(D_i)$ is empty. U is normal to f by Lemma 6 (1).

(2) Suppose $F(D)$ is empty. Then $U = (U \cap A) \otimes B(D)$. Since $U \cap A \in \mathbf{K}_f$ and $W \in \mathbf{K}_f$, $U \in \mathbf{K}_f$ by the free amalgamation property of $\mathbf{K}_f$.

(3) Suppose D is connected. If $B(D)$ is a cycle, then it is already smooth. Suppose $B(D)$ is a path, say $v_0 v_1 \cdots v_l$. Since $F(D)$ is non-empty, there is i such that v_i is adjacent to a leaf in $F(D)$. We can assume that i is the smallest index with this property. Let D' be the substructure $D - \{v_0, v_1, \dots, v_{i-1}\}$ of D. Then $F(D') = F(D)$, D' is smooth, connected and $D = D' \otimes_{v_i} v_0 v_1 \dots v_i$ with path $v_0 v_1 \dots v_i$. We have

$$U = ((U \cap A) \rtimes_{F(D)} D') \otimes_{v_i} v_0 v_1 \dots v_i.$$

(4) Suppose that $(U \cap A) \rtimes_{F(D')} D'$ is normal to f for any smooth connected substructure D' of D.

Let C be a connected component of D. Put $U_C = (U \cap A) \rtimes_{F(C)} C$. By (1), it is enough to show that U_C is normal to f. If $F(C)$ is empty then U' is normal to f by (2). We can assume that $F(C)$ is non-empty. By (3), there is a substructure C' of C such that C' is a smooth connected substructure of C, $F(C') = F(C)$ and U_C is an extension of $(U \cap A) \rtimes_{F(C)} C'$ by paths. By the assumption, $(U \cap A) \rtimes_{F(C)} C'$ is normal to f. U_C is normal to f by Lemma 7 (1). $\square$

Lemma 16. *Recall that $\alpha = m/d < 1$ with relatively prime positive integers m and d. Let W be a twig or a wreath for α, and D a substructure of W which is connected, smooth, and has exactly k defects with $k \geq 0$.*

1. *If $\alpha \leq 1/2$ then $|B(D)| \leq |F(D)| + k$.*
2. *$|B(D)| \leq m(|F(D)| + k)$ in general.*
3. *If $B(D) = B(W)$ then $\delta(D/F(D)) = k\alpha$.*

(4) *If $B(D)$ is a non-empty proper substructure of $B(W)$ then $\delta(D/F(D)) > k\alpha$.*

Proof. Let D' be a substructure of W such that $B(D') = B(D)$ and D' has no defect. We can obtain D' by adding every defect of D to D. Since every vertex in $F(W)$ is a leaf of W, $F(W)$ and $E(B(W), F(W))$ are in one-to-one correspondence by a map sending $f \in F(W)$ to an edge bf of W with $b \in B(W)$. Each $v \in F(D') - F(D)$ corresponds to a defect of D. Therefore, $|F(D')| = |F(D)| + k$.

(1) Suppose $\alpha \leq 1/2$. We have $l\alpha \leq 1$, and $(l+1)\alpha > 1$ for some $l \geq 2$. By the construction of a twig or a wreath, for every vertex b in $B(W)$, there is an edge bf of W with $f \in F(W)$.

Since each vertex b in $B(D') = B(D)$ has an edge bf of D' with $f \in F(D')$, we have $|B(D')| \leq |F(D')|$. Therefore, $|B(D)| \leq |F(D)| + k$.

(2) Suppose $\alpha > 1/2$. In this case, for each b in $B(W)$, there is at most one edge bf of W with $f \in F(W)$.

If W is a twig, then $|B(W)| = m$. Hence, $|B(D)| \leq m$. Since $F(D)$ is non-empty, we have $|B(D)| \leq m \leq m(|F(D)| + k)$.

Suppose W is a wreath associated to s^q where s is the special sequence for α and q a positive integer.

Consider the case $B(D) = B(W)$. In this case, $D' = W$. Since s^q is a periodic sequence of period m, and by the construction of W, for any path in $B(W)$ of length $m-1$ (there are m vertices in this path), there is an edge from a vertex in the path to a vertex in $F(W)$. Therefore, $|B(D)| \leq m|F(W)|$. Since $|F(W)| = |F(D')| = |F(D)| + k$, we have $|B(D)| \leq m(|F(D)| + k)$.

Now, consider the case $B(D)$ is a path $v_0 v_1 \cdots v_{p-1}$ in $B(W)$. Since D is smooth, there is an edge $v_0 f_0$ of D with $f_0 \in F(D)$. Because W is associated to s^q and s^q is a periodic sequence of period m, for any j with v_{jm} in $B(D)$ there is a vertex f_j in $F(W)$ which is adjacent to v_{jm} in W. Each f_j belongs to $F(D')$ and $f_j \neq f_{j'}$ if $j \neq j'$ because each f_j is a leaf of W. Therefore, $|B(D)| = |B(D')| \leq m|F(D')| = m(|F(D) + k)$.

(3) If $B(D') = B(W)$ then $D' = W$. We have $\delta(D'/F(D')) = 0$ by Lemma 12 (1) and Lemma 13 (1). By Fact 1 (1), we have

$$\begin{aligned} \delta(D/F(D)) &= \delta(B(D)) - e(B(D), F(D))\alpha \quad \text{and} \\ \delta(D'/F(D')) &= \delta(B(D')) - e(B(D'), F(D'))\alpha. \end{aligned}$$

Also, we have $B(D') = B(D)$ and $e(B(D'), F(D')) = e(B(D), E(D)) + k$ by the definition of defects. Therefore, $\delta(D/F(D)) = k\alpha$.

(4) Similar to (3). If $B(D')$ is a non-empty proper substructure of $B(W)$, then $\delta(D'/F(D')) > 0$ by Lemma 12 (2) and Lemma 13 (2). Therefore, $\delta(D/F(D)) > k\alpha$. □

Lemma 17. *Let W be a twig or a wreath for α, D a smooth connected substructure of W with 2 or more defects. Let $G = A \rtimes_{F(D)} D$ where A is non-empty and normal to f. Then G is normal to f.*

Proof. Let k be the number of defects of D. Then $\delta(D/F(D)) \geq k\alpha$ by Lemma 16 (3), (4). By Lemma 8 (2), we have

$$\delta(A \rtimes_{F(D)} D) = \delta(A) + \delta(D/F(D)) \geq \delta(A) + k\alpha \geq f(|A|) + km/d \geq f(2^{km}|A|).$$

Case $\alpha \leq 1/2$. We have $|B(D)| \leq |A| + k$ by Lemma 16 (1). So, $|A \rtimes_{F(D)} D| = |A| + |B(D)| \leq 2|A| + k$. We have $2^{km} \geq km + 2$ because $k \geq 2$ and $m \geq 1$. Hence $2^{km}|A| \geq (km+2)|A| \geq 2|A| + k \geq |A \rtimes_{F(D)} D|$. Therefore, $A \rtimes_{F(D)} D$ is normal to f.

Case $\alpha > 1/2$. By Lemma 16 (2), we have $|B(D)| \leq m(|A|+k)$. So, $|A \rtimes_{F(D)} D| = |A| + |B(D)| \leq |A| + m(|A|+k) = (m+1)|A| + km$. Since $\alpha > 1/2$ we have $m \geq 2$, and thus $km \geq 4$. Hence, $2^{km} > 2km+1$. We have

$$2^{km}|A| > (2km+1)|A| > (m+1)|A| + km.$$

Therefore, $A \rtimes_{F(D)} D$ is normal to f. □

Lemma 18. *Let W be a twig or a wreath for α, D a connected substructure of W. Let $G = A \rtimes_{F(D)} D$ where A is non-empty and normal to f, and $B(D) \neq B(W)$. If D has 1 or more defects then G is normal to f.*

Proof. If D has 2 or more defects then G is normal to f by Lemma 17. So, we can assume that D has exactly 1 defect.

By Lemma 15 (2), (3) and Lemma 7 (1), it is enough to show that G is normal to f assuming D is smooth.

Recall that $\alpha = m/d < 1$ with relatively prime positive integers m and d. We have $\delta(A \rtimes_{F(D)} D) > \delta(A) + \alpha$ by Lemma 8 (2) and Lemma 16 (4). Hence,

$$\delta(A \rtimes_{F(D)} D) \geq \delta(A) + \alpha + 1/d \geq f(|A|) + (m+1)/d \geq f(2^{m+1}(|A|)).$$

Case $\alpha \leq 1/2$. By Lemma 16 (1), $|B(D)| \leq |A|+1$. Since $m \geq 1$, we have

$$2^{m+1}|A| > 2|A| + 1 \geq |A| + |B(D)|.$$

Therefore, $A \rtimes_{F(D)} D$ is normal to f.

Case $\alpha > 1/2$. By Lemma 16 (2), $|B(D)| \leq m(|A|+1)$ and $m \geq 2$ as $\alpha > 1/2$. We have $2^{m+1} > 2(m+1)$. Therefore,

$$2^{m+1}|A| > 2(m+1)|A| > |A| + m(|A|+1) \geq |A| + |B(D)|.$$

Hence, $A \rtimes_{F(D)} D$ is normal to f. □

If $\alpha \leq 1/2$, we can drop the assumption that D has 1 or more defects in Lemma 18. This fact will make the proof of Proposition 6 below easy in the case $\alpha \leq 1/2$.

Definition 21. Let W be a twig or a wreath for α, A, C graphs and P a set of isolated vertices of A.

We call C a *canonical extension* of A by W over P if C can be written as $C = A \rtimes_{F(W)} W$ and the following hold:

(1) If $|F(W)| = 2$ then $F(W) \subseteq P$, and if $F(W) \geq 3$ then $F(W)$ contains at least 3 vertices in P.
(2) Whenever $D \subseteq W$, D has no defects, D is connected in W, and $|F(D)| \geq 2$, then $F(D)$ contains a vertex in P.

Note that if $P' \supseteq P$ is another set of isolated vertices of A then C is a canonical extension of A by W over P'. We sometimes omit the reference to P and/or W.

We call C a *semicanonical extension* of A over P if

$$C = C_1 \otimes_A C_2 \otimes_A \cdots \otimes_A C_n$$

where C_i is a canonical extension of A over P for $i = 1, \ldots, n$ with $n \geq 0$. If $n = 0$ then $C = A$ by convention. We call each C_i a *component* of C. Hence, n is the number of components of C. We sometimes omit the reference to P. A canonical extension of A over P is a semicanonical extension of A over P with one component.

Lemma 19. *Let C be a semicanonical extension of A. Then C is a zero-extension of A.*

Proof. Let n be the number of components of C. We prove the statement by induction on n. If $n=0$ then $C=A$. Hence C is a zero-extension of A by definition. Suppose $n=1$. Then C is a canonical extension of A. Hence, $C = A \rtimes_{F(W)} W$ where W is a twig or a wreath. we have $A \leq C$ by Lemmas 9 (2), and 14. We also have $\delta(C/A)=0$ by Lemmas 8 (1), and 14. Therefore, C is a zero-extension of A.

Suppose $n>1$. Then $C = C' \otimes_A C''$ where C' is a semicanonical extension of A with $n-1$ components and C'' a canonical extension of A. Both C' and C'' are zero-extensions of A by the induction hypothesis. Therefore, C is also a canonical extension of A by Fact 6. □

Lemma 20. *Let W be a twig or a wreath for α, A a graph such that $A = A' \otimes P$ where P is a graph with no edges and $|P| > |A'|$. Assume that $|P| \geq 5$. If $|F(W)| \leq |A|$ then $F(W)$ can be embedded in A in a way that $A \rtimes_{F(W)} W$ is a canonical extension of A over P.*

Proof. $B(W)$ can be written as a path $b_0 b_1 \cdots b_{m-1}$ or a cycle $b_0 b_1 \cdots b_{km-1} b_0$ for some k. Enumerate the vertices in $F(W)$ as $f_0, f_1, \ldots, f_{|F(W)|-1}$ in a way that if f_i is adjacent to b_p and f_j is adjacent to b_q with $p<q$ then $i<j$. If $|F(W)| \leq 5$ then embed $F(W)$ into P. We can do this by the assumption that $|P| \geq 5$.

If $|F(W)| \geq 6$, embed each f_i with an even index i into P. We can do this because $|F(W)| \leq |A|$ and more than half of the vertices of A belongs to P. Embed each f_i with an odd index i into the rest of vertices of A in any way. □

Proposition 5. *Let $A = A' \otimes P$ with P a non-empty graph with no edges. If G is a semi-canonical extension of A over P then $A' < G$.*

Proof. Suppose
$$G = C_1 \otimes_A C_2 \otimes_A \cdots \otimes_A C_n$$
where C_i is a canonical extension of A by W_i over P with W_i a twig or a wreath for α for $i = 1, \ldots, n$.

First, note that $A' < A' \otimes P = A$ and $A \leq G$.

Let U be a graph with $A' \subsetneq U \subseteq G$. We can write
$$U = (U \cap C_1) \otimes_{U \cap A} \cdots \otimes_{U \cap A} (U \cap C_n)$$
with $U \cap C_i = (U \cap A) \rtimes_{F(D_i)} D_i$ where D_i is a substructure of W_i for $i = 1, \ldots, n$.

If $B(D_i)$ is empty for $i = 1, \ldots, n$, we have $U = U \cap A$. Hence, $A' \subsetneq U \subseteq A$. So, we have $\delta(A') < \delta(U)$ by $A' < A$.

Otherwise, we can choose i with $1 \leq i \leq n$ such that $B(D_i)$ is non-empty.

We have $\delta(U/U \cap A) = \sum_{j=1}^{n} \delta(U \cap C_j / U \cap A) \geq \delta(U \cap C_i / U \cap A) = \delta(D_i / F(D_i))$ by Fact 2 (1) and Lemma 8 (1).

If $D_i \neq W_i$, we have $\delta(D_i/F(D_i)) > 0$ by Lemma 14 and non-emptiness of $B(D_i)$. Hence, $\delta(U/U \cap A) > 0$ by the inequality above. We have $\delta(U) > \delta(U \cap A) \geq \delta(A')$.

If $D_i = W_i$, then $F(D_i) = F(W_i)$. In this case, we have $F(W_i) \subseteq U \cap A$. Since C_i is a canonical extension of A by W_i, $F(W_i)$ contains an isolated vertex from P. Hence, $A' \subsetneq U \cap A$ and so $\delta(A') < \delta(U \cap A)$. We have $\delta(U \cap A) \leq \delta(U)$ by $A \leq G$. Therefore, $\delta(A') < \delta(U)$. □

Lemma 21. *Let C be a canonical extension of A by a wreath W for α where A and W belong to $\mathbf{K}_f$ and $|F(W)| \geq 3$. Then C belongs to $\mathbf{K}_f$.*

Proof. Let U be a substructure of C. We show that U is normal to f. We can write $U = (U \cap A) \rtimes_{F(D)} D$ with a substructure D of W. By Lemma 15 (2) and (4), it is enough to show that U is normal to f assuming D is smooth and connected, and $F(D)$ is non-empty. Since $F(D) \subseteq U \cap A$, $U \cap A$ is non-empty. Note that $U - (U \cap A) = B(D)$.

Case $B(D) = B(W)$. Since we are assuming that $|F(W)| \geq 3$, $F(W)$ has at least 3 isolated vertices of A by Definition 21. Suppose D has at most 1 defect. Then $F(D)$ has at least 2 isolated vertices in A, and thus $U \cap A$ has 2 isolated vertices. Therefore, $U \cap A$ is $3m|U \cap A|$-normal to f by Lemma 5. By Lemma 16, we have $|B(D)| \leq m(|F(D)|+1) \leq 2m|U \cap A|$. Hence, $U = (U \cap A) \rtimes_{F(D)} D$ is normal to f by Lemma 4. Suppose D has 2 or more defects. Then $U = (U \cap A) \rtimes_{F(D)} D$ is normal to f by Lemma 17.

Case $B(D) \neq B(W)$. If D has a defect then U is normal to f by Lemma 18. So, we can assume that D has no defects. Recall that we are assuming $F(D)$ is non-empty. Suppose $|F(D)| = 1$. Since $B(D) \neq B(W)$, $B(D)$ is a path. So, U is an extension of $U \cap A$ by a paths. Hence, U is normal to f by Lemma 7 (1). Now, we can assume that $|F(D)| \geq 2$. Since D has no defects, $F(D)$ contains an isolated vertex of A by (2) in the definition of a canonical extension of A. Hence, $U \cap A$ is $m|U \cap A|$-normal by Lemma 5. Since D is smooth with no defects, we have $|B(D)| \leq m|F(D)| \leq m|U \cap A|$ by Lemma 16 (2). Thus, U is normal to f. □

Lemma 22. *Let $G = C_0 \otimes_A C_1$ where C_0 is a canonical extension of A by a wreath W_0 for α, and C_1 a canonical extension of A by a wreath W_1 for α. Suppose that A, W_0 and W_1 belong to $\mathbf{K}_f$, $|F(W_0)| \geq 3$ and $|F(W_1)| \geq 3$. Then G belongs to $\mathbf{K}_f$.*

Proof. Suppose $U \subseteq G = C_0 \otimes_A C_1$. We show that U is normal to f. We can write $U = U_0 \otimes_{U \cap A} U_1$ where $U_0 = U \cap C_0$, and $U_1 = U \cap C_1$. We can also write $U_0 = (U \cap A) \rtimes_{F(D_0)} D_0$ with $D_0 \subseteq W_0$ and $U_1 = (U \cap A) \rtimes_{F(D_1)} D_1$ with $D_1 \subseteq W_1$.

If $D_0 \neq W_0$ and $D_1 \neq W_1$ then $U \cap A < U_0$ and $U \cap A < U_1$ by Lemmas 14 and 8 (1). By Lemma 21, C_1 and C_2 belong to $\mathbf{K}_f$. Hence, U_0 and U_1 belong to $\mathbf{K}_f$. Therefore, $U \in \mathbf{K}_f$ by Proposition 1, and thus U is normal to f.

Now, we can assume that $D_0 = W_0$ or $D_1 = W_1$. By symmetry, we can assume that $D_0 = W_0$. We have $U_0 = (U \cap A) \rtimes_{F(W_0)} W_0$. Since $F(W_0) \subseteq V(U \cap A)$, $U \cap A$ has at least 2 isolated vertices by Definition 21. $U \cap A$ is $3m|U \cap A|$-normal to f by Lemma 5. Since $|B(W_0)| \leq m|F(W_0)| \leq m|U \cap A|$, U_0 is normal to f.

Now, our aim is to show that $U = U_0 \rtimes_{F(D_1)} D_1$ is normal to f.

By Lemma 15 (4), we can assume that D_1 is smooth and connected.

Case D_1 has at most 1 defect. Since $F(D_1) \subseteq U \cap A$ and $F(W_0) \subseteq U \cap A$, with Lemma 16, we have $|B(W_0)| + |B(D_1)| \leq m(F(W_0)) + m(|F(D_1)|+1) \leq 3m|U \cap A|$. Therefore, U is normal to f by Lemma 4.

Case D_1 has 2 or more defects. $U = U_0 \rtimes_{F(D_1)} D_1$ is normal to f by Lemma 17. □

Lemma 23. *Let*

$$C_0 = A' \otimes T_1 \otimes T_2 \otimes \cdots \otimes T_k$$

where each T_i is a twig for α for $i = 1, \ldots, k$ and A' a non-empty graph. Put

$$A = A' \otimes F(T_1) \otimes \cdots \otimes F(T_k).$$

Let P be a set of isolated points of A such that $F(T_1) \otimes \cdots \otimes F(T_k) \subseteq P$. Then C_0 is a semicanonical extension of A over P. Let $G = C_0 \otimes_A C_1$ where C_1 is a canonical extension of A by a wreath W for α with $F(W) = V(A)$. Suppose that A' and W belong to $\mathbf{K}_f$ and $|F(W)| \geq 3$. Then G belongs to $\mathbf{K}_f$.

Proof. We show first that C_0 is a semicanonical extension of A over P. Let $C_0^i = A \rtimes_{F(T_i)} T_i$ for $i = 1, \ldots, k$. Then each C_0^i is a canonical extension of A over P by definition. Now, C_0 is a semicanonical extension of A over P where the C_0^i are the components of C_0.

C_0 belongs to $\mathbf{K}_f$ because any twig belongs to $\mathbf{K}_f$ and $\mathbf{K}_f$ has the free amalgamation property. We also have $C_1 \in \mathbf{K}_f$ by Lemma 21. Also, $A \le C_0$, and $A \le C_1$ by Lemma 9 (2) and Fact 2. Hence, $A \le C_0 \otimes_A C_1 = G$ by Fact 2.

By the definition of C_0, we have $|C_0 - A| = km$. Since $F(T_i) \subseteq A$ and $F(T_i)$ is non-empty for each $i = 1, \ldots, k$, we have $|A| > k$. We also have $|C_1 - A| \le m|A|$ by Lemma 16 (2). Hence, $|G - A| \le m(k + |A|) < 2m|A|$.

Let U be a substructure of $G = C_0 \otimes_A C_1$. We show that U is normal to f. We can write $U = U_0 \otimes_{U\cap A} U_1$ where $U_0 = U \cap C_0$, and $U_1 = U \cap C_1$. We can also write $U_1 = (U \cap A) \rtimes_{F(D)} D$ where D is a substructure of W. Then $U = U_0 \rtimes_{F(D)} D$.

By Lemma 15 (2) and (4), it is enough to show that U is normal to f assuming D is smooth and connected, and $F(D)$ is non-empty.

Since $A \le C_0 \otimes_A C_1$, we have $U \cap A \le U_0 \otimes_{U\cap A} U_1$.

By the definition of C_0, by renumbering the indices of the T_i, we can write $U_0 = (U \cap A') \otimes H_1 \otimes \cdots \otimes H_k$ where $H_i = U \cap T_i$ for each i, and $F(H_i)$ is non-empty for $i = 1, \ldots, k_0$, and $F(H_i)$ is empty for $i = k_0 + 1, \ldots, k$. Put $U_0' = (U \cap A') \otimes H_1 \otimes \cdots \otimes H_{k_0}$. Then $U \cap A \subseteq U_0'$ and

$$U_0 = U_0' \otimes B(T_{k_0+1}) \otimes \cdots \otimes B(T_k).$$

Hence,

$$U = (U_0' \rtimes_{F(D)} D) \otimes B(T_{k_0+1}) \otimes \cdots \otimes B(T_k).$$

Note that U_0' and $B(T_i)$ for $i = k_0 + 1, \ldots, k$ belong to $\mathbf{K}_f$ because $U_0 \subseteq C_0$ and $C_0 \in \mathbf{K}_f$ hold. Therefore, in order to show that U is normal to f, it is enough to show that $U_0' \rtimes_{F(D)} D$ is normal to f by the free amalgamation property of $\mathbf{K}_f$.

If $k_0 = 0$, then $U_0' = U \cap A$ and thus $U_0' \otimes_{U\cap A} U_1 = U_1 \subseteq C_1$. Note that $U_0' \otimes_{U\cap A} U_1 = U_0' \rtimes_{F(D)} D$. Hence it is normal to f since $C_1 \in \mathbf{K}_f$.

We can assume that $k_0 \ge 1$. Then $U \cap A$ has at least 1 isolated vertex. If $U \cap A$ has only 1 isolated vertex, then U_0' is an extension of $U \cap A$ by paths. Hence, $U_0' \otimes_{U\cap A} U_1$ is an extension of U_1 by paths. Since U_1 is normal to f, so is $U_0' \otimes_{U\cap A} U_1$.

Now, we can assume that $U \cap A$ has at least 2 isolated vertices. Recall that $U_1 = (U \cap A) \rtimes_{F(D)} D$ where D is a smooth connected substructure of W.

Case D has at most 1 defect. Since $F(H_i) \subseteq U \cap A$ for $i = 1, \ldots, k_0$, and $F(D) \subseteq U \cap A$, with Lemma 16, we have $|U_0' - (U \cap A)| + |B(D)| \le mk_0 + m(|F(D)| + 1) \le 3m|U \cap A|$. Therefore, $U_0' \rtimes_{F(D)} D = U_0' \otimes_{U\cap A} U_1$ is normal to f by Lemma 4.

Case D has 2 or more defects. $U_0' \rtimes_{F(D)} D$ is normal to f by Lemma 17 because U_0' is normal to f. □

Lemma 24. *Let $C_0 \otimes_A C_1$ be a member of $\mathbf{K}_f$ where C_0 is a zero-extension of A, C_1 a canonical extension of A by a wreath W_1 for α with $F(W_1) = V(A)$. Let*

$$G = C_0 \otimes_A C_1 \otimes_A C_2 \otimes_A \cdots \otimes_A C_n$$

where $C_i \cong_A C_1$ for each $i = 2, \ldots, n$. If G is normal to f then $G \in \mathbf{K}_f$.

Proof. Note that $C_0 \otimes_A C_1$ and $C_0 \otimes_A C_j$ for $j \ge 2$ are isomorphic over C_0. So, $C_0 \otimes_A C_j$ belongs to $\mathbf{K}_f$ for any $j \ge 1$.

We have $C_1 = A \rtimes_{F(W_1)} W_1$ with $F(W_1) = V(A)$. Let W_i for $i \ge 2$ be a wreath isomorphic to W_1 such that $C_i = A \rtimes_{F(W_i)} W_i$.

Suppose $U \subseteq G$.

Case $A \subseteq U$. Since G is normal to f, U is normal to f by Lemma 4.

Case $A \not\subseteq U$. Then $U \cap A$ is a proper subset of A. For each i with $0 \le i \le n$, put $U_i = U \cap C_i$. Then for $i \ge 1$, we have $U_i = (U \cap A) \rtimes_{F(D_i)} D_i$ where $F(D_i)$ is a proper subset of $F(W_i) = V(A)$. Hence, $F(D_i) < D_i$ by Lemma 14 for each $i \geq 1$. We have $U \cap C_0 = U_0 < U_0 \rtimes_{F(D_i)} D_i$ by Lemma 9. Put $U_i' = U_0 \rtimes_{F(D_i)} D_i$. Then $U_0 < U_i'$. Note that it is possible that $U_0 = U_i'$. Since $U_0 \rtimes_{F(D_i)} D_i = U_0 \otimes_{U\cap A} U_i$, we have

$$U = U_1' \otimes_{U_0} \cdots \otimes_{U_0} U_n'.$$

Since $U_i' = U_0 \otimes_{U\cap A} U_i$ is a substructure of $C_0 \otimes_A C_i \in \mathbf{K}_f$, U_i' belongs to $\mathbf{K}_f$ for $i = 1, \ldots, n$. Therefore, U belongs to $\mathbf{K}_f$ by the free amalgamation property. □

With the following proposition and Theorem 5, we get Theorem 8.

Proposition 6. *Let A be a graph in $\mathbf{K}_f$. Then there is a graph C in $\mathbf{K}_f$ such that $A < C$ and C is absolutely closed in $\mathbf{K}_f$.*

Proof. Suppose $A \in \mathbf{K}_f$. We can assume that A is non-empty because if we find an absolutely closed structure C in $\mathbf{K}_f$ then we have $\emptyset < C$ anyway. Let l_0 be an integer such that any cycle of length l_0 or more belongs to $\mathbf{K}_f$. Such an integer l_0 exists by Proposition 3. Let l_1 be such that $l_1 m \geq l_0$. Let T_1 be a twig for α. Choose an integer l_2 greater than $|A|$, $l_1|F(T_1)|$ and 5. Let W be a wreath for α such that $|B(W)| = (|A| + l_2)m$. $B(W)$ belongs to $\mathbf{K}_f$ because $|B(W)| > l_2 m > l_1 m \geq l_0$. Hence, W belongs to $\mathbf{K}_f$ and $|F(W)| \geq |A| + l_2$ by Lemma 11.

Let $A_1 = A \otimes P$ where P is a graph with no edges and such that $|F(W)| = |A| + |P|$. Then we have $l_2 \leq |P|$. Therefore, we have $|A| < |P|$, $5 < |P|$, and $l_1|F(T_1)| < |P|$. Also, we have $|F(W)| > l_2 > 5 > 3$.

Let C_1 be a canonical extension of A_1 by W over P. C_1 exists by Lemma 20. Since $A_1 = A \otimes P$ belongs to $\mathbf{K}_f$ by the free amalgamation property, C_1 belongs to $\mathbf{K}_f$ by Lemma 21. Also, C_1 is a zero-extension of A_1 by Lemma 19. Hence, we have $\delta(C_1) = \delta(A_1)$ and $C_1 - A_1 = B(W)$. Therefore, A_1 is $|B(W)|$-normal to f. Let p be a greatest integer u such that A_1 is u-normal to f. This means that A_1 is p-critical to f. We have $|B(W)| \leq p$ since A_1 is $|B(W)|$-normal to f. Put $k = |A| + l_2$. Then $|B(W)| = km$. So, $km \leq p$. Let r and q_0 be integers such that $p = q_0 m + r$ with $0 \leq r < m$. We have $0 < k \leq q_0$ since $km \leq p$. Let r_1 and q_1 be integers such that $q_0 = q_1 k + r_1$ and $0 \leq r_1 < k$. Then $q_0 m = q_1(km) + r_1 m$.

Now, our aim is to show that there is a semicanonical extension of A_1 over P in $\mathbf{K}_f$ with size $|A_1| + q_0 m$. Then it will be r-critical to f by Lemma 2 (5).

Claim 1. *There is a semicanonical extension C_0 of A_1 over P such that $|C_0 - A_1| = r_1 m$ and $C_0 \otimes_{A_1} C_1$ belongs to $\mathbf{K}_f$.*

Case $r_1 m \geq l_0$. Let W_0 be a wreath for α where $B(W_0)$ is a cycle of length $r_1 m$. $B(W_0)$ belongs to $\mathbf{K}_\alpha$ because it has length l_0 or more. Since $|F(W_0)| < |F(W)| = |A_1|$, there is a canonical extension C_0 of A_1 by W_0 over P by Lemma 20. We have $|C_0 - A_1| = |B(W_0)| = r_1 m$. Then $C_0 \otimes_{A_1} C_1$ belongs to $\mathbf{K}_f$ by Lemma 22.

Case $r_1 m < l_0$. If $r_1 = 0$, then we have the claim with $C_0 = A_1$.

Suppose $r_1 > 0$. Since $l_0 \leq l_1 m$, we have $r_1 < l_1$. By the choice of P, we have $r_1|F(T_1)| < l_1|F(T_1)| < |P|$. Let $C_0 = A \otimes P' \otimes T_1 \otimes \cdots \otimes T_{r_1}$ where T_i is a twig for α for $i = 2, \ldots, r_1$, and P' a graph with no edges. Since each $F(T_i)$ consists of isolated vertices, by choosing P' properly, we can assume that $P' \otimes F(T_1) \otimes \cdots \otimes F(T_{r_1}) = P$. Note that each T_i is isomorphic to T_1. C_0 is a semicanonical extension of A_1 over P and $C_0 \otimes_{A_1} C_1$ belongs to $\mathbf{K}_f$ by Lemma 23. Also, $|C_0 - A_1| = r_1|B(T_1)| = r_1 m$. Now, we have the claim.

Let $C = C_0 \otimes_{A_1} C_1 \otimes_{A_1} C_2 \otimes_{A_1} \cdots \otimes_{A_1} C_{q_1}$ where $C_i \cong_{A_1} C_1$ for each $i = 2, \ldots, q_1$. Since C_0 is a semicanonical extension of A_1 over P by Claim and each C_i is a canonical extension of A_1 over P, C is a semicanonical extension of A_1 over P by definition. By the construction,

$|C - A_1| = r_1 m + q_1(km) = q_0 m \leq p$. So, C is normal to f since A_1 is p-normal to f. Hence, C belongs to $\mathbf{K}_f$ by the claim above and Lemma 24. Also, C is a zero-extension of A_1 by Lemma 19. Therefore, C is r-critical by Lemma 2 (5). Since $r < m$, C is absolutely closed in $\mathbf{K}_f$ by Lemma 3.

We also have $A < C$ by Proposition 5. □

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REFERENCES

[1] J.T. Baldwin and K. Holland, Constructing ω-stable structures: model completeness, Ann. Pure Appl. Log. **125**, 159–172 (2004).
[2] J.T. Baldwin and S. Shelah, Randomness and semigenericity, Trans. Am. Math. Soc. **349**, 1359–1376 (1997).
[3] J.T. Baldwin and N. Shi, Stable generic structures, Ann. Pure Appl. Log. **79**, 1–35 (1996).
[4] R. Diestel, *Graph Theory*, Fourth Edition, Springer, New York (2010).
[5] K. Holland, Model completeness of the new strongly minimal sets, J. Symb. Log. **64**, 946–962 (1999).
[6] E. Hrushovski, A stable $\aleph_0$-categorical pseudoplane, preprint (1988).
[7] E. Hrushovski, A new strongly minimal set, Ann. Pure Appl. Log. **62**, 147–166 (1993).
[8] K. Ikeda, H. Kikyo, Model complete generic structures, in *the Proceedings of the 13th Asian Logic Conference*, World Scientific, 114–123 (2015).
[9] H. Kikyo, Model complete generic graphs I, RIMS Kokyuroku **1938**, 15–25 (2015).
[10] H. Kikyo, Balanced Zero-Sum Sequences and Minimal Intrinsic Extensions, to appear in RIMS Kokyuroku.
[11] F.O. Wagner, Relational structures and dimensions, in *Automorphisms of first-order structures*, Clarendon Press, Oxford, 153–181 (1994).
[12] F.O. Wagner, *Simple Theories*, Kluwer, Dordrecht (2000).