Zariski Geometries, Lecture 3

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Assumption

There is an irreducible Zariski curve $J$ with an abelian group structure on it.
Lemma (Lemma 4.2.1)

There is a one-dimensional, smooth, faithful family $N$ of curves on $J^2$ through $(0, 0)$. 
Corollary (Cor 4.2.5)

There are a one-dimensional irreducible manifold $U$ and constructible irreducible ternary relations $P, S \subset U^2$ which determine a partial Z-field structure on $U$.
Lemma (Lem 4.2.6)

There is a non-commutative meta abelian Z-group structure $T(U)$ on $U \times U$.

For some dense open $U' \subseteq U$, there is a Z-embedding of the pre-group $T(U')$ into a connected Z-group $G$ such that

- $\dim G = 2$
- $G$ is a solvable group with finite center.
There is a $\mathbb{Z}$-field in $\mathcal{M}$. 
Start with the Z-group $G$ with the finite center $C(G)$. With some arguments, we may assume that $G$ is centerless.

Consider the commutater subgroup $[G, G]$ of $G$, and $1 \neq a \in G$.

**Claim:** $\dim a^G = 1$.

This is because $a^G$ is a normal subgroup of $[G, G]$ and $0 < \dim a^G < \dim G$. 
If \( b \) is another non-unit element of \([G, G]\), we have

\[
a^G = b^G \quad \text{or} \quad a^G \cap b^G = \emptyset.
\]

- We have a partition of \([G, G]/\{1\}\) into one-dimensional orbits.

By dimension argument, we see that there are only finitely many one-dimensional orbits.
Outline of proof (3)

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[\mathcal{G}, \mathcal{G}] contains a connected normal subgroup \([\mathcal{G}, \mathcal{G}]^\circ\) of dimension 1.

Thus \([\mathcal{G}, \mathcal{G}]^\circ = a^G \cup \{1\}\) for some non-unit \(a \in \mathcal{G}\).

Fix \(a\) for the rest of the proof.
[\[G, G\] is a constructible group of dimension 1.

[\[G, G\] contains a connected normal subgroup [\[G, G\]^\circ] of dimension 1.

Thus [\[G, G\]^\circ = a^G \cup \{1\}] for some non-unit \(a \in G\). Fix \(a\) for the rest of the proof.

Put \(K^+ := [G, G]^\circ\), and write the group operation of \(K^+\) additively.
The group $G$ acts on $K^+$ by conjugation; for $g \in G$ and $x \in K^+$

$$gx := g^{-1}xg$$
Outline of proof (4)

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- Recall we have fixed non-unit $a \in G$. $C(a)$ is a normal subgroup of $G$.

- Notice that $g_1$ and $g_2$ induces the same action if
  \[ g_1^{-1}g_2 \in C(a) \]
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- $K^\times$ acts transitively on $K^+ - \{0\}$.
- $K^\times$ is a connected one-dimensional group, hence commutative.
Outline of proof (6)

For $g, g_1, g_2 \in K^\times$, we have;

- If $ga = g_1a + g_2a$ then $gx = g_1x + g_2x$ for all $x \in K^+ - \{0\}$.
- If $g_1a = -g_2a$ then $g_1x = -g_2x$ for all $x \in K^+ - \{0\}$.

We can identify $g \in K^\times$ with $ga \in K^+ - \{0\}$.

Transfer the multiplication of $K^\times$ to $K^+$.

The manifold $K^+$ gets the addition and the multiplication, both operations are Z-closed, hence $K^+$ is a Z-field!
Example 4.2.9 (1)

Work $V_1 \subseteq (K^+)^*$. Consider a family of curves of the form

$$g(v) = a \cdot v + b.$$ 

We need $g(1) = 1$. Hence, put $b = 1 - a$. Thus,

$$g_a(v) = a \cdot v + 1 - a.$$
Example 4.2.9 (2)

For \( g_a(v) = a \cdot v + 1 - a \), \( g(v) = b \cdot v + 1 - b \), we have

\[
g_a(g_b(v)) = a(b \cdot v + 1 - b) + 1 - a = ab \cdot v + 1 - ab
\]

On the other hand,

\[
g_{ab}(v) = ab \cdot v + 1 - ab
\]

Thus, we have

\[
g_a \circ g_b(v) = g_a(g_b(v)) = g_{ab}(v)
\]
We use the multiplication to introduce

\[
g_a(v) \oplus g_b(v) = (a \cdot v + 1 - a) \cdot (b \cdot v + 1 - b) \\
= ab \cdot v^2 + (a + b - 2ab) \cdot v + ab + 1 - a - b \\
= f(v)
\]

This curve has a derivative at 1; \( f'(1) = a + b \).

Thus \( f(v) \) is tangent to \( g_{a+b} = (a + b) \cdot v + 1 - a - b \) at \( (1, 1) \).

Therefore,

\[
g_a \oplus g_b T g_{a+b}
\]
Theorem (Thm 4.4.1)

Let $M$ be a Zariski structure satisfying (EU) and $C$ be a pre-smooth Zariski curve in $M$. Assume that $C$ is non-linear (i.e., ample). Then there is a non-constant continuous map

$$f : C \to \mathbb{P}^1(K)$$

for some algebraically closed field $K$. 

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Theorem (Thm 4.3.9, generalized Chow Theorem)

*Any closed subset of \( \mathbb{P}^n \) is an algebraic subvariety of \( \mathbb{P}^n \).*

Let \( K \) be the acf interpreted in an ample Zariski geometry \( M \).

Theorem (Thm 4.3.10, purity Theorem)

*Any relation \( R \subseteq K^m \) definable in the natural language of \( M \) is constructible i.e., a boolean combination of algebraic sets.*
Simple Zariski groups are algebraic

Theorem (Thm 4.4.6)

- $G$ is a simple Z-group satisfying (EU)
- Some one-dim irred Z-subset $C \subseteq G$ is pre-smooth.

Then $G$ is Z-isomorphic to an algebraic group $\hat{G}(K)$ for some algebraically closed field $K$. 
Z-meromorphic functions

Given a Zariski set $N$ and an algebraically closed field $K$.

**Definition (Def 4.4.2)**

A continuous function $g : N \rightarrow K$ with the domain containing an open subset of $N$ is called Z-meromorphic.

**Proposition**

The set of all Z-meromorphic functions from $N$ to $K$ forms a field $K_Z(N)$.

**Definition (p. 101)**

$K_Z(N)$ is called the set of all Z-meromorphic functions from $N$ to $K$. 
Let $m = \text{rk}(G)$.

- (General property) For any $G$, simple group of finite Morley rank, $\text{Th}(G)$ is almost strongly-minimal, hence $\aleph_1$-categorical.
Let $C$ be a strongly minimal definable subset of $G$. There is a definable relation $F \subseteq G \times C^m$ giving a finite-to-finite correspondence between a subset $R \subseteq G$ and a subset $D \subseteq C^m$. 
Outline of proof (2)

- Let $C$ be a strongly minimal definable subset of $G$. There is a definable relation $F \subseteq G \times C^m$ giving a finite-to-finite correspondence between a subset $R \subseteq G$ and a subset $D \subseteq C^m$.
- For $G$ in the theorem, there exists a non-constant $\mathbb{Z}$-meromorphic function $G \rightarrow K$. 

Consider the field $K_Z(G)$ of $Z$-meromorphic functions from $G$ to $K$. 
Outline of proof (3)

- Consider the field $K_Z(G)$ of $\mathbb{Z}$-meromorphic functions from $G$ to $K$.
- Each $g \in G$ acts on $K_Z(G)$ by $f(x) \mapsto f(g \cdot c)$ giving a representation of $G$ as a group of automorphisms of $K_Z(G)$. 
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By Rosenlicht’s Theorem and the purity theorem, we see that there is a $G$-invariant finite-dimensional $K$-subspace $V$ of $K_Z(G)$. 
Outline of proof (4)

- $G$ can be represented as a definable subgroup $\hat{G}(K)$ of $GL(V)$. 
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- $G$ can be represented as a definable subgroup $\hat{G}(K)$ of $GL(V)$.
- By the purity theorem again, the definable subgroup $\hat{G}(K)$ is algebraic.
- The representation is isomorphism since $G$ is simple.
Hrushovski proved Geometric Mordel-Lang conjecture using Zariski geometry argument.
A recent application by Zilber, 2010

Zilber reproved a theorem of Bogomorph, Korotaev and Tschinkel with the help of Zariski geometry.

**Theorem**

*For any algebraically closed fields* $K_1$ *and* $K_2$ *there is a field isomorphism* $\beta : K_1 \to K_2$ *inducing an isomorphism of pairs and a bijective isogeny* $\psi$ *such that*

$$\alpha = \psi \circ \beta : (J_1; C_1, +) \to (J_2 : C_2, +)$$