The difficulty of marriage

Makoto Fujiwara

Tohoku university

2011.12.10 数学基礎論若手の会

joint work with

Kojiro Higuchi and Takayuki Kihara, Tohoku University
Contents

1 Introduction
   1 What is Marriage Problem?
   2 Marriage Theorems and Recursive Graph Theory
   3 What is Reverse Mathematics?

2 Reverse Mathematics of Marriage Theorems
   1 Previous Research by Hirst
   2 Marriage Theorems with Expanding Hall Condition

3 Indexed Marriage Problem and Recursive Solution
There are boys and girls and each boy is acquainted with finitely many girls. Under what condition is it possible that all boys marry one of his acquaintances?

Actually the following condition is the answer of this problem.

**Hall condition**

For all $n$, $n$ boys know at least $n$ girls.

This problem can be written in terms of graph theory.
What is Marriage Problem?

Marriage problem

There are boys and girls and each boy is acquainted with finitely many girls. Under what condition is it possible that all boys marry one of his acquaintances?

Actually the following condition is the answer of this problem.

Hall condition

For all \( n \), \( n \) boys know at least \( n \) girls.

This problem can be written in terms of graph theory.
What is Marriage Problem?

**Marriage problem**

There are boys and girls and each boy is acquainted with finitely many girls. Under what condition is it possible that all boys marry one of his acquaintances?

Actually the following codition is the answer of this problem.

**Hall condition**

For all $n$, $n$ boys know at least $n$ girls.

This problem can be written in terms of graph theory.
Marriage problem

Let \((B, G; R)\) be a bipartite graph such that each vertex in \(B\) is adjacent to finitely many vertices in \(G\). Under what condition is it possible that there exists a matching of \(B\) ?

- A graph is **bipartite** if the vertices can be divided so that every edges consist of two another kind of vertices.
- A **matching of** \(B\) is a set of independent edges such that every vertices in \(B\) is incident with an edge in it.

**Theorem (P. Hall, 1935)**

Let \(G = (B, G; R)\) be a finite bipartite graph. Then there is a matching of \(B\) if \(G\) satisfies Hall condition.

- Hall condition (for \(B\)) : for all \(X \subset_{\text{fin}} B\), \(|N(X)| \geq |X|\).

Here \(N(X)\) is the set of vertices in \(G\) which are adjacent to the vertices in \(X\). \(|N(X)| \geq |X|\) means that \(n\) boys know at least \(n\) girls.
Marriage problem

Let \((B, G; R)\) be a bipartite graph such that each vertex in \(B\) is adjacent to finitely many vertices in \(G\). Under what condition is it possible that there exists a matching of \(B\)?

- A graph is **bipartite** if the vertices can be divided so that every edges consist of two another kind of vertices.
- A **matching of** \(B\) is a set of independent edges such that every vertices in \(B\) is incident with an edge in it.

**Theorem (P. Hall, 1935)**

Let \(G = (B, G; R)\) be a finite bipartite graph.

Then there is a matching of \(B\) if \(G\) satisfies Hall condition.

- Hall condition (for \(B\)) : for all \(X \subset_{fin} B\), \(|N(X)| \geq |X|\).

Here \(N(X)\) is the set of vertices in \(G\) which are adjacent to the vertices in \(X\). \(|N(X)| \geq |X|\) means that **n boys know at least n girls.**
This theorem can be extended to infinite graph.

**Theorem (M. Hall, 1948)**

Let $\mathbf{G} = (B, G; R)$ be a bipartite graph and $B$-locally finite. Then $\mathbf{G}$ has a solution if $\mathbf{G}$ satisfies Hall condition.

- $B$-locally finite: for all $b \in B$, $|N(b)| < \infty$.
- Solution of $\mathbf{G}$: injection $M : B \rightarrow G$ s.t. $M \subset R$.

That is, $B$-locally finite means that each boy is acquainted with finitely many girls and the solution means the appropriate marriage.

Note that if “$B$-locally finite” is dropped, the above statement does not hold.
Let $G$ be a recursive bipartite (countable) graph and $G$ is $B$-locally finite. Then if $G$ satisfies Hall condition, $G$ has a solution by the previous theorem. Now can we take the solution recursively? The answer is “no”.

**Theorem (A.Manaster and J.Rosenstein, 1972)**

There exists a recursive bipartite graph which is $B$-locally finite and satisfies Hall condition, but has no recursive solution.

Can we modify this to have a recursive solution?

- $B$-strong locally finite: the function $p(b) = |N(b)|$ is recursive.
- $G$-strong locally finite: the function $p(g) = |N(g)|$ is recursive.

**Theorem (A.Manaster and J.Rosenstein, 1972)**

There exists a recursive bipartite graph which is $B, G$-strong locally finite and satisfies Hall condition, but has no recursive solution.
Let $G$ be a recursive bipartite (countable) graph and $G$ is $B$-locally finite. Then if $G$ satisfies Hall condition, $G$ has a solution by the previous theorem. Now can we take the solution recursively? The answer is “no”.

**Theorem (A. Manaster and J. Rosenstein, 1972)**

*There exists a recursive bipartite graph which is $B$-locally finite and satisfies Hall condition, but has no recursive solution.*

Can we modify this to have a recursive solution?

- $B$-strong locally finite: the function $p(b) = |N(b)|$ is recursive.
- $G$-strong locally finite: the function $p(g) = |N(g)|$ is recursive.

**Theorem (A. Manaster and J. Rosenstein, 1972)**

*There exists a recursive bipartite graph which is $B$, $G$-strong locally finite and satisfies Hall condition, but has no recursive solution.*
Let $G$ be a \textit{recursive} bipartite (countable) graph and $G$ is $B$-locally finite. Then if $G$ satisfies Hall condition, $G$ has a solution by the previous theorem. Now can we take the solution \textit{recursively}? The answer is “no”.

\textbf{Theorem (A. Manaster and J. Rosenstein, 1972)}

\textit{There exists a recursive bipartite graph which is $B$-locally finite and satisfies Hall condition, but has no recursive solution.}

Can we modify this to have a recursive solution?

- $B$-strong locally finite: the function $p(b) = |N(b)|$ is recursive.
- $G$-strong locally finite: the function $p(g) = |N(g)|$ is recursive.

\textbf{Theorem (A. Manaster and J. Rosenstein, 1972)}

\textit{There exists a recursive bipartite graph which is $B, G$-strong locally finite and satisfies Hall condition, but has no recursive solution.}
Let $G$ be a recursive bipartite (countable) graph and $G$ is $B$-locally finite. Then if $G$ satisfies Hall condition, $G$ has a solution by the previous theorem. Now can we take the solution recursively? The answer is “no”.

Theorem (A. Manaster and J. Rosenstein, 1972)

There exists a recursive bipartite graph which is $B$-locally finite and satisfies Hall condition, but has no recursive solution.

Can we modify this to have a recursive solution?

- $B$-strong locally finite: the function $p(b) = |N(b)|$ is recursive.
- $G$-strong locally finite: the function $p(g) = |N(g)|$ is recursive.

Theorem (A. Manaster and J. Rosenstein, 1972)

There exists a recursive bipartite graph which is $B$, $G$-strong locally finite and satisfies Hall condition, but has no recursive solution.
Marriage Theorems and Recursive Graph Theory

Let $G$ be a recursive bipartite (countable) graph and $G$ is $B$-locally finite. Then if $G$ satisfies Hall condition, $G$ has a solution by the previous theorem. Now can we take the solution recursively? The answer is “no”.

Theorem (A.Manaster and J.Rosenstein, 1972)

There exists a recursive bipartite graph which is $B$-locally finite and satisfies Hall condition, but has no recursive solution.

Can we modify this to have a recursive solution?

- $B$-strong locally finite: the function $p(b) = |N(b)|$ is recursive.
- $G$-strong locally finite: the function $p(g) = |N(g)|$ is recursive.

Theorem (A.Manaster and J.Rosenstein, 1972)

There exists a recursive bipartite graph which is $B$, $G$-strong locally finite and satisfies Hall condition, but has no recursive solution.
Expanding Hall Condition

There is a function $h_B$ s.t.

$$h_B(0) = 0 \& \forall n \forall X \in \text{fin} B(|X| \geq h_B(n) \rightarrow |N(X)| - |X| \geq n).$$

That is, expanding Hall condition requires that if the number of boys is greater than $h_B(n)$, they knows extra $n$ girls.

Theorem (H.Kierstead, 1983)

If $G$ is recursive bipartite graph which is $B, G$-strong locally finite and satisfies expanding Hall condition with a recursive $h_B$, then $G$ has a recursive solution.

Theorem (H.Kierstead, 1983)

There exists a recursive bipartite graph which is $B, G$-strong locally finite and satisfies expanding Hall condition, but has no recursive solution.
Expanding Hall Condition

There is a function $h_B$ s.t.

$$h_B(0) = 0 \& \forall n \forall X \subseteq \text{fin} \, B(|X| \geq h_B(n) \rightarrow |N(X)| - |X| \geq n).$$

That is, expanding Hall condition requires that if the number of boys is greater than $h_B(n)$, they knows extra $n$ girls.

Theorem (H.Kierstead, 1983)

If $G$ is recursive bipartite graph which is $B, G$-strong locally finite and satisfies expanding Hall condition with a recursive $h_B$, then $G$ has a recursive solution.

Theorem (H.Kierstead, 1983)

There exists a recursive bipartite graph which is $B, G$-strong locally finite and satisfies expanding Hall condition, but has no recursive solution.
**Expanding Hall Condition**

There is a function \( h_B \) s.t.

\[
h_B(0) = 0 \& \forall n \forall X \subset_{fin} B(|X| \geq h_B(n) \rightarrow |N(X)| - |X| \geq n).
\]

That is, expanding Hall condition requires that if the number of boys is greater than \( h_B(n) \), they know extra \( n \) girls.

**Theorem (H.Kierstead, 1983)**

*If* \( G \) *is recursive bipartite graph which is* \( B, G \)-strong locally finite and satisfies expanding Hall condition with a recursive \( h_B \), then* \( G \) *has a recursive solution.*

**Theorem (H.Kierstead, 1983)**

*There exists a recursive bipartite graph which is* \( B, G \)-strong locally finite and satisfies expanding Hall condition, but has no recursive solution.*
Kierstead also indicated that a bipartite graph which satisfies expanding Hall condition has a solution even if “$B$-locally finite” is dropped.

Fact.
If $G$ is a bipartite graph which satisfies expanding Hall condition, then $G$ has a solution.
The aim of reverse mathematics is classifying the mathematical theorems by the difficulty.

For that reason, we look for the set existence axiom which is exactly needed to prove each theorem. That is, we check which set existence axiom is necessary and sufficient to the theorem over base system.

In reverse mathematics, we prove not only a theorem form axioms but also an axiom from the theorem. That is the reason why this research program is called “reverse mathematics”.

Although there are so many mathematical theorems, most of ordinary mathematical theorems are classified to the following five classes in this method.
What is Reverse Mathematics?

- The aim of reverse mathematics is classifying the mathematical theorems by the difficulty.
- For that reason, we look for the set existence axiom which is exactly needed to prove each theorem. That is, we check which set existence axiom is necessary and sufficient to the theorem over base system.
- In reverse mathematics, we prove not only a theorem form axioms but also an axiom from the theorem. That is the reason why this research program is called “reverse mathematics”.
- Although there are so many mathematical theorems, most of ordinary mathematical theorems are classified to the following five classes in this method.
What is Reverse Mathematics?

- The aim of reverse mathematics is classifying the mathematical theorems by the difficulty.
- For that reason, we look for the set existence axiom which is exactly needed to prove each theorem. That is, we check which set existence axiom is necessary and sufficient to the theorem over base system.
- In reverse mathematics, we prove not only a theorem form axioms but also an axiom from the theorem. That is the reason why this research program is called “reverse mathematics”.
- Although there are so many mathematical theorems, most of ordinary mathematical theorems are classified to the following five classes in this method.
What is Reverse Mathematics?

- The aim of reverse mathematics is classifying the mathematical theorems by the difficulty.
- For that reason, we look for the set existence axiom which is exactly needed to prove each theorem. That is, we check which set existence axiom is necessary and sufficient to the theorem over base system.
- In reverse mathematics, we prove not only a theorem form axioms but also an axiom from the theorem. That is the reason why this research program is called “reverse mathematics”.
- Although there are so many mathematical theorems, most of ordinary mathematical theorems are classified to the following five classes in this method.
Now we analyze how difficult each marriage problem is!

Marriage theorems provides an typical example of reverse mathematics. In my talk, only $\text{RCA}_0$ and $\text{WKL}_0$ and $\text{ACA}_0$ appear.
strictly $\Pi^1_1$-CA$_0$ : RCA$_0$ + $\Pi^1_1$ set existence axiom
stronger ATR$_0$ : RCA$_0$+ arithmetical transfinite recursion
   $\uparrow$
ACA$_0$ : RCA$_0$+ arithmetical set existence axiom
   $\uparrow$
WKL$_0$ : RCA$_0$+ weak König’s lemma
   discrete ordered semiring + $\Sigma^0_1$-induction
RCA$_0$ : + recursive set existence axiom

RCA$_0$ is our base formal system. It just guarantees the existence of recursive sets.

Now we analyze how difficult each marriage problem is!

Marriage theorems provides an typical example of reverse mathematics. In my talk, only RCA$_0$ and WKL$_0$ and ACA$_0$ appear.
\[ \text{strictly } \Pi^1_1-CA_0 : \text{RCA}_0 + \Pi^1_1 \text{ set existence axiom} \]
\[ \text{stronger } \text{ATR}_0 : \text{RCA}_0 + \text{arithmetical transfinite recursion} \]
\[ \uparrow \text{ACA}_0 : \text{RCA}_0 + \text{arithmetical set existence axiom} \]
\[ \uparrow \text{WKL}_0 : \text{RCA}_0 + \text{weak König’s lemma} \]
\[ \text{RCA}_0 : \text{discrete ordered semiring } + \Sigma^0_1\text{-induction} \]
\[ + \text{recursive set existence axiom} \]

\( \text{RCA}_0 \) is our base formal system. It just guarantees the existence of recursive sets.

Now we analyze how difficult each marriage problem is!

Marriage theorems provides an typical example of reverse mathematics. In my talk, only \( \text{RCA}_0 \) and \( \text{WKL}_0 \) and \( \text{ACA}_0 \) appear.
Hirst analyzed the strength of several marriage theorems with respect to reverse mathematics.

**Theorem (J.Hirst, 1990)**

The following is provable within RCA₀.

*If $G = (B, G; R)$ is a bipartite graph which satisfies Hall condition and $B$ is finite, then $G$ has a solution.*

This theorem is used throughout as a basic tool to analyze the strength of marriage theorems.
Theorem (J. Hirst, 1990)

- RCA\(_0\) \vdash B'_H GM \iff ACA\(_0\)
- RCA\(_0\) \vdash B''_H GM \iff WKL\(_0\)

\(B'_H GM\): If \(G = (B, G; R)\) is a bipartite graph which is \(B\)-locally finite and satisfies Hall condition, then \(G\) has a solution.

\(B''_H GM\): If \(G = (B, G; R)\) is a bipartite graph which is \(B\)-strong locally finite and satisfies Hall condition, then \(G\) has a solution.

“\(B\)-strong locally finite” is written as follows within RCA\(_0\).

\[\exists p : B \to \mathbb{N} \text{ s.t. } \forall b, \, g((b, g) \in R \to g < p(b))\]
Hirst also analyzed the strength of symmetric marriage theorems.

Theorem (J. Hirst, 1990)

- $\text{RCA}_0 \vdash B'_H G'_H M_s \iff \text{ACA}_0$
- $\text{RCA}_0 \vdash B''_H G''_H M_s \iff \text{WKL}_0$

$B'_H G'_H M_s$: If $G = (B, G; R)$ is a bipartite graph which is $B, G$-locally finite and satisfies Hall condition for $B$ and $G$, then $G$ has a symmetric solution.

$B''_H G''_H M_s$: If $G = (B, G; R)$ is a bipartite graph which is $B, G$-strong locally finite and satisfies Hall condition for $B$ and $G$, then $G$ has a symmetric solution.

- “$G$-locally finite” and “$G$-strong locally finite” and “Hall Condition for $G$” are defined in the same manner for $B$.
- A symmetric solution is a bijective solution. That is, the symmetric solution requires any girl also can marry.
“Expanding Hall condition” and “expanding Hall condition with recursive $h_B$” are written as follows within RCA$_0$.

- $H'$:
  \[ \forall n \exists m \forall X \subseteq_{\text{fin}} B(|N(X)| - |X| \geq 0 \land |X| \geq m \rightarrow |N(X)| - |X| \geq n) \]

- $H''$:
  \[ \exists h_B : B \rightarrow \mathbb{N} \text{ s.t.} \]
  \[ \forall n \forall X \subseteq_{\text{fin}} B(|N(X)| - |X| \geq 0 \land |X| \geq h_B(n) \rightarrow |N(X)| - |X| \geq n) \]

We analyze the strength of all the considerable marriage theorems and symmetric marriage theorems with respect to reverse mathematics.
## The strength of marriage theorems

<table>
<thead>
<tr>
<th>Condition</th>
<th>Expanding Hall condition</th>
<th>Strongly expanding Hall condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACA₀</td>
<td>Bₜ,GM</td>
<td>Bₜ,GM</td>
</tr>
<tr>
<td></td>
<td>*</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Bₜ,G’M</td>
<td>Bₜ,G’M</td>
</tr>
<tr>
<td></td>
<td>*</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Bₜ,G’’M</td>
<td>Bₜ,G’’M (¬ RCA₀ + Σ³⁻IND)</td>
</tr>
<tr>
<td>BₜGM (Hirst)</td>
<td>Bₜ,G’M</td>
<td>Bₜ,G’M</td>
</tr>
<tr>
<td>BₜG’M</td>
<td>Bₜ,G’M</td>
<td>Bₜ,G’M</td>
</tr>
<tr>
<td>BₜG’’M</td>
<td>Bₜ,G’’M</td>
<td>Bₜ,G’’M (¬ RCA₀ + Σ³⁻IND)</td>
</tr>
<tr>
<td>WKL₀</td>
<td>Bₜ’GM (Hirst)</td>
<td>Bₜ’GM</td>
</tr>
<tr>
<td></td>
<td>Bₜ’GM</td>
<td>Bₜ’GM</td>
</tr>
<tr>
<td></td>
<td>Bₜ’G’M</td>
<td>Bₜ’G’M</td>
</tr>
<tr>
<td></td>
<td>Bₜ’G’’M</td>
<td>Bₜ’G’’M (Kierstead)</td>
</tr>
<tr>
<td>RCA₀</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* : false  \( X' \) : \( X \)-locally finite  \( X'' \) : \( X \)-strong locally finite
Corollary.
If $G$ is recursive bipartite graph which is $G$-strong locally finite and satisfies expanding Hall condition with a recursive $h_B$, then $G$ has a recursive solution.

$\therefore \text{RCA}_0 + \Sigma^0_3\text{-IND} \vdash B_{H''}G''M.$

Corollary.
There exists a recursive bipartite graph which is $B$-strong locally finite and $G$-locally finite and satisfies expanding Hall condition with a recursive $h_B$, but does not have a recursive solution.

$\therefore \text{RCA}_0 \vdash B_{H''}'G'M \rightarrow \text{WKL}_0.$

The reasonable question is whether $\Sigma^0_3$ induction is essential for proving $B_{H''}G''M$ and $B_{H''}'G''M$.

Conjecture
$\text{RCA}_0 \not\vdash B_{H''}G''M.$
Corollary.

If $G$ is recursive bipartite graph which is $G$-strong locally finite and satisfies expanding Hall condition with a recursive $h_B$, then $G$ has a recursive solution.

$\therefore \text{RCA}_0 + \Sigma^0_3\text{-IND} \vdash B_{\overline{H}}''G''M.$

Corollary.

There exists a recursive bipartite graph which is $B$-strong locally finite and $G$-locally finite and satisfies expanding Hall condition with a recursive $h_B$, but does not have a recursive solution.

$\therefore \text{RCA}_0 \vdash B''_{\overline{H}}G'M \rightarrow WKL_0.$

The reasonable question is whether $\Sigma^0_3$ induction is essential for proving $B_{\overline{H}}''G''M$ and $B'_{\overline{H}}''G''M$.

Conjecture

$\text{RCA}_0 \not\vdash B_{\overline{H}}''G''M.$
The strength of symmetric marriage theorems

<table>
<thead>
<tr>
<th>$B_H G_H M_s$</th>
<th>$B_H G'_H M_s$</th>
<th>$B_H G''_H M_s$</th>
<th>$B'_H G'_H M_s$</th>
<th>$B'_H G''_H M_s$</th>
<th>$B''_H G'_H M_s$</th>
<th>$B''_H G''_H M_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_H G_H M_s$</td>
<td>$B_H G'_H M_s$</td>
<td>$B_H G''_H M_s$</td>
<td>$B'_H G'_H M_s$</td>
<td>$B'_H G''_H M_s$</td>
<td>$B''_H G'_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
</tr>
<tr>
<td>$B_H G'_H M_s$</td>
<td>$B_H G''_H M_s$</td>
<td>$B_H G''_H M_s$</td>
<td>$B'_H G'_H M_s$</td>
<td>$B'_H G''_H M_s$</td>
<td>$B''_H G'_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
</tr>
<tr>
<td>$B'_H G_H M_s$</td>
<td>$B'_H G'_H M_s$</td>
<td>$B'_H G''_H M_s$</td>
<td>$B'_H G'_H M_s$</td>
<td>$B'_H G''_H M_s$</td>
<td>$B''_H G'_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
</tr>
<tr>
<td>$B'_H G'_H M_s (Hirst)$</td>
<td>$B'_H G'_H M_s$</td>
<td>$B'_H G''_H M_s$</td>
<td>$B'_H G'_H M_s$</td>
<td>$B'_H G''_H M_s$</td>
<td>$B''_H G'_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
</tr>
<tr>
<td>$B'_H G''_H M_s$</td>
<td>$B'_H G''_H M_s$</td>
<td>$B'_H G''_H M_s$</td>
<td>$B'_H G''_H M_s$</td>
<td>$B'_H G''_H M_s$</td>
<td>$B''_H G'_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
</tr>
<tr>
<td>$B''_H G_H M_s$</td>
<td>$B''_H G'_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
</tr>
<tr>
<td>$B''_H G'_H M_s$</td>
<td>$B''_H G'_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
</tr>
<tr>
<td>$B''_H G''_H M_s (Hirst)$</td>
<td>$B''_H G''_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
</tr>
<tr>
<td>$B''_H G''_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
<td>$B''_H G''_H M_s$</td>
</tr>
</tbody>
</table>

False   ACA$_0$   WKL$_0$   RCA$_0$

- It does not make sense for each assertion to consider the assertion which assumptions for $B$ and $G$ are interchanged, for those are pairwise equivalent clearly. Therefore the above table lists strength of all considerable symmetric marriage theorems.

- There is a counter example of the green assertions.

Makoto Fujiwara (Tohoku university)
Corollary.

If $G$ is recursive bipartite graph which is $B, G$-strong locally finite and satisfies expanding H.c. for $B$ and $G$ with a recursive $h_B$ and $h_G$, then $G$ has a recursive sym. solution.

$\therefore$ RCA$_0 \vdash B''^r G''^r M_s$.

Corollary.

There exists a recursive bipartite graph which is $B, G$-strong locally finite and satisfies expanding H.c. for $B$ and $G$ with a recursive $h_G$, but does not have a recursive sym. solution.

$\therefore$ RCA$_0 \vdash B''^r G''^r M_s \rightarrow$ WKL$_0$.

Corollary.

There exists a recursive bipartite graph which is $B$-strong locally finite and $G$-locally finite and satisfies expanding H.c. for $B$ and $G$ with a rec. $h_B$ and $h_G$, but does not have a recursive sym. solution.

$\therefore$ RCA$_0 \vdash B''^r G''^r M_s \rightarrow$ ACA$_0$. 

Makoto Fujiwara (Tohoku university) 

The difficulty of marriage 2011.12.10 数学基礎論若手の会 18 / 25
Theorem (A. Manaster and J. Rosenstein, 1972)

*There exists a recursive bipartite graph which is \( B,G \)-strong locally finite and satisfies Hall condition, but has no recursive solution.*

As the method of modifying the above marriage problem to have a recursive solution, the following two ways are considerable.

1. **Strengthening the condition about locally finite**
2. **Strengthening the Hall condition** → by Kierstead.

We go about the modification by the first method.
Indexed Marriage Theorem

Let $B = G = \omega$ and $R \subset \omega \times \omega$ and there is a function $q : \omega \rightarrow \omega$ s.t. $\forall (i, j) \in R(j \leq i + q(i))$. Then $G = (B, G; R)$ has a solution if $G$ satisfies Hall condition.

- We regard the function $q$ as the parameter and consider the indexed marriage theorems dependent on $q$.
- By the result of Manaster-Rosenstein, in general we can not take the solution of this indexed marriage problem recursively even if the function $q$ is recursive.
- But if $q(i) \equiv 0$, clearly we can take the solution recursively.

⇒ Where is the boundary?
Indexed Marriage Theorem

Let $B = G = \omega$ and $R \subset \omega \times \omega$ and there is a function $q : \omega \to \omega$ s.t. $\forall (i, j) \in R(j \leq i + q(i))$. Then $G = (B, G; R)$ has a solution if $G$ satisfies Hall condition.

- We regard the function $q$ as the parameter and consider the indexed marriage theorems dependent on $q$.
- By the result of Manaster-Rosenstein, in general we can not take the solution of this indexed marriage problem recursively even if the function $q$ is recursive.
- But if $q(i) \equiv 0$, clearly we can take the solution recursively.

⇒ Where is the boundary?
Indexed Marriage Theorem

Let $B = G = \omega$ and $R \subset \omega \times \omega$ and there is a function $q : \omega \to \omega$ s.t. $\forall (i, j) \in R (j \leq i + q(i))$. Then $G = (B, G; R)$ has a solution if $G$ satisfies Hall condition.

- We regard the function $q$ as the parameter and consider the indexed marriage theorems dependent on $q$.
- By the result of Manaster-Rosenstein, in general we can not take the solution of this indexed marriage problem recursively even if the function $q$ is recursive.
- But if $q(i) \equiv 0$, clearly we can take the solution recursively.

$\Rightarrow$ Where is the boundary?
Indexed Marriage Theorem

Let \( B = G = \omega \) and \( R \subseteq \omega \times \omega \) and there is a function \( q : \omega \rightarrow \omega \) s.t. \( \forall (i, j) \in R(j \leq i + q(i)) \).

Then \( G = (B, G; R) \) has a solution if \( G \) satisfies Hall condition.

- We regard the function \( q \) as the parameter and consider the indexed marriage theorems dependent on \( q \).
- By the result of Manaster-Rosenstein, in general we can not take the solution of this indexed marriage problem recursively even if the function \( q \) is recursive.
- But if \( q(i) \equiv 0 \), clearly we can take the solution recursively.

\( \Rightarrow \) Where is the boundary?
Indexed Marriage Theorem

Let \( B = G = \omega \) and \( R \subseteq \omega \times \omega \) and there is a function \( q : \omega \to \omega \) s.t. \( \forall (i, j) \in R(j \leq i + q(i)) \).

Then \( G = (B, G; R) \) has a solution if \( G \) satisfies Hall condition.

- We regard the function \( q \) as the parameter and consider the indexed marriage theorems dependent on \( q \).
- By the result of Manaster-Rosenstein, in general we can not take the solution of this indexed marriage problem recursively even if the function \( q \) is recursive.
- But if \( q(i) \equiv 0 \), clearly we can take the solution recursively.

⇒ Where is the boundary?
Theorem

\[ \text{RCA}_0 \vdash (q : \mathbb{N} \to \mathbb{N} \land \forall i(q(i) \leq \bar{k})) \to \text{IM}^{(q)}. \]

\[ \text{IM}^{(q)} : \text{ If } G = (\mathbb{N}, \mathbb{N}; R) \text{ is a bipartite graph which satisfies Hall condition and } \forall (i, j) \in R(j \leq i + q(i)), \text{ then } G \text{ has a solution.} \]

Corollary.

Suppose \( q \) is a function from \( \omega \) to \( \omega \) and bounded by some constant.

If \( G = (\omega, \omega; R) \) be a recursive bipartite countable graph which satisfies Hall condition and \( \forall (i, j) \in R(j \leq i + q(i)) \), then \( G \) has a recursive solution.

Remark.

\[ \text{RCA}_0 + \Pi^0_2\text{-IND} \vdash (q : \mathbb{N} \to \mathbb{N} \land \exists k \forall i(q(i) \leq k)) \to \text{IM}^{(q)}. \]

Question.

\[ \text{RCA}_0 \vdash (q : \mathbb{N} \to \mathbb{N} \land \exists k \forall i(q(i) \leq k)) \to \text{IM}^{(q)}? \text{ or not?} \]
Theorem

\[ \text{RCA}_0 \vdash (q : \mathbb{N} \to \mathbb{N} \land \forall i(q(i) \leq \bar{k})) \to \text{IM}^{(q)}. \]

\[ \text{IM}^{(q)} : \quad \text{If } G = (\mathbb{N}, \mathbb{N}; R) \text{ is a bipartite graph which satisfies Hall condition and } \forall (i, j) \in R(j \leq i + q(i)), \text{ then } G \text{ has a solution.} \]

Corollary.

Suppose \( q \) is a function from \( \omega \) to \( \omega \) and bounded by some constant.

If \( G = (\omega, \omega; R) \) be a recursive bipartite countable graph which satisfies Hall condition and \( \forall (i, j) \in R(j \leq i + q(i)) \), then \( G \) has a recursive solution.

Remark.

\[ \text{RCA}_0 + \Pi_2^0-\text{IND} \vdash (q : \mathbb{N} \to \mathbb{N} \land \exists k \forall i(q(i) \leq k)) \to \text{IM}^{(q)}. \]

Question.

\[ \text{RCA}_0 \vdash (q : \mathbb{N} \to \mathbb{N} \land \exists k \forall i(q(i) \leq k)) \to \text{IM}^{(q)}? \text{ or not?} \]
**Theorem**

\[ \text{RCA}_0 \vdash [U(q) \rightarrow \text{IM}^{(q)}] \iff \text{WKL}_0, \]

where \( U(q) \) means

\[ q : \mathbb{N} \rightarrow \mathbb{N} \land \forall k \exists i(q(i) > i + k) \land \forall i, i'(i < i' \rightarrow q(i) \leq q(i')) \]

**Corollary.**

Suppose \( q \) is a recursive function from \( \omega \) to \( \omega \) and unbounded and nondecreasing.

Then there exists a recursive bipartite graph \( G = (\omega, \omega; R) \) which satisfies Hall condition and \( \forall (i, j) \in R(j \leq i + q(i)) \), but has no recursive solution.
We also analyze the symmetric indexed marriage problem.

**Theorem**

\[ \text{RCA}_0 \vdash (q : \mathbb{N} \to \mathbb{N} \land \forall i(q(i) \leq \bar{k})) \to \text{IM}^{(q)}_s. \]

**IM}^{(q)}_s :** If \( G = (\mathbb{N}, \mathbb{N}; R) \) is a bipartite graph which satisfies symmetric Hall condition and \( \forall (i, j) \in R(j \leq i + q(i) \land i \leq j + q(j)) \), then \( G \) has a symmetric solution.

**Corollary.**

Suppose \( q \) is a function from \( \omega \) to \( \omega \) and bounded by some constant.

If \( G = (\omega, \omega; R) \) be a recursive bipartite countable graph which satisfies symmetric Hall condition and \( \forall (i, j) \in R(j \leq i + q(i) \land i \leq j + q(j)) \), then \( G \) has a recursive symmetric solution.

**Remark.**

\[ \text{RCA}_0 + \Pi^0_2\text{-IND} \vdash (q : \mathbb{N} \to \mathbb{N} \land \exists k \forall i(q(i) \leq k)) \to \text{IM}^{(q)}_s. \]


Thank you for your attention.

...Who should I marry?