The Simple Regression Model

$y = \beta_0 + \beta_1 x + u$

Ch.2 The simple regression model

1. Definition of the simple regression model
2. Deriving the OLS estimates
3. Mechanics of OLS
4. Units of measurement & functional form
5. Expected values & variances of OLSE
6. Regression through the origin

2.1 Definition of the model

- Equation (2.1), $y = \beta_0 + \beta_1 x + u$, defines the *Simple Regression model*.
- In the model, we typically refer to
  - y as the Dependent Variable
  - x as the Independent Variable
  - β s as parameters, and
  - u as the error term.

TABLE 2.1

Terminology for Simple Regression

y	*x*
Dependent variable	Independent variable
Explained variable	Explanatory variable
Response variable	Control variable
Predicted variable	Predictor variable
Regressand	Regressor

The Concept of Error Term

- u represents factors other than x that affect y.
- If the other factors in u are held fixed, so that $\Delta u = 0$, then $\Delta y = \beta_1 \Delta x$.
- Ex. 2.1: $yield = \beta_0 + \beta_1 fertilizer + u$ (2.3)
  - u includes land quality, rainfall, etc.
- Ex. 2.2: $wage = \beta_0 + \beta_1 educ + u$ (2.4)
  - u includes experience, ability, tenure, etc.

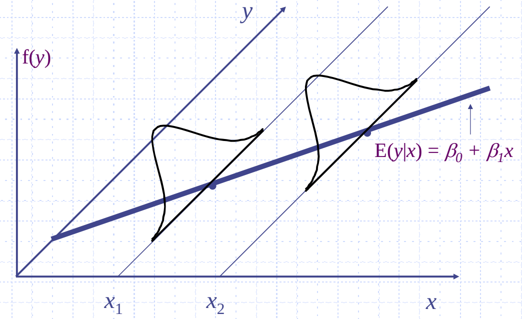
A Simple Assumption for *u*

- The average value of u, the error term, in the population is 0. That is, $E(u) = 0$.
  - This is not a restrictive assumption, since we can always use β_0 to normalize $E(u)$ to 0.
- To draw ceteris paribus conclusions about how x affects y, we have to hold all other factors (in u) fixed.

Zero Conditional Mean

- We need to make a crucial assumption about how u and x are related.
- We want it to be the case that knowing something about x does not give us any information about u, so that they are completely unrelated. That is, that
  - $E(u|x) = E(u) = 0$ (2.5&2.6), which implies
  - $E(y|x) = \beta_0 + \beta_1 x$ **(PRF)** (2.8)

E(y|x) as a linear function of x, where for any x the distribution of y is centered about E(y|x)

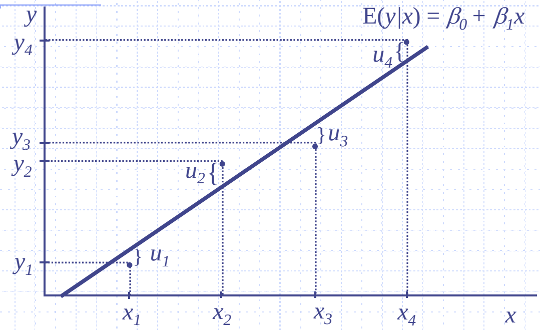


2.2 Deriving the OLSE

- Basic idea of regression is to estimate the ***population parameters*** from a ***sample***.
- Let $\{(x_i, y_i): i = 1, \ldots, n\}$ denote a random sample of size n from the population.
- For each observation in this sample, it will be the case that

$$y_i = \beta_0 + \beta_1 x_i + u_i. \qquad (2.9)$$

Population regression line, sample data points and the associated error terms



Deriving OLSE using MM

- To derive the OLS estimates, we need to realize that our main assumption of
  - $E(u|x) = E(u) = 0$ also implies that
  - $Cov(x,u) = E(xu) = 0$
    - Because $Cov(X,Y) = E(XY) - E(X)E(Y)$ (B.27)
- Now we prepare 2 restrictions to estimate βs.
  - $E(u) = 0$ (2.10)
  - $E(xu) = 0$ (2.11)

Cont. Deriving OLSE using MM

- Since $u = y - \beta_0 - \beta_1 x$, we can rewrite;

$$E(u) = E(y - \beta_0 - \beta_1 x) = 0 \qquad (2.12)$$

$$E(xu) = E[x(y - \beta_0 - \beta_1 x)] = 0 \qquad (2.13)$$

- These are called moment restrictions
  - The approach to estimation implies imposing the population moment restrictions on the sample moments. It means, a sample estimator of E(X), the mean of a population distribution, is simply the arithmetic mean of the sample.

More Derivation of OLS

- We want to choose values of the parameters that will ensure that the sample versions of our moment restrictions are true
- The sample versions are as follows:

$$n^{-1}\sum_{i=1}^{n}\left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i\right) = 0 \quad (2.14)$$

$$n^{-1}\sum_{i=1}^{n} x_i\left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i\right) = 0 \quad (2.15)$$

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Cont. More Derivation of OLS

- Given the definition of a sample mean, and properties of summation, we can rewrite the first condition as follows

$$\bar{y} = \hat{\beta}_0 + \hat{\beta}_1\bar{x} \quad (2.16) \quad \text{or} \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x} \quad (2.17)$$

- So the OLS estimated slope is

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \quad (2.19)$$

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Summary of OLS slope estimate

- The slope estimate is the sample covariance between *x* and *y* divided by the sample variance of *x*.
- If *x* and *y* are positively (negatively) correlated, the slope will be positive (negative).
- *x* needs to vary in our sample.
  - See (2.18) & Figure (2.3)

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More OLS

- Intuitively, OLS is fitting a line through the sample points such that the ***sum of squared residuals*** is as small as possible, hence the term is called least squares.
- The ***residual***, $\hat{u}$, is an estimate of the error term, u, and is the difference between the fitted line (sample regression function) and the sample point.

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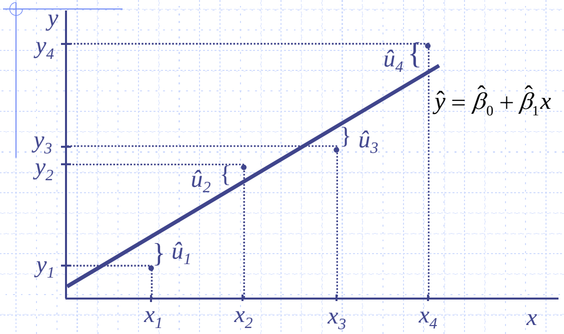
Sample regression line, sample data points and the associated estimated error terms



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Alternate approach to derivation

- Given the intuitive idea of fitting a line, we can set up a formal minimization problem.

$$\sum_{i=1}^{n}(\hat{u}_i)^2 = \sum_{i=1}^{n}\left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i\right)^2 \quad (2.22)$$

- The first order conditions, which are the almost same as (2.14) & (2.15),

$$\sum_{i=1}^{n}\left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i\right) = 0, \quad \sum_{i=1}^{n} x_i\left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i\right) = 0$$

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2.3 Properties of OLS

Algebraic Properties of OLS

1. The sum of the OLS residuals is zero. Thus, the sample average of the OLS residuals is zero as well.

$$\sum_{i=1}^{n} \hat{u}_i = 0 \text{ and thus, } \frac{1}{n}\sum_{i=1}^{n} \hat{u}_i = 0 \quad (2.30)$$

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Cont. Algebraic Properties

2. The sample covariance between the regressors and the OLS residuals is zero

$$\sum_{i=1}^{n} x_i \hat{u}_i = 0 \quad (2.31)$$

3. The OLS regression line always goes through the mean of the sample

$$\bar{y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{x}$$

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Cont. Algebraic Properties

We can think of each observation as being made up of an explained part, and an unexplained part, $y_i = \hat{y}_i + \hat{u}_i$ (2.32) Then we define the following:

$$\sum (y_i - \bar{y})^2 \equiv \text{SST} \quad (2.33)$$

$$\sum (\hat{y}_i - \bar{y})^2 \equiv \text{SSE} \quad (2.34)$$

$$\sum \hat{u}_i^2 \equiv \text{SSR} \quad (2.35)$$

Then, $\text{SST} = \text{SSE} + \text{SSR}$ (2.36)

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Goodness-of-Fit

- It's useful we think about how well the sample regression line fits sample data.
- From (2.36),

$$R^2 \equiv \frac{SSE}{SST} = 1 - \frac{SSR}{SST} \quad (2.38).$$

- R^2 indicates the fraction of the sample variation in y_i that is explained by the model.

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2.4 Measurement Units & Function Form

- If we use the model $y^* = \beta_0^* + \beta_1^* x^* + u^*$ instead of $y = \beta_0 + \beta_1 x + u$, we get

$$\hat{\beta}_0^* = c\hat{\beta}_0 \text{ and } \hat{\beta}_1^* = \frac{c}{d}\hat{\beta}_1$$

where $y^* = c\, y$ and $x^* = d\, x$. Similarly,

$$\hat{\beta}_1^* = \frac{\partial y}{\partial x} \cdot \frac{x}{y} \Leftrightarrow \hat{\beta}_1 = \frac{\partial y}{\partial x}$$

where $y^* = \ln y$ and $x^* = \ln x$.

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2.5 Means & Variance of OLSE

- Now, we view $\hat{\beta}_i$ as estimators for the parameters β_i that appears in the population, which means properties of the distributions of $\hat{\beta}_i$ over different random samples from the population.

Unbiasedness of OLS

- **Unbiased estimator:** An estimator whose expected value (or mean of its sampling distribution) equals the population value (regardless of the population value).

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Cont. Unbiasedness of OLS

Assumption for unbiasedness

1. Linear in parameters as $y = \beta_0 + \beta_1 x + u$
2. Random sampling $\{(x_i, y_i): i = 1, 2, \ldots, n\}$,

   Thus, $y_i = \beta_0 + \beta_1 x_i + u_i$
3. Sample variation in the x_i, thus

   $$\sum (x_i - \bar{x})^2 > 0$$
4. Zero conditional mean, $E(u|x) = 0$

Cont. Unbiasedness of OLS

- In order to think about unbiasedness, we need to rewrite our estimator in terms of the population parameter.

$$\hat{\beta}_1 = \frac{\sum (x_i - \bar{x}) y_i}{\sum (x_i - \bar{x})^2} = \beta_1 + \frac{\sum (x_i - \bar{x}) u_i}{\sum (x_i - \bar{x})^2} \quad (2.49), (2.52)$$

$$\text{then } E(\hat{\beta}_1) = \beta_1 + \frac{\sum (x_i - \bar{x})}{\sum (x_i - \bar{x})^2} \cdot E(u_i \mid x) = \beta_1 \quad (2.53)$$

* we can also get $E(\hat{\beta}_0) = \beta_0$ in the same way.

Unbiasedness Summary

- The OLS estimates of β_1 and β_0 are unbiased.
- Proof of unbiasedness depends on our 4 assumptions – if any assumption fails, then OLS is not necessarily unbiased.
- Remember unbiasedness is a description of the ***estimator*** – in a given sample our ***estimate*** may be “near” or “far” from the true parameter.

Variances of the OLS Estimators

- Now we know that the sampling distribution of our estimate is centered around the true parameter.
  - We want to think about how spread out this distribution is.
  - It is much easier to think about this variance under an additional assumption, so assume

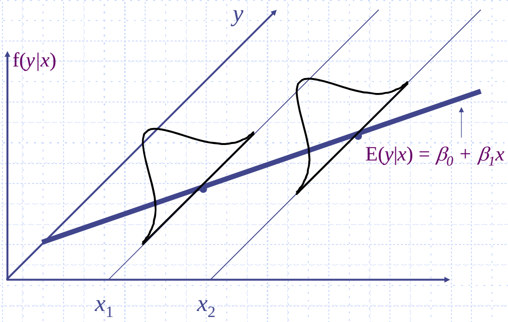
5. $\text{Var}(u|x) = \sigma^2$ (Homoskedasticity)

Cont. Variance of OLSE

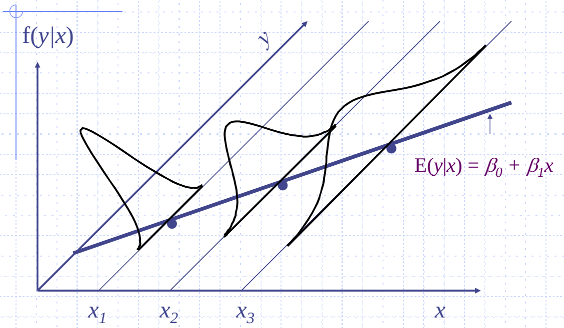
- σ^2 is also the unconditional variance, called the ***error variance***, since
  - $\text{Var}(u|x) = E(u^2|x) - [E(u|x)]^2$
  - $E(u|x) = 0$, so $\sigma^2 = E(u^2|x) = E(u^2) = \text{Var}(u)$
  - And σ, the square root of the error variance, is called the ***standard deviation of the error***.
- Then we can say

  $E(y|x) = \beta_0 + \beta_1 x$ and $\text{Var}(y|x) = \sigma^2$

Homoskedastic Case



Heteroskedastic Case



Cont. Variance of OLSE

$$Var\ (\hat{\beta}_1) = \frac{\sigma^2}{\sum (x_i - \bar{x})^2} \quad (2.57)$$

- The larger the error variance, σ^2, the larger the variance of the slope estimate.
- The larger the variability in the x_i, the smaller the variance of the slope estimate.
- As a result, a larger sample size should decrease the variance of the slope estimate.

Estimating the Error Variance

- We don't know what is the error variance, σ^2, because we don't observe the errors, u_i.
- What we observe are only the residuals, $\hat{u}_i$, not the errors, u_i.
- So we can use the residuals to form an estimate of the error variance.

Cont. Error Variance Estimate

$$\begin{aligned}\hat{u}_i &= y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \\ &= (\beta_0 + \beta_1 x_i + u_i) - \hat{\beta}_0 - \hat{\beta}_1 x_i \\ &= u_i - (\hat{\beta}_0 - \beta_0) - (\hat{\beta}_1 - \beta_1) x_i\end{aligned}$$

Then, an unbiased estimator of σ^2 is

$$\hat{\sigma}^2 = \frac{1}{(n-2)} \sum \hat{u}_i^2 \quad (2.61)$$

Cont. Error Variance Estimate

$\hat{\sigma} = \sqrt{\hat{\sigma}^2}$ = Standard error of the regression

recall that s.d.$(\hat{\beta}) = \sqrt{Var(\hat{\beta})}$

If we substitute $\hat{\sigma}$ for σ, then we have

the standard error of $\hat{\beta}_1$,

$$\text{se}(\hat{\beta}_1) = \sqrt{\frac{\hat{\sigma}^2}{\sum (x_i - \bar{x})^2}}$$

2.6 Regression through the Origin

- Now, consider the model without a intercept:

$$\tilde{y} = \tilde{\beta}_1 x \quad (2.63).$$

- Solving the FOC to the minimization problem, OLS estimated slope is

$$\tilde{\beta}_1 = \frac{\sum x_i y_i}{\sum x_i^2} \quad (2.66).$$

* Recall that a intercept can always normalize E(u) to 0 in the model with β_0.