Forcing iterations and Cichon’s diagram

Diego Alejandro Mejía Guzmán

Graduate School of System Informatics
Kobe University
December 11th, 2011
1. Cichon’s diagram
   - Reals
   - Measure
   - Category
   - Cichon’s diagram

2. Forcing and iterations
   - Forcing
   - Iteration of c.c.c. forcing
   - Examples on the Cichon’s diagram

3. Consistency results with 3 or more cardinals
   - Left side of the Cichon’s diagram
   - Right side of the Cichon’s diagram

4. Questions
The following are the typical spaces considered for the analysis of the real line:

- **The Cantor space** \(2^\omega = \prod_{n<\omega} 2 = \{f / f : \omega \to 2\}\) (recall that \(2 = \{0, 1\}\)).

- **The Baire space** \(\omega^\omega = \prod_{n<\omega} \omega = \{f / f : \omega \to \omega\}\).

- \(\mathbb{R}\) the set of real numbers.

- \([0, 1]\) the unit interval in \(\mathbb{R}\).

These are perfect *Polish spaces* (second countable completely metrizable topological spaces).

- We refer as a *real* to an element of one of those spaces.

- Those spaces, as every perfect Polish space, have size \(c := 2^{\aleph_0}\), the *size of the continuum.*
The following are the typical spaces considered for the analysis of the real line

- **The Cantor space** $2^\omega = \prod_{n<\omega} 2 = \{f / f : \omega \rightarrow 2\}$ (recall that $2 = \{0, 1\}$).
- **The Baire space** $\omega^\omega = \prod_{n<\omega} \omega = \{f / f : \omega \rightarrow \omega\}$.
- $\mathbb{R}$ the set of real numbers.
- $[0, 1]$ the unit interval in $\mathbb{R}$.

These are perfect *Polish spaces* (second countable completely metrizable topological spaces).

- We refer as a *real* to an element of one of those spaces.
- Those spaces, as every perfect Polish space, have size $c := 2^{\aleph_0}$, the *size of the continuum*. 
The following are the typical spaces considered for the analysis of the real line

- **The Cantor space** $2^\omega = \prod_{n<\omega} 2 = \{f / f : \omega \to 2\}$ (recall that $2 = \{0, 1\}$).
- **The Baire space** $\omega^\omega = \prod_{n<\omega} \omega = \{f / f : \omega \to \omega\}$.

- $\mathbb{R}$ the set of real numbers.
- $[0, 1]$ the unit interval in $\mathbb{R}$.

These are perfect *Polish spaces* (second countable completely metrizable topological spaces).

- We refer as a real to an element of one of those spaces.
- Those spaces, as every perfect Polish space, have size $\mathfrak{c} := 2^{\aleph_0}$, the size of the continuum.
The following are the typical spaces considered for the analysis of the real line:

- *The Cantor space* $2^\omega = \prod_{n<\omega} 2 = \{f / f : \omega \to 2\}$ (recall that $2 = \{0, 1\}$).
- *The Baire space* $\omega^\omega = \prod_{n<\omega} \omega = \{f / f : \omega \to \omega\}$.
- $\mathbb{R}$ the set of real numbers.
- $[0, 1]$ the unit interval in $\mathbb{R}$.

These are perfect *Polish spaces* (second countable completely metrizable topological spaces).

- We refer as a *real* to an element of one of those spaces.
- Those spaces, as every perfect Polish space, have size $c := 2^{\aleph_0}$, the *size of the continuum*.
The following are the typical spaces considered for the analysis of the real line:

- **The Cantor space** $2^\omega = \prod_{n<\omega} 2 = \{ f / f : \omega \to 2 \}$ (recall that $2 = \{0, 1\}$).
- **The Baire space** $\omega^\omega = \prod_{n<\omega} \omega = \{ f / f : \omega \to \omega \}$.
- $\mathbb{R}$ the set of real numbers.
- $[0, 1]$ the unit interval in $\mathbb{R}$.

These are perfect *Polish spaces* (second countable completely metrizable topological spaces).

- We refer as a *real* to an element of one of those spaces.
- Those spaces, as every perfect Polish space, have size $c := 2^{\aleph_0}$, the *size of the continuum*. 
The following are the typical spaces considered for the analysis of the real line

- **The Cantor space** \(2^\omega = \prod_{n<\omega} 2 = \{ f \mid f : \omega \to 2 \}\) (recall that \(2 = \{0, 1\}\)).
- **The Baire space** \(\omega^\omega = \prod_{n<\omega} \omega = \{ f \mid f : \omega \to \omega \}\).
- \(\mathbb{R}\) the set of real numbers.
- \([0, 1]\) the unit interval in \(\mathbb{R}\).

These are perfect *Polish spaces* (second countable completely metrizable topological spaces).

- We refer as a *real* to an element of one of those spaces.
- Those spaces, as every perfect Polish space, have size \(c := 2^{\aleph_0}\), the *size of the continuum*. 
The following are the typical spaces considered for the analysis of the real line:

- **The Cantor space** $2^\omega = \prod_{n<\omega} 2 = \{f / f : \omega \to 2\}$ (recall that $2 = \{0, 1\}$).
- **The Baire space** $\omega^\omega = \prod_{n<\omega} \omega = \{f / f : \omega \to \omega\}$.
- $\mathbb{R}$ the set of real numbers.
- $[0, 1]$ the unit interval in $\mathbb{R}$.

These are perfect *Polish spaces* (second countable completely metrizable topological spaces).

- We refer as a *real* to an element of one of those spaces.
- Those spaces, as every perfect Polish space, have size $c := 2^{\aleph_0}$, the *size of the continuum*. 
The following are the typical spaces considered for the analysis of the real line:

- **The Cantor space** $2^\omega = \prod_{n<\omega} 2 = \{ f / f: \omega \to 2 \}$ (recall that $2 = \{0, 1\}$).
- **The Baire space** $\omega^\omega = \prod_{n<\omega} \omega = \{ f / f: \omega \to \omega \}$.
- $\mathbb{R}$ the set of real numbers.
- $[0, 1]$ the unit interval in $\mathbb{R}$.

These are perfect *Polish spaces* (second countable completely metrizable topological spaces).

- We refer as a *real* to an element of one of those spaces.
- Those spaces, as every perfect Polish space, have size $c := 2^{\aleph_0}$, the *size of the continuum*.
On a perfect polish space $X$,

- $\mathcal{B}(X)$ is the $\sigma$-algebra generated by the open sets of $X$. A *borel subset of $X$* is a member of $\mathcal{B}(X)$.

On the cantor space $2^\omega$ is defined a standard *measure* $\mu : \mathcal{B}(2^\omega) \rightarrow [0,1]$.

- $N \subseteq 2^\omega$ is *null* if $N \subseteq B$ for some borel set such that $\mu(B) = 0$.
- $\mathcal{L}(2^\omega)$ is the $\sigma$-algebra generated by the open sets and the null sets of $2^\omega$. An object in that family is called *(Lebesgue)* measurable.
- The measure $\mu$ can be extended to a *complete measure* on $\mathcal{L}(2^\omega)$, which we still denote by $\mu$. Here, $N \subseteq 2^\omega$ is null iff $\mu(N) = 0$.
- $\mathcal{N}(2^\omega)$ denotes the $\sigma$-ideal of null sets in $2^\omega$. 
On a perfect polish space $X$, 

- $\mathcal{B}(X)$ is the $\sigma$-algebra generated by the open sets of $X$. A borel subset of $X$ is a member of $\mathcal{B}(X)$.

On the cantor space $2^\omega$ is defined a standard measure $\mu : \mathcal{B}(2^\omega) \to [0, 1]$.

- $N \subseteq 2^\omega$ is null if $N \subseteq B$ for some borel set such that $\mu(B) = 0$.
- $\mathcal{L}(2^\omega)$ is the $\sigma$-algebra generated by the open sets and the null sets of $2^\omega$. An object in that family is called (Lebesgue) measurable.

- The measure $\mu$ can be extended to a complete measure on $\mathcal{L}(2^\omega)$, which we still denote by $\mu$. Here, $N \subseteq 2^\omega$ is null iff $\mu(N) = 0$.
- $\mathcal{N}(2^\omega)$ denotes the $\sigma$-ideal of null sets in $2^\omega$. 
Measure

On a perfect polish space $X$,

- $\mathcal{B}(X)$ is the $\sigma$-algebra generated by the open sets of $X$. A *borel subset of $X$* is a member of $\mathcal{B}(X)$.

On the cantor space $2^\omega$ is defined a standard *measure* $\mu : \mathcal{B}(2^\omega) \to [0, 1]$.

- $N \subseteq 2^\omega$ is *null* if $N \subseteq B$ for some borel set such that $\mu(B) = 0$.
- $\mathcal{L}(2^\omega)$ is the $\sigma$-algebra generated by the open sets and the null sets of $2^\omega$. An object in that family is called *(Lebesgue)* measurable.
- The measure $\mu$ can be extended to a *complete measure* on $\mathcal{L}(2^\omega)$, which we still denote by $\mu$. Here, $N \subseteq 2^\omega$ is null iff $\mu(N) = 0$.
- $\mathcal{N}(2^\omega)$ denotes the $\sigma$-ideal of null sets in $2^\omega$. 
Measure

On a perfect polish space $X$,

- $\mathcal{B}(X)$ is the $\sigma$-algebra generated by the open sets of $X$. A **borel subset of $X$** is a member of $\mathcal{B}(X)$.

On the cantor space $2^\omega$ is defined a standard **measure** $\mu : \mathcal{B}(2^\omega) \to [0, 1]$.

- $N \subseteq 2^\omega$ is **null** if $N \subseteq B$ for some borel set such that $\mu(B) = 0$.
- $\mathcal{L}(2^\omega)$ is the $\sigma$-algebra generated by the open sets and the null sets of $2^\omega$. An object in that family is called (Lebesgue) **measurable**.
- The measure $\mu$ can be extended to a **complete measure** on $\mathcal{L}(2^\omega)$, which we still denote by $\mu$. Here, $N \subseteq 2^\omega$ is null iff $\mu(N) = 0$.
- $\mathcal{N}(2^\omega)$ denotes the $\sigma$-ideal of null sets in $2^\omega$. 
Measure

On a perfect polish space $X$,
- $\mathcal{B}(X)$ is the $\sigma$-algebra generated by the open sets of $X$. A \textit{borel subset of $X$} is a member of $\mathcal{B}(X)$.

On the cantor space $2^\omega$ is defined a standard \textit{measure} $\mu : \mathcal{B}(2^\omega) \rightarrow [0, 1]$.
- $N \subseteq 2^\omega$ is \textit{null} if $N \subseteq B$ for some borel set such that $\mu(B) = 0$.
- $\mathcal{L}(2^\omega)$ is the $\sigma$-algebra generated by the open sets and the null sets of $2^\omega$. An object in that family is called \textit{(Lebesgue) measurable}.
- The measure $\mu$ can be extended to a \textit{complete measure} on $\mathcal{L}(2^\omega)$, which we still denote by $\mu$. Here, $N \subseteq 2^\omega$ is null iff $\mu(N) = 0$.
- $\mathcal{N}(2^\omega)$ denotes the $\sigma$-ideal of null sets in $2^\omega$. 

Diego Alejandro Mejía Guzmán

Forcing iterations and Cichon’s diagram
On a perfect polish space $X$,

- $\mathcal{B}(X)$ is the $\sigma$-algebra generated by the open sets of $X$. A *borel subset of $X$* is a member of $\mathcal{B}(X)$.

On the cantor space $2^\omega$ is defined a standard *measure* $\mu : \mathcal{B}(2^\omega) \rightarrow [0, 1]$.

- $N \subseteq 2^\omega$ is *null* if $N \subseteq B$ for some borel set such that $\mu(B) = 0$.

- $\mathcal{L}(2^\omega)$ is the $\sigma$-algebra generated by the open sets and the null sets of $2^\omega$. An object in that family is called *Lebesgue measurable*.

- The measure $\mu$ can be extended to a *complete measure* on $\mathcal{L}(2^\omega)$, which we still denote by $\mu$. Here, $N \subseteq 2^\omega$ is null iff $\mu(N) = 0$.

- $\mathcal{N}(2^\omega)$ denotes the $\sigma$-ideal of null sets in $2^\omega$. 
On a perfect polish space $X$,

- $\mathcal{B}(X)$ is the $\sigma$-algebra generated by the open sets of $X$. A *borel subset of $X$* is a member of $\mathcal{B}(X)$.

On the cantor space $2^\omega$ is defined a standard *measure* $\mu : \mathcal{B}(2^\omega) \to [0, 1]$.

- $N \subseteq 2^\omega$ is *null* if $N \subseteq B$ for some borel set such that $\mu(B) = 0$.

- $\mathcal{L}(2^\omega)$ is the $\sigma$-algebra generated by the open sets and the null sets of $2^\omega$. An object in that family is called *(Lebesgue)* measurable.

- The measure $\mu$ can be extended to a *complete measure* on $\mathcal{L}(2^\omega)$, which we still denote by $\mu$. Here, $N \subseteq 2^\omega$ is null iff $\mu(N) = 0$.

- $\mathcal{N}(2^\omega)$ denotes the $\sigma$-ideal of null sets in $2^\omega$. 

On a perfect polish space $X$,

- $\mathcal{B}(X)$ is the $\sigma$-algebra generated by the open sets of $X$. A *borel subset of $X$* is a member of $\mathcal{B}(X)$.

On the cantor space $2^\omega$ is defined a standard *measure* $\mu : \mathcal{B}(2^\omega) \to [0, 1]$.

- $N \subseteq 2^\omega$ is *null* if $N \subseteq B$ for some borel set such that $\mu(B) = 0$.

- $\mathcal{L}(2^\omega)$ is the $\sigma$-algebra generated by the open sets and the null sets of $2^\omega$. An object in that family is called *(Lebesgue)* measurable.

- The measure $\mu$ can be extended to a *complete measure* on $\mathcal{L}(2^\omega)$, which we still denote by $\mu$. Here, $N \subseteq 2^\omega$ is null iff $\mu(N) = 0$.

- $\mathcal{N}(2^\omega)$ denotes the $\sigma$-ideal of null sets in $2^\omega$. 

Diego Alejandro Mejía Guzmán
On a perfect polish space $X$,

- $\mathcal{B}(X)$ is the $\sigma$-algebra generated by the open sets of $X$. A \textit{borel subset of $X$} is a member of $\mathcal{B}(X)$.

On the cantor space $2^\omega$ is defined a standard \textit{measure} $\mu : \mathcal{B}(2^\omega) \rightarrow [0, 1]$.

- $N \subseteq 2^\omega$ is \textit{null} if $N \subseteq B$ for some borel set such that $\mu(B) = 0$.
- $\mathcal{L}(2^\omega)$ is the $\sigma$-algebra generated by the open sets and the null sets of $2^\omega$. An object in that family is called \textit{(Lebesgue) measurable}.

- The measure $\mu$ can be extended to a \textit{complete measure} on $\mathcal{L}(2^\omega)$, which we still denote by $\mu$. Here, $N \subseteq 2^\omega$ is null iff $\mu(N) = 0$.
- $\mathcal{N}(2^\omega)$ denotes the $\sigma$-ideal of null sets in $2^\omega$. 
Some cardinal invariants for measure

We define the following cardinal numbers.

- **add($\mathcal{N}$)**  The *additivity of the null ideal* is the least size of a family $\mathcal{F} \subseteq \mathcal{N}(2^\omega)$ which union is not null.
- **cov($\mathcal{N}$)** The *covering of the null ideal* is the least size of a family $\mathcal{F} \subseteq \mathcal{N}(2^\omega)$ which union covers all the reals, i.e., $\bigcup \mathcal{F} = 2^\omega$.
- **non($\mathcal{N}$)** The *uniformity of the null ideal* is the least size of a non-null set of reals.
- **cof($\mathcal{N}$)** The *cofinality of the null ideal* is the least size of a cofinal subfamily of $\mathcal{N}(2^\omega)$. $\mathcal{F} \subseteq \mathcal{N}(2^\omega)$ is cofinal in $\mathcal{N}(2^\omega)$ if for every $A \in \mathcal{N}(2^\omega)$ there exists a $B \in \mathcal{F}$ such that $A \subseteq B$. 
Some cardinal invariants for measure

We define the following cardinal numbers.

\( \text{add}(\mathcal{N}) \)  The \textit{additivity of the null ideal} is the least size of a family \( \mathcal{F} \subseteq \mathcal{N}(2^\omega) \) which union is not null.

\( \text{cov}(\mathcal{N}) \)  The \textit{covering of the null ideal} is the least size of a family \( \mathcal{F} \subseteq \mathcal{N}(2^\omega) \) which union covers all the reals, i.e., \( \bigcup \mathcal{F} = 2^\omega \).

\( \text{non}(\mathcal{N}) \)  The \textit{uniformity of the null ideal} is the least size of a non-null set of reals.

\( \text{cof}(\mathcal{N}) \)  The \textit{cofinality of the null ideal} is the least size of a cofinal subfamily of \( \mathcal{N}(2^\omega) \). \( \mathcal{F} \subseteq \mathcal{N}(2^\omega) \) is \textit{cofinal} in \( \mathcal{N}(2^\omega) \) if for every \( A \in \mathcal{N}(2^\omega) \) there exists a \( B \in \mathcal{F} \) such that \( A \subseteq B \).
We define the following cardinal numbers.

- **add(\(\mathcal{N}\))** The *additivity of the null ideal* is the least size of a family \(\mathcal{F} \subseteq \mathcal{N}(2^{\omega})\) which union is not null.

- **cov(\(\mathcal{N}\))** The *covering of the null ideal* is the least size of a family \(\mathcal{F} \subseteq \mathcal{N}(2^{\omega})\) which union covers all the reals, i.e., \(\bigcup \mathcal{F} = 2^{\omega}\).

- **non(\(\mathcal{N}\))** The *uniformity of the null ideal* is the least size of a non-null set of reals.

- **cof(\(\mathcal{N}\))** The *cofinality of the null ideal* is the least size of a cofinal subfamily of \(\mathcal{N}(2^{\omega})\). \(\mathcal{F} \subseteq \mathcal{N}(2^{\omega})\) is *cofinal* in \(\mathcal{N}(2^{\omega})\) if for every \(A \in \mathcal{N}(2^{\omega})\) there exists a \(B \in \mathcal{F}\) such that \(A \subseteq B\).
We define the following cardinal numbers.

**add(\mathcal{N})** The *additivity of the null ideal* is the least size of a family $\mathcal{F} \subseteq \mathcal{N}(2^\omega)$ which union is not null.

**cov(\mathcal{N})** The *covering of the null ideal* is the least size of a family $\mathcal{F} \subseteq \mathcal{N}(2^\omega)$ which union covers all the reals, i.e., $\bigcup \mathcal{F} = 2^\omega$.

**non(\mathcal{N})** The *uniformity of the null ideal* is the least size of a non-null set of reals.

**cof(\mathcal{N})** The *cofinality of the null ideal* is the least size of a cofinal subfamily of $\mathcal{N}(2^\omega)$. $\mathcal{F} \subseteq \mathcal{N}(2^\omega)$ is cofinal in $\mathcal{N}(2^\omega)$ if for every $A \in \mathcal{N}(2^\omega)$ there exists a $B \in \mathcal{F}$ such that $A \subseteq B$. 

Some cardinal invariants for measure

We define the following cardinal numbers.

\[ \text{add}(\mathcal{N}) \quad \text{The additivity of the null ideal} \quad \text{is the least size of a family} \quad \mathcal{F} \subseteq \mathcal{N}(2^\omega) \text{ which union is not null.} \]

\[ \text{cov}(\mathcal{N}) \quad \text{The covering of the null ideal} \quad \text{is the least size of a family} \quad \mathcal{F} \subseteq \mathcal{N}(2^\omega) \text{ which union covers all the reals, i.e.,} \quad \bigcup \mathcal{F} = 2^\omega. \]

\[ \text{non}(\mathcal{N}) \quad \text{The uniformity of the null ideal} \quad \text{is the least size of a non-null set of reals.} \]

\[ \text{cof}(\mathcal{N}) \quad \text{The cofinality of the null ideal} \quad \text{is the least size of a cofinal subfamily of} \quad \mathcal{N}(2^\omega). \quad \mathcal{F} \subseteq \mathcal{N}(2^\omega) \text{ is cofinal in} \quad \mathcal{N}(2^\omega) \text{ if for every} \quad A \in \mathcal{N}(2^\omega) \text{ there exists a} \quad B \in \mathcal{F} \text{ such that} \quad A \subseteq B. \]
Some cardinal invariants for measure

We define the following cardinal numbers.

- **add(\(\mathcal{N}\))** The *additivity of the null ideal* is the least size of a family \(\mathcal{F} \subseteq \mathcal{N}(2^\omega)\) which union is not null.

- **cov(\(\mathcal{N}\))** The *covering of the null ideal* is the least size of a family \(\mathcal{F} \subseteq \mathcal{N}(2^\omega)\) which union covers all the reals, i.e., \(\bigcup \mathcal{F} = 2^\omega\).

- **non(\(\mathcal{N}\))** The *uniformity of the null ideal* is the least size of a non-null set of reals.

- **cof(\(\mathcal{N}\))** The *cofinality of the null ideal* is the least size of a cofinal subfamily of \(\mathcal{N}(2^\omega)\). \(\mathcal{F} \subseteq \mathcal{N}(2^\omega)\) is *cofinal* in \(\mathcal{N}(2^\omega)\) if for every \(A \in \mathcal{N}(2^\omega)\) there exists a \(B \in \mathcal{F}\) such that \(A \subseteq B\).
Likewise, a Lebesgue measure can be defined for the spaces $\omega^\omega$, $\mathbb{R}$ and $[0, 1]$. The value of each of these cardinals doesn’t change if we use one of those spaces. The following diagram represents the order relation between these cardinals.
Let $X$ be a topological space.

- $A \subseteq X$ is *nowhere dense* (n.w.d.) if, for every $G$ non-empty open in $X$, there exists a non-empty open $H \subseteq G$ that doesn’t intersect $A$.
- $M \subseteq X$ is *meager* if it is the countable union of n.w.d. sets.
- $\mathcal{M}(X)$ denotes the $\sigma$-ideal of meager sets in $X$.

**The Baire Category Theorem**

In a complete metrizable space, every nonempty open set is non-meager.
Let $X$ be a topological space.

- $A \subseteq X$ is **nowhere dense (n.w.d.)** if, for every $G$ non-empty open in $X$, there exists a non-empty open $H \subseteq G$ that doesn’t intersect $A$.
- $M \subseteq X$ is **meager** if it is the countable union of n.w.d. sets.
- $\mathcal{M}(X)$ denotes the $\sigma$-ideal of meager sets in $X$.

**The Baire Category Theorem**

In a complete metrizable space, every nonempty open set is non-meager.
Let $X$ be a topological space.

- $A \subseteq X$ is *nowhere dense (n.w.d.)* if, for every $G$ non-empty open in $X$, there exists a non-empty open $H \subseteq G$ that doesn’t intersect $A$.

- $M \subseteq X$ is *meager* if it is the countable union of n.w.d. sets.

- $\mathcal{M}(X)$ denotes the $\sigma$-ideal of meager sets in $X$.

### The Baire Category Theorem

In a complete metrizable space, every nonempty open set is non-meager.
Let $X$ be a topological space.

- $A \subseteq X$ is *nowhere dense (n.w.d.*) if, for every $G$ non-empty open in $X$, there exists a non-empty open $H \subseteq G$ that doesn’t intersect $A$.

- $M \subseteq X$ is *meager* if it is the countable union of n.w.d. sets.

- $\mathcal{M}(X)$ denotes the $\sigma$-ideal of meager sets in $X$.

**The Baire Category Theorem**

In a complete metrizable space, every nonempty open set is non-meager.
Let $X$ be a topological space.

- $A \subseteq X$ is *nowhere dense* (n.w.d.) if, for every $G$ non-empty open in $X$, there exists a non-empty open $H \subseteq G$ that doesn’t intersect $A$.

- $M \subseteq X$ is *meager* if it is the countable union of n.w.d. sets.

- $\mathcal{M}(X)$ denotes the $\sigma$-ideal of meager sets in $X$.

**The Baire Category Theorem**

In a complete metrizable space, every nonempty open set is non-meager.
Some cardinal invariants for category

Like in the case of measure, for a perfect Polish space $X$, we define the following cardinal numbers.

- $\text{add}(\mathcal{M})$: The *additivity of the meager ideal* is the least size of a family $\mathcal{F} \subseteq \mathcal{M}(X)$ which union is not meager.

- $\text{cov}(\mathcal{M})$: The *covering of the meager ideal* is the least size of a family $\mathcal{F} \subseteq \mathcal{M}(X)$ which union covers $X$.

- $\text{non}(\mathcal{M})$: The *uniformity of the meager ideal* is the least size of a non-meager subset of $X$.

- $\text{cof}(\mathcal{M})$: The *cofinality of the meager ideal* is the least size of a cofinal subfamily of $\mathcal{M}(X)$. 
Some cardinal invariants for category

Like in the case of measure, for a perfect Polish space $X$, we define the following cardinal numbers.

- $\text{add}(\mathcal{M})$ The \textit{additivity of the meager ideal} is the least size of a family $\mathcal{F} \subseteq \mathcal{M}(X)$ which union is not meager.

- $\text{cov}(\mathcal{M})$ The \textit{covering of the meager ideal} is the least size of a family $\mathcal{F} \subseteq \mathcal{M}(X)$ which union covers $X$.

- $\text{non}(\mathcal{M})$ The \textit{uniformity of the meager ideal} is the least size of a non-meager subset of $X$.

- $\text{cof}(\mathcal{M})$ The \textit{cofinality of the meager ideal} is the least size of a cofinal subfamily of $\mathcal{M}(X)$. 
Some cardinal invariants for category

Like in the case of measure, for a perfect Polish space $X$, we define the following cardinal numbers.

- $\text{add}(\mathcal{M})$ The *additivity of the meager ideal* is the least size of a family $\mathcal{F} \subseteq \mathcal{M}(X)$ which union is not meager.

- $\text{cov}(\mathcal{M})$ The *covering of the meager ideal* is the least size of a family $\mathcal{F} \subseteq \mathcal{M}(X)$ which union covers $X$.

- $\text{non}(\mathcal{M})$ The *uniformity of the meager ideal* is the least size of a non-meager subset of $X$.

- $\text{cof}(\mathcal{M})$ The *cofinality of the meager ideal* is the least size of a cofinal subfamily of $\mathcal{M}(X)$. 
Some cardinal invariants for category

Like in the case of measure, for a perfect Polish space $X$, we define the following cardinal numbers.

- $\text{add}(\mathcal{M})$ The *additivity of the meager ideal* is the least size of a family $\mathcal{F} \subseteq \mathcal{M}(X)$ which union is not meager.

- $\text{cov}(\mathcal{M})$ The *covering of the meager ideal* is the least size of a family $\mathcal{F} \subseteq \mathcal{M}(X)$ which union covers $X$.

- $\text{non}(\mathcal{M})$ The *uniformity of the meager ideal* is the least size of a non-meager subset of $X$.

- $\text{cof}(\mathcal{M})$ The *cofinality of the meager ideal* is the least size of a cofinal subfamily of $\mathcal{M}(X)$. 
Some cardinal invariants for category

Like in the case of measure, for a perfect Polish space $X$, we define the following cardinal numbers.

**add($\mathcal{M}$)** The *additivity of the meager ideal* is the least size of a family $\mathcal{F} \subseteq \mathcal{M}(X)$ which union is not meager.

**cov($\mathcal{M}$)** The *covering of the meager ideal* is the least size of a family $\mathcal{F} \subseteq \mathcal{M}(X)$ which union covers $X$.

**non($\mathcal{M}$)** The *uniformity of the meager ideal* is the least size of a non-meager subset of $X$.

**cof($\mathcal{M}$)** The *cofinality of the meager ideal* is the least size of a cofinal subfamily of $\mathcal{M}(X)$.
Some cardinal invariants for category

Like in the case of measure, the value of these invariants doesn’t depend on the perfect Polish space used. The following diagram represents the order relation between these cardinals.

\[
\begin{array}{c}
\aleph_1 \quad \text{add}(\mathcal{M}) \quad \text{cov}(\mathcal{M}) \quad \text{cof}(\mathcal{M}) \quad c \\
\text{non}(\mathcal{M})
\end{array}
\]
Two more cardinal invariants

In $\omega^\omega$, define

- For $f, g \in \omega^\omega$, $f \leq^* g$ means that $f(n) \leq g(n)$ for all but finitely many $n < \omega$. Here we say that $f$ is dominated by $g$.

- $\mathcal{F} \subseteq \omega^\omega$ is a dominating family if every real is dominated by some member of $\mathcal{F}$.

- The unbounding number is the least size of a $\leq^*$-unbounded family of reals.

- The dominating number is the least size of a dominating family.
Two more cardinal invariants

In $\omega^\omega$, define

- For $f, g \in \omega^\omega$, $f \leq^* g$ means that $f(n) \leq g(n)$ for all but finitely many $n < \omega$. Here we say that $f$ is dominated by $g$.

- $F \subseteq \omega^\omega$ is a dominating family if every real is dominated by some member of $F$.

  - The unbounding number is the least size of a $\leq^*$-unbounded family of reals.
  - The dominating number is the least size of a dominating family.
Two more cardinal invariants

In $\omega^\omega$, define

- For $f, g \in \omega^\omega$, $f \leq^* g$ means that $f(n) \leq g(n)$ for all but finitely many $n < \omega$. Here we say that $f$ is dominated by $g$.

- $F \subseteq \omega^\omega$ is a dominating family if every real is dominated by some member of $F$.

  - The unbounding number is the least size of a $\leq^*$-unbounded family of reals.
  - The dominating number is the least size of a dominating family.
Two more cardinal invariants

In $\omega^\omega$, define

- For $f, g \in \omega^\omega$, $f \leq^* g$ means that $f(n) \leq g(n)$ for all but finitely many $n < \omega$. Here we say that $f$ is dominated by $g$.

- $\mathcal{F} \subseteq \omega^\omega$ is a dominating family if every real is dominated by some member of $\mathcal{F}$.

- The unbounding number is the least size of a $\leq^*$-unbounded family of reals.

- The dominating number is the least size of a dominating family.
Two more cardinal invariants

In $\omega^\omega$, define

- For $f, g \in \omega^\omega$, $f \leq^* g$ means that $f(n) \leq g(n)$ for all but finitely many $n < \omega$. Here we say that $f$ is dominated by $g$.

- $\mathcal{F} \subseteq \omega^\omega$ is a dominating family if every real is dominated by some member of $\mathcal{F}$.

  - The unbounding number is the least size of a $\leq^*$-unbounded family of reals.
  - The dominating number is the least size of a dominating family.
Two more cardinal invariants

In $\omega^\omega$, define

- For $f, g \in \omega^\omega$, $f \leq^* g$ means that $f(n) \leq g(n)$ for all but finitely many $n < \omega$. Here we say that $f$ is dominated by $g$.

- $\mathcal{F} \subseteq \omega^\omega$ is a dominating family if every real is dominated by some member of $\mathcal{F}$.

  - The unbounding number is the least size of a $\leq^*$-unbounded family of reals.
  - The dominating number is the least size of a dominating family.
In ZFC,

\[
\begin{align*}
\text{cov}(\mathcal{N}) & \quad \text{non}(\mathcal{M}) & \quad \text{cof}(\mathcal{M}) & \quad \text{cof}(\mathcal{N}) \\
\text{add}(\mathcal{N}) & \quad \text{add}(\mathcal{M}) & \quad \text{cov}(\mathcal{M}) & \quad \text{non}(\mathcal{N})
\end{align*}
\]

Also \( \text{add}(\mathcal{M}) = \min\{b, \text{cov}(\mathcal{M})\} \) and \( \text{cof}(\mathcal{M}) = \max\{d, \text{non}(\mathcal{M})\} \).
In ZFC,

\[
\begin{align*}
\text{cov}(\mathcal{N}) & \quad \text{non}(\mathcal{M}) & \quad \text{cof}(\mathcal{M}) & \quad \text{cof}(\mathcal{N}) \\
\text{add}(\mathcal{N}) & \quad \text{add}(\mathcal{M}) & \quad \text{cov}(\mathcal{M}) & \quad \text{non}(\mathcal{N}) \\
\end{align*}
\]

Also \add(M) = \min\{b, \text{cov}(M)\} and \cof(M) = \max\{d, \text{non}(M)\}. 
A **forcing notion** or **p.o. set** is a system \( \langle P, \leq, 1 \rangle \) where \( \leq \) is a reflexive and transitive relation in \( P \) and \( 1 \in P \) is a maximum element. Elements in \( P \) are called **conditions** and \( 1 \) is the **trivial condition**.

From a model \( M \) of ZFC and \( G \) a \( P \)-generic set over \( M \), \( M[G] \) is defined as a model of ZFC that extends \( M \) and contains \( G \). In this context, we call \( M \) the **ground model** and \( M[G] \) the **generic extension**. Every element in \( M[G] \) has a \( P \)-name in the ground model that “codes” it.
Fast review forcing

- A **forcing notion** or **p.o. set** is a system \( \langle P, \leq, 1 \rangle \) where \( \leq \) is a reflexive and transitive relation in \( P \) and \( 1 \in P \) is a maximum element. Elements in \( P \) are called **conditions** and \( 1 \) is the **trivial condition**.

From a model \( M \) of ZFC and \( G \) a \( P \)-generic set over \( M \), \( M[G] \) is defined as a model of ZFC that extends \( M \) and contains \( G \). In this context, we call \( M \) the **ground model** and \( M[G] \) the **generic extension**. Every element in \( M[G] \) has a \( P \)-name in the ground model that “codes” it.
Fast review forcing

A forcing notion or p.o. set is a system $\langle P, \leq, 1 \rangle$ where $\leq$ is a reflexive and transitive relation in $P$ and $1 \in P$ is a maximum element. Elements in $P$ are called conditions and $1$ is the trivial condition.

From a model $M$ of ZFC and $G$ a $P$-generic set over $M$, $M[G]$ is defined as a model of ZFC that extends $M$ and contains $G$. In this context, we call $M$ the ground model and $M[G]$ the generic extension. Every element in $M[G]$ has a $P$-name in the ground model that “codes” it.
A forcing notion or p.o. set is a system \( \langle P, \leq, 1 \rangle \) where \( \leq \) is a reflexive and transitive relation in \( P \) and \( 1 \in P \) is a maximum element. Elements in \( P \) are called conditions and \( 1 \) is the trivial condition.

From a model \( M \) of ZFC and \( G \) a \( P \)-generic set over \( M \), \( M[G] \) is defined as a model of ZFC that extends \( M \) and contains \( G \). In this context, we call \( M \) the ground model and \( M[G] \) the generic extension. Every element in \( M[G] \) has a \( P \)-name in the ground model that “codes” it.
Forcing

A forcing notion or p.o. set is a system \( \langle P, \leq, 1 \rangle \) where \( \leq \) is a reflexive and transitive relation in \( P \) and \( 1 \in P \) is a maximum element. Elements in \( P \) are called conditions and \( 1 \) is the trivial condition.

From a model \( M \) of ZFC and \( G \) a \( P \)-generic set over \( M \), \( M[G] \) is defined as a model of ZFC that extends \( M \) and contains \( G \). In this context, we call \( M \) the ground model and \( M[G] \) the generic extension. Every element in \( M[G] \) has a \( P \)-name in the ground model that “codes” it.
Fast review forcing

- A **forcing notion** or p.o. set is a system \( \langle P, \leq, 1 \rangle \) where \( \leq \) is a reflexive and transitive relation in \( P \) and \( 1 \in P \) is a maximum element. Elements in \( P \) are called **conditions** and \( 1 \) is the **trivial condition**.

From a model \( M \) of ZFC and \( G \) a \( P \)-generic set over \( M \), \( M[G] \) is defined as a model of ZFC that extends \( M \) and contains \( G \). In this context, we call \( M \) the **ground model** and \( M[G] \) the **generic extension**. Every element in \( M[G] \) has a \( P \)-name in the ground model that “codes” it.
Examples of forcing notions

1. **Trivial forcing.** \( \mathbb{1} = \{0\} \). Any generic extension is the same ground model.

2. **Cohen forcing.** The conditions are finite partial functions from \( \omega \) to \( \omega \), ordered by \( \supseteq \). This forcing adds a real in the generic extension that is not in any borel meager set coded in the ground model. Such a real is called *cohen real over the ground model*.

3. **Random forcing.** The conditions are borel non-null subsets of \( 2^\omega \), ordered by inclusion. This forcing adds a real that is not in any borel null set coded in the ground model. Such a real is called *random real over the ground model*. 
Examples of forcing notions

1. **Trivial forcing.** \( \mathbb{1} = \{0\} \). Any generic extension is the same ground model.

2. **Cohen forcing.** The conditions are finite partial functions from \( \omega \) to \( \omega \), ordered by \( \supseteq \). This forcing adds a real in the generic extension that is not in any borel meager set coded in the ground model. Such a real is called *cohen real over the ground model*.

3. **Random forcing.** The conditions are borel non-null subsets of \( 2^\omega \), ordered by inclusion. This forcing adds a real that is not in any borel null set coded in the ground model. Such a real is called *random real over the ground model*. 
Examples of forcing notions

1. **Trivial forcing.** \( 1 = \{0\} \). Any generic extension is the same ground model.

2. **Cohen forcing.** The conditions are finite partial functions from \( \omega \) to \( \omega \), ordered by \( \supseteq \). This forcing adds a real in the generic extension that is not in any borel meager set coded in the ground model. Such a real is called a **Cohen real over the ground model**.

3. **Random forcing.** The conditions are borel non-null subsets of \( 2^\omega \), ordered by inclusion. This forcing adds a real that is not in any borel null set coded in the ground model. Such a real is called a **random real over the ground model**.
Examples of forcing notions

1. **Trivial forcing.** \(\mathbb{1} = \{0\}\). Any generic extension is the same ground model.

C. **Cohen forcing.** The conditions are finite partial functions from \(\omega\) to \(\omega\), ordered by \(\supseteq\). This forcing adds a real in the generic extension that is not in any borel meager set coded in the ground model. Such a real is called *cohen real over the ground model*.

B. **Random forcing.** The conditions are borel non-null subsets of \(2^\omega\), ordered by inclusion. This forcing adds a real that is not in any borel null set coded in the ground model. Such a real is called *random real over the ground model*. 
Examples of forcing notions

1. **Trivial forcing.** \(1 = \{0\}\). Any generic extension is the same ground model.

C. **Cohen forcing.** The conditions are finite partial functions from \(\omega\) to \(\omega\), ordered by \(\supseteq\). This forcing adds a real in the generic extension that is not in any borel meager set coded in the ground model. Such a real is called a **Cohen real over the ground model**.

B. **Random forcing.** The conditions are borel non-null subsets of \(2^\omega\), ordered by inclusion. This forcing adds a real that is not in any borel null set coded in the ground model. Such a real is called a **random real over the ground model**.
Examples of forcing notions

1 Trivial forcing. Let $\mathbb{1} = \{0\}$. Any generic extension is the same ground model.

C Cohen forcing. The conditions are finite partial functions from $\omega$ to $\omega$, ordered by $\supseteq$. This forcing adds a real in the generic extension that is not in any borel meager set coded in the ground model. Such a real is called *cohen real over the ground model*.

B Random forcing. The conditions are borel non-null subsets of $2^\omega$, ordered by inclusion. This forcing adds a real that is not in any borel null set coded in the ground model. Such a real is called *random real over the ground model*. 
Examples of forcing notions

1. **Trivial forcing.** $1 = \{0\}$. Any generic extension is the same ground model.

C. **Cohen forcing.** The conditions are finite partial functions from $\omega$ to $\omega$, ordered by $\supseteq$. This forcing adds a real in the generic extension that is not in any borel meager set coded in the ground model. Such a real is called *cohen real over the ground model*.

B. **Random forcing.** The conditions are borel non-null subsets of $2^\omega$, ordered by inclusion. This forcing adds a real that is not in any borel null set coded in the ground model. Such a real is called *random real over the ground model*. 
Examples of forcing notions

1. **Trivial forcing.** \( \mathbb{1} = \{0\} \). Any generic extension is the same ground model.

C. **Cohen forcing.** The conditions are finite partial functions from \( \omega \) to \( \omega \), ordered by \( \supseteq \). This forcing adds a real in the generic extension that is not in any borel meager set coded in the ground model. Such a real is called *cohen real over the ground model*.

B. **Random forcing.** The conditions are borel non-null subsets of \( 2^\omega \), ordered by inclusion. This forcing adds a real that is not in any borel null set coded in the ground model. Such a real is called *random real over the ground model*. 
Examples of forcing notions

**A. Amoeba forcing.** The conditions are open subsets of $2^\omega$ of measure less than $\frac{1}{2}$, ordered by $\supseteq$. This forcing adds a borel null set coded in the extension that covers all the borel null sets coded in the ground model.

**D. Hechler forcing.** This forcing adds a function in $\omega^\omega$ that dominates all the functions in $\omega^\omega$ in the ground model. Such a real is called *dominating real over the ground model*.

**E. Eventually different real forcing** This forcing adds a function $e \in \omega^\omega$ such that, for every $f \in \omega^\omega$ in the ground model, $e(n) \neq f(n)$ for all but finitely many $n < \omega$. Such a real is called *eventually different real over the ground model*.

All these p.o. sets have the *countable chain condition (c.c.c.)*.
Examples of forcing notions

A Amoeba forcing. The conditions are open subsets of $2^\omega$ of measure less than $\frac{1}{2}$, ordered by $\supseteq$. This forcing adds a borel null set coded in the extension that covers all the borel null sets coded in the ground model.

D Hechler forcing. This forcing adds a function in $\omega^\omega$ that dominates all the functions in $\omega^\omega$ in the ground model. Such a real is called dominating real over the ground model.

E Eventually different real forcing This forcing adds a function $e \in \omega^\omega$ such that, for every $f \in \omega^\omega$ in the ground model, $e(n) \neq f(n)$ for all but finitely many $n < \omega$. Such a real is called eventually different real over the ground model.

All these p.o. sets have the countable chain condition (c.c.c.).
Examples of forcing notions

**A Amoeba forcing.** The conditions are open subsets of $2^\omega$ of measure less than $\frac{1}{2}$, ordered by $\supseteq$. This forcing adds a borel null set coded in the extension that covers all the borel null sets coded in the ground model.

**D Hechler forcing.** This forcing adds a function in $\omega^\omega$ that dominates all the functions in $\omega^\omega$ in the ground model. Such a real is called *dominating real over the ground model*.

**E Eventually different real forcing** This forcing adds a function $e \in \omega^\omega$ such that, for every $f \in \omega^\omega$ in the ground model, $e(n) \neq f(n)$ for all but finitely many $n < \omega$. Such a real is called *eventually different real over the ground model*.

All these p.o. sets have the *countable chain condition* (c.c.c.).
Examples of forcing notions

**A Amoeba forcing.** The conditions are open subsets of $2^\omega$ of measure less than $\frac{1}{2}$, ordered by $\supseteq$. This forcing adds a borel null set coded in the extension that covers all the borel null sets coded in the ground model.

**D Hechler forcing.** This forcing adds a function in $\omega^\omega$ that dominates all the functions in $\omega^\omega$ in the ground model. Such a real is called *dominating real over the ground model*.

**E Eventually different real forcing** This forcing adds a function $e \in \omega^\omega$ such that, for every $f \in \omega^\omega$ in the ground model, $e(n) \neq f(n)$ for all but finitely many $n < \omega$. Such a real is called *eventually different real over the ground model*.

All these p.o. sets have the *countable chain condition (c.c.c.)*. 
Examples of forcing notions

A Amoeba forcing. The conditions are open subsets of $2^\omega$ of measure less than $\frac{1}{2}$, ordered by $\supseteq$. This forcing adds a borel null set coded in the extension that covers all the borel null sets coded in the ground model.

D Hechler forcing. This forcing adds a function in $\omega^\omega$ that dominates all the functions in $\omega^\omega$ in the ground model. Such a real is called *dominating real over the ground model*.

E Eventually different real forcing. This forcing adds a function $e \in \omega^\omega$ such that, for every $f \in \omega^\omega$ in the ground model, $e(n) \neq f(n)$ for all but finitely many $n < \omega$. Such a real is called *eventually different real over the ground model*.

All these p.o. sets have the countable chain condition (c.c.c.).
Examples of forcing notions

A Amoeba forcing. The conditions are open subsets of $2^\omega$ of measure less than $\frac{1}{2}$, ordered by $\supseteq$. This forcing adds a borel null set coded in the extension that covers all the borel null sets coded in the ground model.

D Hechler forcing. This forcing adds a function in $\omega^\omega$ that dominates all the functions in $\omega^\omega$ in the ground model. Such a real is called dominating real over the ground model.

E Eventually different real forcing. This forcing adds a function $e \in \omega^\omega$ such that, for every $f \in \omega^\omega$ in the ground model, $e(n) \neq f(n)$ for all but finitely many $n < \omega$. Such a real is called eventually different real over the ground model.

All these p.o. sets have the countable chain condition (c.c.c.).
Examples of forcing notions

A Amoeba forcing. The conditions are open subsets of $2^\omega$ of measure less than $\frac{1}{2}$, ordered by $\supseteq$. This forcing adds a borel null set coded in the extension that covers all the borel null sets coded in the ground model.

D Hechler forcing. This forcing adds a function in $\omega^\omega$ that dominates all the functions in $\omega^\omega$ in the ground model. Such a real is called dominating real over the ground model.

E Eventually different real forcing This forcing adds a function $e \in \omega^\omega$ such that, for every $f \in \omega^\omega$ in the ground model, $e(n) \neq f(n)$ for all but finitely many $n < \omega$. Such a real is called eventually different real over the ground model.

All these p.o. sets have the countable chain condition (c.c.c.).
Iteration of c.c.c. forcing notions

For an ordinal $\delta$ a \textit{finite support iteration (f.s.i.) of length $\delta$}
\[ P_\delta = \langle P_\alpha, \dot{Q}_\alpha \rangle_{\alpha < \delta} \]
is defined recursively, where

(a) $P_\alpha$ is a p.o. set for $\alpha \leq \delta$, which represents the $\alpha$-stage of the iteration.

(b) $\dot{Q}_\alpha$ is a $P_\alpha$-name for a p.o. for $\alpha < \delta$.

(c) For $\alpha \leq \delta$, each condition in $P_\alpha$ is a sequence $\langle \dot{q}_\xi \rangle_{\xi < \alpha}$ such that

\begin{enumerate}
  \item $\dot{q}_\xi$ is a $P_\xi$-name for a condition in $\dot{Q}_\xi$.
  \item For all but finitely many $\xi < \alpha$, $\dot{q}_\xi$ is the trivial condition.
\end{enumerate}

We are going to consider only f.s.i. of c.c.c. forcing notions, i.e., we only use $\dot{Q}_\alpha$ as $P_\alpha$-names of c.c.c. forcing notion. The f.s.i. of c.c.c. forcing notions is always c.c.c.
Iteration of c.c.c. forcing notions

For an ordinal \( \delta \) a \textit{finite support iteration (f.s.i.) of length} \( \delta \)
\[ P_\delta = \langle P_\alpha, \dot{Q}_\alpha \rangle_{\alpha < \delta} \]
is defined recursively, where

(a) \( P_\alpha \) is a p.o. set for \( \alpha \leq \delta \), which represents the \( \alpha \)-stage of the iteration.

(b) \( \dot{Q}_\alpha \) is a \( P_\alpha \)-name for a p.o. for \( \alpha < \delta \).

(c) For \( \alpha \leq \delta \), each condition in \( P_\alpha \) is a sequence \( \langle \dot{q}_\xi \rangle_{\xi < \alpha} \) such that

(I) \( \dot{q}_\xi \) is a \( P_\xi \)-name for a condition in \( \dot{Q}_\xi \).

(II) For all but finitely many \( \xi < \alpha \), \( \dot{q}_\xi \) is the trivial condition.

We are going to consider only f.s.i. of c.c.c. forcing notions, i.e., we only use \( \dot{Q}_\alpha \) as \( P_\alpha \)-names of c.c.c. forcing notion. The f.s.i. of c.c.c. forcing notions is always c.c.c.
Iteration of c.c.c. forcing notions

For an ordinal $\delta$ a **finite support iteration (f.s.i.)** of length $\delta$

$P_\delta = \langle P_\alpha, Q_\alpha \rangle_{\alpha<\delta}$ is defined recursively, where

(a) $P_\alpha$ is a p.o. set for $\alpha \leq \delta$, which represents the $\alpha$-stage of the iteration.

(b) $Q_\alpha$ is a $P_\alpha$-name for a p.o. for $\alpha < \delta$.

(c) For $\alpha \leq \delta$, each condition in $P_\alpha$ is a sequence $\langle q_\xi \rangle_{\xi<\alpha}$ such that

(1) $q_\xi$ is a $P_\xi$-name for a condition in $Q_\xi$.

(II) For all but finitely many $\xi < \alpha$, $q_\xi$ is the trivial condition.

We are going to consider only f.s.i. of c.c.c. forcing notions, i.e., we only use $Q_\alpha$ as $P_\alpha$-names of c.c.c. forcing notion. The f.s.i. of c.c.c. forcing notions is always c.c.c.
Iteration of c.c.c. forcing notions

For an ordinal $\delta$ a finite support iteration (f.s.i.) of length $\delta$
$P_\delta = \langle P_\alpha, \dot{Q}_\alpha \rangle_{\alpha < \delta}$ is defined recursively, where

(a) $P_\alpha$ is a p.o. set for $\alpha \leq \delta$, which represents the $\alpha$-stage of the iteration.

(b) $\dot{Q}_\alpha$ is a $P_\alpha$-name for a p.o. for $\alpha < \delta$.

(c) For $\alpha \leq \delta$, each condition in $P_\alpha$ is a sequence $\langle \dot{q}_\xi \rangle_{\xi < \alpha}$ such that

1. $\dot{q}_\xi$ is a $P_\xi$-name for a condition in $\dot{Q}_\xi$.
2. For all but finitely many $\xi < \alpha$, $\dot{q}_\xi$ is the trivial condition.

We are going to consider only f.s.i. of c.c.c. forcing notions, i.e., we only use $\dot{Q}_\alpha$ as $P_\alpha$-names of c.c.c. forcing notion. The f.s.i. of c.c.c. forcing notions is always c.c.c.
Iteration of c.c.c. forcing notions

For an ordinal $\delta$ a \textit{finite support iteration (f.s.i.) of length $\delta$}

$P_\delta = \langle P_\alpha, \dot{Q}_\alpha \rangle_{\alpha < \delta}$ is defined recursively, where

(a) $P_\alpha$ is a p.o. set for $\alpha \leq \delta$, which represents the $\alpha$-stage of the iteration.

(b) $\dot{Q}_\alpha$ is a $P_\alpha$-name for a p.o. for $\alpha < \delta$.

(c) For $\alpha \leq \delta$, each condition in $P_\alpha$ is a sequence $\langle \dot{q}_\xi \rangle_{\xi < \alpha}$ such that

1. $\dot{q}_\xi$ is a $P_\xi$-name for a condition in $\dot{Q}_\xi$.

2. For all but finitely many $\xi < \alpha$, $\dot{q}_\xi$ is the trivial condition.

We are going to consider only f.s.i. of c.c.c. forcing notions, i.e., we only use $\dot{Q}_\alpha$ as $P_\alpha$-names of c.c.c. forcing notion. The f.s.i. of c.c.c. forcing notions is always c.c.c.
Iteration of c.c.c. forcing notions

For an ordinal $\delta$ a **finite support iteration (f.s.i.) of length $\delta$**

$$P_\delta = \langle P_\alpha, \dot{Q}_\alpha \rangle_{\alpha<\delta}$$ is defined recursively, where

(a) $P_\alpha$ is a p.o. set for $\alpha \leq \delta$, which represents the $\alpha$-stage of the iteration.

(b) $\dot{Q}_\alpha$ is a $P_\alpha$-name for a p.o. for $\alpha < \delta$.

(c) For $\alpha \leq \delta$, each condition in $P_\alpha$ is a sequence $\langle \dot{q}_\xi \rangle_{\xi<\alpha}$ such that

   (i) $\dot{q}_\xi$ is a $P_\xi$-name for a condition in $\dot{Q}_\xi$.

   (ii) For all but finitely many $\xi < \alpha$, $\dot{q}_\xi$ is the trivial condition.

We are going to consider only f.s.i. of c.c.c. forcing notions, i.e., we only use $\dot{Q}_\alpha$ as $P_\alpha$-names of c.c.c. forcing notion. The f.s.i. of c.c.c. forcing notions is always c.c.c.
For an ordinal $\delta$ a \textit{finite support iteration (f.s.i.) of length $\delta$} \linebreak \mathbb{P}_\delta = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle_{\alpha < \delta}$ is defined recursively, where \linebreak

(a) $\mathbb{P}_\alpha$ is a p.o. set for $\alpha \leq \delta$, which represents the $\alpha$-stage of the iteration. \linebreak

(b) $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a p.o. for $\alpha < \delta$. \linebreak

(c) For $\alpha \leq \delta$, each condition in $\mathbb{P}_\alpha$ is a sequence $\langle \dot{q}_\xi \rangle_{\xi < \alpha}$ such that \linebreak

\begin{enumerate}
\item $\dot{q}_\xi$ is a $\mathbb{P}_\xi$-name for a condition in $\dot{\mathbb{Q}}_\xi$.
\item For all but finitely many $\xi < \alpha$, $\dot{q}_\xi$ is the trivial condition.
\end{enumerate}

We are going to consider only f.s.i. of c.c.c. forcing notions, i.e., we only use $\dot{\mathbb{Q}}_\alpha$ as $\mathbb{P}_\alpha$-names of c.c.c. forcing notion. The f.s.i. of c.c.c. forcing notions is always c.c.c.
Iteration of c.c.c. forcing notions

For an ordinal $\delta$ a *finite support iteration (f.s.i.) of length $\delta* $ $P_\delta = \langle P_\alpha, \dot{Q}_\alpha \rangle_{\alpha<\delta}$ is defined recursively, where

(a) $P_\alpha$ is a p.o. set for $\alpha \leq \delta$, which represents the $\alpha$-stage of the iteration.

(b) $\dot{Q}_\alpha$ is a $P_\alpha$-name for a p.o. for $\alpha < \delta$.

(c) For $\alpha \leq \delta$, each condition in $P_\alpha$ is a sequence $\langle \dot{q}_\xi \rangle_{\xi<\alpha}$ such that

1. $\dot{q}_\xi$ is a $P_\xi$-name for a condition in $\dot{Q}_\xi$.
2. For all but finitely many $\xi < \alpha$, $\dot{q}_\xi$ is the trivial condition.

We are going to consider only f.s.i. of c.c.c. forcing notions, i.e., we only use $\dot{Q}_\alpha$ as $P_\alpha$-names of c.c.c. forcing notion. The f.s.i. of c.c.c. forcing notions is always c.c.c.
Let $V$ be a model of ZFC, $G_\delta$ a $\mathbb{P}_\delta$-generic over $V$. This generic restricted to $\mathbb{P}_\alpha \ (\alpha \leq \delta)$ is denoted by $G_\alpha$, put $V_\alpha = V[G_\alpha]$. Here, $V_0 = V$ (the ground model) and $\langle V_\alpha \rangle_{\alpha \leq \delta}$ is an increasing sequence of models of ZFC such that $V_{\alpha+1}$ is a $Q_\alpha$-generic extension of $V_\alpha$.

$$
\begin{array}{cccc}
V & \subseteq & V_1 & \subseteq & \cdots & \subseteq & V_\alpha & \subseteq & V_{\alpha+1} & \subseteq & \cdots & \subseteq & V_\delta
\end{array}
$$

**Fact**

Assume $\delta$ is a limit ordinal with uncountable cofinality. In the context of a c.c.c. forcing iteration of length $\delta$, any real in $V_\delta$ is in $V_\alpha$ for some $\alpha < \delta$. 
Let $V$ be a model of ZFC, $G_\delta$ a $P_\delta$-generic over $V$. This generic restricted to $P_\alpha$ ($\alpha \leq \delta$) is denoted by $G_\alpha$, put $V_\alpha = V[G_\alpha]$. Here, $V_0 = V$ (the ground model) and $\langle V_\alpha \rangle_{\alpha \leq \delta}$ is an increasing sequence of models of ZFC such that $V_{\alpha+1}$ is a $Q_\alpha$-generic extension of $V_\alpha$.

$$V \subseteq V_{Q_0\text{-ext.}} \subseteq V_{Q_1\text{-ext.}} \subseteq \cdots \subseteq V_\alpha \subseteq V_{Q_\alpha\text{-ext.}} \subseteq V_{\alpha+1} \subseteq \cdots \subseteq V_\delta$$

**Fact**

Assume $\delta$ is a limit ordinal with uncountable cofinality. In the context of a c.c.c. forcing iteration of length $\delta$, any real in $V_\delta$ is in $V_\alpha$ for some $\alpha < \delta$. 
Iteration of c.c.c. forcing notions

Let $V$ be a model of ZFC, $G_\delta$ a $\mathbb{P}_\delta$-generic over $V$. This generic restricted to $\mathbb{P}_\alpha$ ($\alpha \leq \delta$) is denoted by $G_\alpha$, put $V_\alpha = V[G_\alpha]$. Here, $V_0 = V$ (the ground model) and $\langle V_\alpha \rangle_{\alpha \leq \delta}$ is an increasing sequence of models of ZFC such that $V_{\alpha+1}$ is a $\mathbb{Q}_\alpha$-generic extension of $V_\alpha$.

\[
V \quad Q_0\text{-ext.} \quad V_1 \quad Q_1\text{-ext.} \quad \cdots \quad V_\alpha \quad Q_\alpha\text{-ext.} \quad V_{\alpha+1} \quad Q_{\alpha+1}\text{-ext.} \quad \cdots \quad V_\delta
\]

Fact

Assume $\delta$ is a limit ordinal with uncountable cofinality. In the context of a c.c.c. forcing iteration of length $\delta$, any real in $V_\delta$ is in $V_\alpha$ for some $\alpha < \delta$. 
Let $V$ be a model of ZFC, $G_\delta$ a $\mathbb{P}_\delta$-generic over $V$. This generic restricted to $\mathbb{P}_\alpha$ ($\alpha \leq \delta$) is denoted by $G_\alpha$, put $V_\alpha = V[G_\alpha]$. Here, $V_0 = V$ (the ground model) and $\langle V_\alpha \rangle_{\alpha \leq \delta}$ is an increasing sequence of models of ZFC such that $V_{\alpha+1}$ is a $Q_\alpha$-generic extension of $V_\alpha$.

$$V \subseteq V_1 \subseteq \cdots \subseteq V_\alpha \subseteq V_{\alpha+1} \subseteq \cdots \subseteq V_\delta$$

**Fact**

Assume $\delta$ is a limit ordinal with uncountable cofinality. In the context of a c.c.c. forcing iteration of length $\delta$, any real in $V_\delta$ is in $V_\alpha$ for some $\alpha < \delta$. 
Assume that $\lambda$ is an uncountable regular cardinal and that $V$ is a model of $\text{ZFC} + \text{GCH}$.

**Iteration of length $\lambda$ using amoeba forcing $\mathbb{A}$**

![Cichon's diagram](image-url)
Fact

In a c.c.c. forcing iteration, cohen reals are generated at limit stages. In other words, the iteration generates a cohen real in $V_{\alpha+\omega}$ over $V_\alpha$. 
In a c.c.c. forcing iteration, cohen reals are generated at limit stages. In other words, the iteration generates a cohen real in $V_{\alpha+\omega}$ over $V_\alpha$. 
Examples on the Cichon’s diagram

Iteration of length $\lambda$ using hechler forcing $D$
Iteration of length $\lambda$ using e.d. forcing $E$

![Diagram of Cichon's diagram with iteration of length $\lambda$ using e.d. forcing $E$.]
Examples on the Cichon’s diagram

Iteration of length $\lambda$ using cohen forcing $C$

Diagram showing the Cichon's diagram with points such as $\mathbb{N}_1$, $\text{add}(\mathcal{N})$, $\text{non}(\mathcal{M})$, $\text{cof}(\mathcal{M})$, $\text{cof}(\mathcal{N})$, and $c$. Yellow circles highlight specific points or values.
Assume that $\lambda$ is an uncountable regular cardinal and that $V$ is a model of $\text{ZFC} + \text{add}(\mathcal{N}) = \mathfrak{c} = \lambda$.

**Iteration of length $\omega_1$ using amoeba forcing $\mathbb{A}$**
Dual case of the previous examples

Iteration of length $\omega_1$ using random forcing $\mathbb{B}$
Dual case of the previous examples

Iteration of length $\omega_1$ using hechler forcing $\mathbb{D}$

![Diagram of Cichon's diagram with new elements added: $\lambda$, $\mathcal{N}_1$, and $b$.]
Dual case of the previous examples

Iteration of length $\omega_1$ using e.d. forcing $E$

Diagram:

- $\mathcal{N}_1$
- $\operatorname{cov}(\mathcal{N})$
- $\operatorname{non}(\mathcal{M})$
- $\operatorname{cof}(\mathcal{M})$
- $\operatorname{cof}(\mathcal{N})$
- $\operatorname{add}(\mathcal{N})$
- $\operatorname{add}(\mathcal{M})$
- $\operatorname{cov}(\mathcal{M})$
- $\operatorname{non}(\mathcal{N})$
- $\lambda$
- $b$
- $\mathbb{N}$
One example for the left side of the Cichon’s diagram

Theorem

Assume $\kappa_0 \leq \kappa_1 \leq \kappa_2 \leq \kappa_3$ are uncountable regular cardinals. It is consistent with $\text{ZFC}$ that $\aleph_1 \leq \kappa_0 = \text{add}(\mathcal{N}) \leq \kappa_1 = \text{cov}(\mathcal{N}) \leq \kappa_2 = b = \text{add}(\mathcal{M}) \leq \kappa_3 = \text{non}(\mathcal{M}) = \text{cov}(\mathcal{N}) = c$. 
How about the dual case?

- The c.c.c. forcing iteration technique works to get similar consistency results on the left side of the Cichon’s diagram.
- The usual c.c.c. iteration technique doesn’t seem to work on the dual case, that is, when we look at the right hand side of the Cichon’s diagram.
- To get consistency results on the case when we look at the right hand side of the Cichon’s diagram and we use two uncountable regular cardinals $\kappa, \lambda$ (greater than $\aleph_1$), a new approach on the iteration technique is needed.
How about the dual case?

- The c.c.c. forcing iteration technique works to get similar consistency results on the left side of the Cichon’s diagram.
- The usual c.c.c. iteration technique doesn’t seem to work on the dual case, that is, when we look at the right hand side of the Cichon’s diagram.
- To get consistency results on the case when we look at the right hand side of the Cichon’s diagram and we use two uncountable regular cardinals $\kappa, \lambda$ (greater than $\aleph_1$), a new approach on the iteration technique is needed.
How about the dual case?

- The c.c.c. forcing iteration technique works to get similar consistency results on the left side of the Cichon’s diagram.
- The usual c.c.c. iteration technique doesn’t seem to work on the dual case, that is, when we look at the right hand side of the Cichon’s diagram.
- To get consistency results on the case when we look at the right hand side of the Cichon’s diagram and we use two uncountable regular cardinals $\kappa, \lambda$ (greater than $\aleph_1$), a new approach on the iteration technique is needed.
For δ, γ ordinals, we consider a matrix iteration
\langle \langle P_{\alpha,\xi}, \dot{Q}_{\alpha,\xi} \rangle_{\xi<\gamma} \rangle_{\alpha \leq \delta} defined by the following conditions.

1. \( P_{\delta,0} = \langle P_{\alpha,0}, \dot{C} \rangle_{\alpha<\delta} \) (f.s.i. of cohen forcing).
2. For a fixed \( \alpha \leq \delta \), \( P_{\alpha,\gamma} = \langle P_{\alpha,\xi}, \dot{Q}_{\alpha,\xi} \rangle_{\xi<\gamma} \) is a f.s.i. of c.c.c forcing notions (closely related to those defined on the examples).
3. For \( \alpha < \beta \leq \delta, \xi < \gamma \), \( \dot{Q}_{\alpha,\xi} \subseteq \dot{Q}_{\beta,\xi} \) in the (\( \beta, \xi \)) extension.
4. For \( \alpha < \beta \leq \delta, \xi < \gamma \), maximal antichains of \( \dot{Q}_{\alpha,\xi} \) in the (\( \alpha, \xi \)) extension are preserved in \( \dot{Q}_{\beta,\xi} \).
Matrix of iterations of c.c.c. forcing notions
(Shelah-Blass, 1984)

For $\delta, \gamma$ ordinals, we consider a matrix iteration

$$\langle \langle P_{\alpha,\xi}, \dot{Q}_{\alpha,\xi} \rangle_{\xi<\gamma} \rangle_{\alpha \leq \delta}$$

defined by the following conditions.

1. $P_{\delta,0} = \langle P_{\alpha,0}, \dot{C} \rangle_{\alpha<\delta}$ (f.s.i. of cohen forcing).

2. For a fixed $\alpha \leq \delta$, $P_{\alpha,\gamma} = \langle P_{\alpha,\xi}, \dot{Q}_{\alpha,\xi} \rangle_{\xi<\gamma}$ is a f.s.i. of c.c.c forcing notions (closely related to those defined on the examples).

3. For $\alpha < \beta \leq \delta, \xi < \gamma$, $\dot{Q}_{\alpha,\xi} \subseteq \dot{Q}_{\beta,\xi}$ in the $(\beta, \xi)$ extension.

4. For $\alpha < \beta \leq \delta, \xi < \gamma$, maximal antichains of $\dot{Q}_{\alpha,\xi}$ in the $(\alpha, \xi)$ extension are preserved in $\dot{Q}_{\beta,\xi}$. 

Diego Alejandro Mejía Guzmán

Forcing iterations and Cichon’s diagram
Matrix of iterations of c.c.c. forcing notions
(Shelah-Blass, 1984)

For $\delta, \gamma$ ordinals, we consider a matrix iteration
\[ \langle \langle P_{\alpha,\xi}, \dot{Q}_{\alpha,\xi} \rangle_{\xi < \gamma} \rangle_{\alpha \leq \delta} \]
defined by the following conditions.

1. $P_{\delta,0} = \langle \langle P_{\alpha,0}, \dot{C} \rangle_{\alpha < \delta} \rangle$ (f.s.i. of cohen forcing).

2. For a fixed $\alpha \leq \delta$, $P_{\alpha,\gamma} = \langle \langle P_{\alpha,\xi}, \dot{Q}_{\alpha,\xi} \rangle_{\xi < \gamma} \rangle$ is a f.s.i. of c.c.c forcing notions (closely related to those defined on the examples).

3. For $\alpha < \beta \leq \delta, \xi < \gamma$, $\dot{Q}_{\alpha,\xi} \subseteq \dot{Q}_{\beta,\xi}$ in the $(\beta, \xi)$ extension.

4. For $\alpha < \beta \leq \delta, \xi < \gamma$, maximal antichains of $\dot{Q}_{\alpha,\xi}$ in the $(\alpha, \xi)$ extension are preserved in $\dot{Q}_{\beta,\xi}$. 
Matrix of iterations of c.c.c. forcing notions
(Shelah-Blass, 1984)

For $\delta, \gamma$ ordinals, we consider a matrix iteration
$\langle \langle P_{\alpha,\xi}, \dot{Q}_{\alpha,\xi} \rangle_{\xi < \gamma} \rangle_{\alpha \leq \delta}$ defined by the following conditions.

1. $P_{\delta,0} = \langle P_{\alpha,0}, \dot{C} \rangle_{\alpha < \delta}$ (f.s.i. of cohen forcing).
2. For a fixed $\alpha \leq \delta$, $P_{\alpha,\gamma} = \langle P_{\alpha,\xi}, \dot{Q}_{\alpha,\xi} \rangle_{\xi < \gamma}$ is a f.s.i. of c.c.c forcing notions (closely related to those defined on the examples).
3. For $\alpha < \beta \leq \delta, \xi < \gamma$, $\dot{Q}_{\alpha,\xi} \subseteq \dot{Q}_{\beta,\xi}$ in the $(\beta, \xi)$ extension.
4. For $\alpha < \beta \leq \delta, \xi < \gamma$, maximal antichains of $\dot{Q}_{\alpha,\xi}$ in the $(\alpha, \xi)$ extension are preserved in $\dot{Q}_{\beta,\xi}$. 
Matrix of iterations of c.c.c. forcing notions
(Shelah-Blass, 1984)

For $\delta, \gamma$ ordinals, we consider a matrix iteration
$$\langle \langle P_{\alpha, \xi}, \dot{Q}_{\alpha, \xi} \rangle_{\xi < \gamma} \rangle_{\alpha \leq \delta}$$
defined by the following conditions.

(1) $P_{\delta, 0} = \langle P_{\alpha, 0}, \dot{C} \rangle_{\alpha < \delta}$ (f.s.i. of cohen forcing).

(2) For a fixed $\alpha \leq \delta$, $P_{\alpha, \gamma} = \langle P_{\alpha, \xi}, \dot{Q}_{\alpha, \xi} \rangle_{\xi < \gamma}$ is a f.s.i. of c.c.c forcing notions (closely related to those defined on the examples).

(3) For $\alpha < \beta \leq \delta, \xi < \gamma$, $\dot{Q}_{\alpha, \xi} \subseteq \dot{Q}_{\beta, \xi}$ in the $(\beta, \xi)$ extension.

(4) For $\alpha < \beta \leq \delta, \xi < \gamma$, maximal antichains of $\dot{Q}_{\alpha, \xi}$ in the $(\alpha, \xi)$ extension are preserved in $\dot{Q}_{\beta, \xi}$. 
Like in the case of “linear” iterations, if $G_{\delta,\gamma}$ is $\mathbb{P}_{\delta,\gamma}$-generic over $V$, we consider the model $V_{\alpha,\xi}$ for $\alpha \leq \delta, \xi \leq \gamma$ as a $\mathbb{P}_{\alpha,\xi}$-extension. $V_{0,0} = V$ and the generic extensions can be seen as in the figure.
One application

**Theorem**

*For* $\kappa \leq \lambda$ *uncountable regular cardinals, it is consistent with ZFC that* $\aleph_1 = \text{cov}(\mathcal{M}) = \text{non}(\mathcal{M}) \leq \kappa = \delta = \text{cof}(\mathcal{M}) \leq \lambda = \text{non}(\mathcal{N}) = \text{cof}(\mathcal{N}) = \mathfrak{c}$. 

![Diagram of Cichon's diagram](attachment:image.png)
Sketched proof

Start with $V$ model of $\text{ZFC} + \text{add}(\mathcal{N}) = c = \lambda$. Let $f : \kappa \cdot \omega_1 \to \kappa$ such that $f(\xi) = \delta$, where $\xi = \kappa \cdot \epsilon + 2\delta + k$ for (unique) $\epsilon < \omega_1$, $\delta < \kappa$ and $k < 2$.

Perform a matrix iteration $\langle \langle P_{\alpha, \xi}, \dot{Q}_{\alpha, \xi} \rangle_{\xi < \kappa \cdot \omega_1} \rangle_{\alpha \leq \kappa}$ as explained, such that, for $\xi < \kappa \cdot \omega_1$,

1. If $\xi \equiv 0 \mod 2$, make $\dot{Q}_{\alpha, \xi} = \dot{E}$ ($P_{\alpha, \xi}$-name).
2. If $\xi \equiv 1 \mod 2$, make

$$\dot{Q}_{\alpha, \xi} = \begin{cases} 1, & \text{if } \alpha \leq f(\xi), \\ \dot{D}_\xi, & \text{if } \alpha > f(\xi), \end{cases}$$

where $\dot{D}_\xi$ is the $P_{f(\xi), \xi}$-name for $D$.

In the extension $V_{\kappa, \kappa \cdot \omega_1}$, $\text{cov}(\mathcal{M}) = \text{non}(\mathcal{M}) = \aleph_1$ because of the cofinally many e.d. and cohen reals added in the iteration. $\text{non}(\mathcal{N}) = c = \lambda$ by preservation properties. $d = \text{cof}(\mathcal{M}) = \kappa$ because of the dominating reals added by each $\dot{D}_\xi$ ($\xi < \kappa \cdot \omega_1$) and by the preservation of the unbounding (cohen) reals added at each $(\alpha + 1, 0)$ step.
Start with $V$ model of $\text{ZFC} + \text{add} (\mathcal{N}) = c = \lambda$. Let $f : \kappa \cdot \omega_1 \to \kappa$ such that $f (\xi) = \delta$, where $\xi = \kappa \cdot \epsilon + 2\delta + k$ for (unique) $\epsilon < \omega_1$, $\delta < \kappa$ and $k < 2$. Perform a matrix iteration $\langle \langle P_\alpha, \xi, \dot{Q}_\alpha, \xi \rangle \xi < \kappa \cdot \omega_1 \rangle_{\alpha \leq \kappa}$ as explained, such that, for $\xi < \kappa \cdot \omega_1$,

1. If $\xi \equiv 0 \mod 2$, make $\dot{Q}_\alpha, \xi = \dot{E} (P_\alpha, \xi \text{-name})$.
2. If $\xi \equiv 1 \mod 2$, make $\dot{Q}_\alpha, \xi = \{1, \text{ if } \alpha \leq f (\xi), \dot{D}_\xi, \text{ if } \alpha > f (\xi), \}$

where $\dot{D}_\xi$ is the $P_{f (\xi), \xi \text{-name}}$ for $\mathcal{D}$.

In the extension $V_{\kappa, \kappa \cdot \omega_1}$, $\text{cov} (\mathcal{M}) = \text{non} (\mathcal{M}) = \aleph_1$ because of the cofinally many e.d. and cohen reals added in the iteration. $\text{non} (\mathcal{N}) = c = \lambda$ by preservation properties. $\vartheta = \text{cof} (\mathcal{M}) = \kappa$ because of the dominating reals added by each $\dot{D}_\xi (\xi < \kappa \cdot \omega_1)$ and by the preservation of the unbounding (cohen) reals added at each $(\alpha + 1, 0)$ step.
Sketched proof

Start with $V$ model of $ZFC + \text{add}(\mathcal{N}) = c = \lambda$. Let $f : \kappa \cdot \omega_1 \to \kappa$ such that $f(\xi) = \delta$, where $\xi = \kappa \cdot \epsilon + 2\delta + k$ for (unique) $\epsilon < \omega_1$, $\delta < \kappa$ and $k < 2$.

Perform a matrix iteration $\langle \langle P_{\alpha,\xi}, \dot{Q}_{\alpha,\xi} \rangle_{\xi < \kappa \cdot \omega_1} \rangle_{\alpha \leq \kappa}$ as explained, such that, for $\xi < \kappa \cdot \omega_1$,

1. If $\xi \equiv 0 \mod 2$, make $\dot{Q}_{\alpha,\xi} = \dot{E}$ ($P_{\alpha,\xi}$-name).
2. If $\xi \equiv 1 \mod 2$, make

$$
\dot{Q}_{\alpha,\xi} = \begin{cases} 
1, & \text{if } \alpha \leq f(\xi), \\
\dot{D}_\xi, & \text{if } \alpha > f(\xi), 
\end{cases}
$$

where $\dot{D}_\xi$ is the $\mathbb{P}_{f(\xi),\xi}$-name for $D$.

In the extension $V_{\kappa,\kappa \cdot \omega_1}$, $\text{cov}(\mathcal{M}) = \text{non}(\mathcal{M}) = \aleph_1$ because of the cofinally many e.d. and cohen reals added in the iteration. $\text{non}(\mathcal{N}) = c = \lambda$ by preservation properties. $d = \text{cof}(\mathcal{M}) = \kappa$ because of the dominating reals added by each $\dot{D}_\xi$ ($\xi < \kappa \cdot \omega_1$) and by the preservation of the unbounding (cohen) reals added at each $(\alpha + 1, 0)$ step.
Sketched proof

Start with $V$ model of $\text{ZFC} + \text{add}(\mathcal{N}) = c = \lambda$. Let $f : \kappa \cdot \omega_1 \to \kappa$ such that $f(\xi) = \delta$, where $\xi = \kappa \cdot \epsilon + 2\delta + k$ for (unique) $\epsilon < \omega_1$, $\delta < \kappa$ and $k < 2$.

Perform a matrix iteration $\langle \langle \dot{P}_{\alpha, \xi}, \dot{Q}_{\alpha, \xi} \rangle_{\xi < \kappa \cdot \omega_1} \rangle_{\alpha \leq \kappa}$ as explained, such that, for $\xi < \kappa \cdot \omega_1$,

1. If $\xi \equiv 0 \text{ mod } 2$, make $\dot{Q}_{\alpha, \xi} = \dot{E}$ (\(\dot{P}_{\alpha, \xi}\)-name).

2. If $\xi \equiv 1 \text{ mod } 2$, make

$$\dot{Q}_{\alpha, \xi} = \begin{cases} 1, & \text{if } \alpha \leq f(\xi), \\ \dot{D}_\xi, & \text{if } \alpha > f(\xi), \end{cases}$$

where $\dot{D}_\xi$ is the $\dot{P}_{f(\xi), \xi}$-name for $D$.

In the extension $V_{\kappa, \kappa \cdot \omega_1}$, $\text{cov}(\mathcal{M}) = \text{non}(\mathcal{M}) = \aleph_1$ because of the cofinally many e.d. and cohen reals added in the iteration. $\text{non}(\mathcal{N}) = c = \lambda$ by preservation properties. $\vartheta = \text{cof}(\mathcal{M}) = \kappa$ because of the dominating reals added by each $\dot{D}_\xi$ ($\xi < \kappa \cdot \omega_1$) and by the preservation of the unbounding (cohen) reals added at each $(\alpha + 1, 0)$ step.
Start with $V$ model of $\text{ZFC} + \text{add}(\mathcal{N}) = c = \lambda$. Let $f : \kappa \cdot \omega_1 \rightarrow \kappa$ such that $f(\xi) = \delta$, where $\xi = \kappa \cdot \epsilon + 2\delta + k$ for (unique) $\epsilon < \omega_1$, $\delta < \kappa$ and $k < 2$.

Perform a matrix iteration $\langle \langle \mathbb{P}_{\alpha, \xi}, \dot{\mathbb{Q}}_{\alpha, \xi} \rangle_{\xi < \kappa \cdot \omega_1} \rangle_{\alpha \leq \kappa}$ as explained, such that, for $\xi < \kappa \cdot \omega_1$,

1. If $\xi \equiv 0\text{mod}2$, make $\dot{\mathbb{Q}}_{\alpha, \xi} = \dot{E}(\mathbb{P}_{\alpha, \xi}\text{-name})$.

2. If $\xi \equiv 1\text{mod}2$, make

$$\dot{Q}_{\alpha, \xi} = \begin{cases} 1, & \text{if } \alpha \leq f(\xi), \\ \hat{D}_\xi, & \text{if } \alpha > f(\xi), \end{cases}$$

where $\hat{D}_\xi$ is the $\mathbb{P}_{f(\xi), \xi}\text{-name for } \mathbb{D}$.

In the extension $V_{\kappa, \kappa \cdot \omega_1}$, $\text{cov}(\mathcal{M}) = \text{non}(\mathcal{M}) = \aleph_1$ because of the cofinally many e.d. and cohen reals added in the iteration. $\text{non}(\mathcal{N}) = c = \lambda$ by preservation properties. $\mathfrak{d} = \text{cof}(\mathcal{M}) = \kappa$ because of the dominating reals added by each $\hat{D}_\xi (\xi < \kappa \cdot \omega_1)$ and by the preservation of the unbounding (cohen) reals added at each $(\alpha + 1,0)$ step.
Sketched proof

Start with \( V \) model of \( \text{ZFC} + \text{add}(\mathcal{N}) = c = \lambda \). Let \( f : \kappa \cdot \omega_1 \to \kappa \) such that \( f(\xi) = \delta \), where \( \xi = \kappa \cdot \epsilon + 2\delta + k \) for (unique) \( \epsilon < \omega_1 \), \( \delta < \kappa \) and \( k < 2 \).

Perform a matrix iteration \( \langle \langle P_{\alpha, \xi}, \dot{Q}_{\alpha, \xi} \rangle_{\xi < \kappa \cdot \omega_1} \rangle_{\alpha \leq \kappa} \) as explained, such that, for \( \xi < \kappa \cdot \omega_1 \),

1. If \( \xi \equiv 0 \mod 2 \), make \( \dot{Q}_{\alpha, \xi} = \dot{E} (P_{\alpha, \xi} \text{-name}) \).

2. If \( \xi \equiv 1 \mod 2 \), make

\[
\dot{Q}_{\alpha, \xi} = \begin{cases} 
1, & \text{if } \alpha \leq f(\xi), \\
\dot{D}_\xi, & \text{if } \alpha > f(\xi),
\end{cases}
\]

where \( \dot{D}_\xi \) is the \( P_{f(\xi), \xi} \)-name for \( D \).

In the extension \( V_{\kappa, \kappa \cdot \omega_1} \), \( \text{cov}(\mathcal{M}) = \text{non}(\mathcal{M}) = \aleph_1 \) because of the cofinally many e.d. and cohen reals added in the iteration. \( \text{non}(\mathcal{N}) = c = \lambda \) by preservation properties. \( d = \text{cof}(\mathcal{M}) = \kappa \) because of the dominating reals added by each \( \dot{D}_\xi \) (\( \xi < \kappa \cdot \omega_1 \)) and by the preservation of the unbounding (cohen) reals added at each (\( \alpha + 1, 0 \)) step.
Sketched proof

Start with $V$ model of $\text{ZFC} + \text{add} (\mathcal{N}) = c = \lambda$. Let $f : \kappa \cdot \omega_1 \to \kappa$ such that $f (\xi) = \delta$, where $\xi = \kappa \cdot \epsilon + 2 \delta + k$ for (unique) $\epsilon < \omega_1$, $\delta < \kappa$ and $k < 2$.

Perform a matrix iteration $\langle \langle P_{\alpha, \xi}, \dot{Q}_{\alpha, \xi} \rangle_{\xi < \kappa \cdot \omega_1} \rangle_{\alpha \leq \kappa}$ as explained, such that, for $\xi < \kappa \cdot \omega_1$,

1. If $\xi \equiv 0 \text{mod} 2$, make $\dot{Q}_{\alpha, \xi} = \dot{E} (P_{\alpha, \xi} \text{-name})$.
2. If $\xi \equiv 1 \text{mod} 2$, make

$$
\dot{Q}_{\alpha, \xi} = \begin{cases} 
1, & \text{if } \alpha \leq f (\xi), \\
\dot{D}_\xi, & \text{if } \alpha > f (\xi),
\end{cases}
$$

where $\dot{D}_\xi$ is the $P_{f (\xi), \xi} \text{-name}$ for $D$.

In the extension $V_{\kappa, \kappa \cdot \omega_1}$, $\text{cov} (\mathcal{M}) = \text{non} (\mathcal{M}) = \aleph_1$ because of the cofinally many e.d. and cohen reals added in the iteration. $\text{non} (\mathcal{N}) = c = \lambda$ by preservation properties. $\vartheta = \text{cof} (\mathcal{M}) = \kappa$ because of the dominating reals added by each $\dot{D}_\xi$ ($\xi < \kappa \cdot \omega_1$) and by the preservation of the unbounding (cohen) reals added at each $(\alpha + 1, 0)$ step.
Sketched proof

Start with \( V \) model of \( \text{ZFC} + \text{add}(\mathcal{N}) = c = \lambda \). Let \( f : \kappa \cdot \omega_1 \rightarrow \kappa \) such that \( f(\xi) = \delta \), where \( \xi = \kappa \cdot \epsilon + 2\delta + k \) for (unique) \( \epsilon < \omega_1, \delta < \kappa \) and \( k < 2 \).

Perform a matrix iteration \( \langle \langle P_{\alpha, \xi}, \dot{Q}_{\alpha, \xi} \rangle_{\xi < \kappa \cdot \omega_1} \rangle_{\alpha \leq \kappa} \) as explained, such that, for \( \xi < \kappa \cdot \omega_1 \),

1. If \( \xi \equiv 0 \text{mod} 2 \), make \( \dot{Q}_{\alpha, \xi} = \dot{E} \) (\( P_{\alpha, \xi} \)-name).
2. If \( \xi \equiv 1 \text{mod} 2 \), make

\[
\dot{Q}_{\alpha, \xi} = \begin{cases} 
1, & \text{if } \alpha \leq f(\xi), \\
\dot{D}_\xi, & \text{if } \alpha > f(\xi),
\end{cases}
\]

where \( \dot{D}_\xi \) is the \( P_{f(\xi), \xi} \)-name for \( D \).

In the extension \( V_{\kappa, \kappa \cdot \omega_1} \), \( \text{cov}(M) = \text{non}(M) = \aleph_1 \) because of the cofinally many e.d. and cohen reals added in the iteration. \( \text{non}(\mathcal{N}) = c = \lambda \) by preservation properties. \( \vartheta = \text{cof}(M) = \kappa \) because of the dominating reals added by each \( \dot{D}_\xi \) (\( \xi < \kappa \cdot \omega_1 \)) and by the preservation of the unbounding (cohen) reals added at each \((\alpha + 1, 0)\) step.
Sketched proof

Start with $V$ model of $\text{ZFC} + \text{add}(\mathcal{N}) = \mathfrak{c} = \lambda$. Let $f : \kappa \cdot \omega_1 \to \kappa$ such that $f(\xi) = \delta$, where $\xi = \kappa \cdot \epsilon + 2\delta + k$ for (unique) $\epsilon < \omega_1$, $\delta < \kappa$ and $k < 2$. Perform a matrix iteration $\langle \langle P_{\alpha, \xi}, \dot{Q}_{\alpha, \xi} \rangle_{\xi < \kappa \cdot \omega_1} \rangle_{\alpha \leq \kappa}$ as explained, such that, for $\xi < \kappa \cdot \omega_1$,

1. If $\xi \equiv 0 \text{mod}2$, make $\dot{Q}_{\alpha, \xi} = \dot{E}$ ($P_{\alpha, \xi}$-name).
2. If $\xi \equiv 1 \text{mod}2$, make

$$\dot{Q}_{\alpha, \xi} = \begin{cases} 1, & \text{if } \alpha \leq f(\xi), \\ \dot{D}_\xi, & \text{if } \alpha > f(\xi), \end{cases}$$

where $\dot{D}_\xi$ is the $P_{f(\xi), \xi}$-name for $D$.

In the extension $V_{\kappa, \kappa \cdot \omega_1}$, $\text{cov}(\mathcal{M}) = \text{non}(\mathcal{M}) = \aleph_1$ because of the cofinally many e.d. and cohen reals added in the iteration. $\text{non}(\mathcal{N}) = \mathfrak{c} = \lambda$ by preservation properties. $\vartheta = \text{cof}(\mathcal{M}) = \kappa$ because of the dominating reals added by each $\dot{D}_\xi$ ($\xi < \kappa \cdot \omega_1$) and by the preservation of the unbounding (cohen) reals added at each $(\alpha + 1, 0)$ step.
Similarly, we can obtain the consistency with ZFC of
More applications

Diego Alejandro Mejía Guzmán
Forcing iterations and Cichon’s diagram
Question 1

If $\aleph_1 < \kappa_0 < \kappa_1 < \kappa_2$ for $\kappa_0, \kappa_1, \kappa_2$ regular cardinals, is it consistent with ZFC that $\aleph_1 = \text{non}(\mathcal{M}) = \text{cov}(\mathcal{M}) < \kappa_0 = d = \text{cof}(\mathcal{M}) < \kappa_1 = \text{non}(\mathcal{N}) < \kappa_2 = \text{cof}(\mathcal{N}) = c$?
Question 2

If $\aleph_1 < \kappa < \lambda$ for $\kappa, \lambda$ regular cardinals, is it consistent with $\text{ZFC}$ that $\aleph_1 = b < \kappa = \text{cov}(\mathcal{N}) < \lambda = \text{non}(\mathcal{M}) = \text{cov}(\mathcal{M}) = c$. 
References


Thank you!