GENERALIZED STATIONARY REFLECTION AND CARDINAL ARITHMETIC

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Abstract. The reflection of stationary subsets of $\mathcal{P}_{\omega_1}(H)$ for all sets $H \supseteq \omega_1$, which we denote by SR_{ω_1}, is known to imply that $\lambda^\omega = \lambda$ for all regular cardinal $\lambda \geq \omega_2$. In particular, it implies $2^\omega \leq \omega_2$ and the Singular Cardinal Hypothesis. For a regular cardinal $\kappa \geq \omega_2$, the reflection of stationary subsets of $\mathcal{P}_\kappa(H)$ for all $H \supseteq \kappa$ is inconsistent with ZFC. But its restriction to stationary sets consisting of internally approachable sets, which we denote by $\mathsf{SR}_\kappa \restriction \mathrm{IA}$, is consistent with ZFC. In this paper, we study consequences of $\mathsf{SR}_\kappa \restriction \mathrm{IA}$ on cardinal arithmetic.

We prove that $\mathsf{SR}_\kappa \restriction \mathrm{IA}$ does not give any bound on 2^{ω_1}, while it implies $\lambda^\omega = \lambda$ for all regular cardinal $\lambda \geq \kappa^+$. We also prove that $\mathsf{SR}_\kappa \restriction \mathrm{IA}_{>\omega}$ does not give any bound on 2^ω and does not imply the Singular Cardinal Hypothesis, where $\mathsf{SR}_\kappa \restriction \mathrm{IA}_{>\omega}$ denotes the reflection of stationary subsets of $\mathcal{P}_\kappa(H)$ consisting of internally approachable sets of uncountable cofinalities.

1. Introduction

So far, the reflection of stationary subsets of $\mathcal{P}_{\omega_1}(H)$ for $H \supseteq \omega_1$ has been extensively studied by many set theorists. First, we recall this.

For a set $H \supseteq \omega_1$, let $\mathsf{SR}_{\omega_1}(H)$ be the following stationary reflection principle:

> Foe every stationary $X \subseteq \mathcal{P}_{\omega_1}(H)$, there is $R \subseteq H$ such that $|R| = \omega_1 \subseteq R$ and $X \cap \mathcal{P}_{\omega_1}(R)$ is stationary in $\mathcal{P}_{\omega_1}(R)$.

Let SR_{ω_1} be the assertion that $\mathsf{SR}_{\omega_1}(H)$ holds for every set $H \supseteq \omega_1$. SR_{ω_1} is often called the Weak Reflection Principle.

Foreman-Magidor-Shelah [7] proved that SR_{ω_1} follows from Martin's Maximum (MM). Moreover, many interesting consequences of MM follows from SR. For example, SR_{ω_1} implies that NS_{ω_1} is presaturated ([7]), Chang's Conjecture holds ([7]), $2^\omega \leq \omega_2$ (Todorčević) and the Singular Cardinal Hypothesis (SCH) holds (Shelah [18]). By the latter two consequences, $\lambda^\omega = \lambda$ for all regular cardinal $\lambda \geq \omega_2$ under SR_{ω_1}.

In this paper, we study consequences of generalization of SR_{ω_1} on cardinal arithmetic.

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Recall that the following straightforward generalization of SR_{ω_1} is inconsistent with ZFC. For a regular uncountable cardinal κ and a set $H \supseteq \kappa$, let $\mathsf{SR}_\kappa(H)$ be the following statement:

> Foe every stationary $X \subseteq \mathcal{P}_\kappa(H)$, there is $R \subseteq H$ such that $|R| = \kappa \subseteq R$ and $X \cap \mathcal{P}_\kappa(R)$ is stationary in $\mathcal{P}_\kappa(R)$.

Let SR_κ be the assertion that $\mathsf{SR}_\kappa(H)$ holds for every set $H \supseteq \kappa$. Feng-Magidor [4] and Foreman-Magidor [6] proved that SR_κ fails for any regular cardinal $\kappa \geq \omega_2$. Also, Shelah-Shioya [19] proved that $\mathsf{SR}_\kappa(\kappa^+)$ fails.

On the other hand, it was proved in [7] that the restriction of SR_κ to stationary sets consisting of internally approachable sets is consistent. Let us recall this restriction of SR_κ.

Let IA be the class of all internally approachable (i.a. for short) sets. (See §3 for the definition of internally approachability.) Suppose κ is a regular uncountable cardinal, and C is a subclass of IA. For a regular carddinal $\lambda \geq \kappa$, let $\mathsf{SR}_\kappa(\mathcal{H}_\lambda) \restriction C$ be the following statement:

> For every stationary $X \subseteq \mathcal{P}_\kappa(\mathcal{H}_\lambda) \cap C$, there is $R \subseteq \mathcal{H}_\lambda$ such that $|R| = \kappa \subseteq R$ and $X \cap \mathcal{P}_\kappa(R)$ is stationary in $\mathcal{P}_\kappa(R)$.

Also, let $\mathsf{SR}^*_\kappa(\mathcal{H}_\lambda) \restriction C$ be the statement obtained from $\mathsf{SR}_\kappa(\mathcal{H}_\lambda) \restriction C$ by requiring R to be i.a. of length κ in addition. Let $\mathsf{SR}_\kappa \restriction C$ ($\mathsf{SR}^*_\kappa \restriction C$ resp.) be the assertion that $\mathsf{SR}_\kappa(\mathcal{H}_\lambda) \restriction C$ ($\mathsf{SR}^*_\kappa(\mathcal{H}_\lambda) \restriction C$ resp.) holds for every regular $\lambda \geq \kappa$.

It is not hard to see that $\mathrm{SR}_{\omega_1} \restriction \mathrm{IA}$ is equivalent to SR_{ω_1}. (See §3.) So $\mathrm{SR}_\kappa \restriction \mathrm{IA}$ can be seen as a natural generalization of SR_{ω_1}. In [6], it was proved that for any regular uncountable cardinal κ, if a supercompact cardinal $> \kappa$ is Lévy collapsed to κ^+, then $\mathrm{SR}_\kappa \restriction \mathrm{IA}$ holds. In fact, κ^+ is generically supercompact with respect to $<\kappa$-closed forcings in this model, and this generic supercompactness of κ^+ implies $\mathrm{SR}^*_\kappa \restriction \mathrm{IA}$. (See §4.)

We study consequences of $\mathsf{SR}_\kappa \restriction \mathrm{IA}$ on cardinal arithmetic. We also discuss $\mathsf{SR}_\kappa \restriction C$ for $C = \mathrm{IA}_\omega, \mathrm{IA}_{>\omega}$, where IA_ω and $\mathrm{IA}_{>\omega}$ denote the class of all i.a. sets of cofinality ω and of cofinality $> \omega$, respectively. (See §3 for the definitions of IA_ω and $\mathrm{IA}_{>\omega}$.)

First, in §4, we show that the generic supercompactness of κ^+ with respect to $<\kappa$-closed forcings does not give any bound on 2^μ for a regular $\mu \geq \kappa$ (Proposition 4.3). So we have the following.

(I) $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}$ does not give any bound on 2^μ for a regular $\mu \geq \kappa$.

In §4, we also prove that if κ^+ is generically supercompact with respect to $<\kappa$-closed forcings, then $\lambda^{<\kappa} = \lambda$ for every regular $\lambda \geq \kappa$. In particular, this generic supercompactness of κ^+ implies that $2^{<\kappa} = \kappa$ and SCH holds above κ.

Next, in §5, we prove the following. (Theorem 5.1)

(II) $\mathsf{SR}_\kappa \restriction \mathrm{IA}_\omega$ implies that $\lambda^\omega = \lambda$ for all regular cardinal $\lambda > \kappa$. In particular, it implies that $2^\omega \leq \kappa^+$ and SCH holds above κ.

This can be proved by a straightforward generalization of the proof of the fact that $\lambda^\omega = \lambda$ for all regular $\lambda \geq \omega_2$ under SR_{ω_1}.

Since the generic supercompactness of κ^+ with respect to $< \kappa$-closed forcings imply that $2^{<\kappa} = \kappa$, it is natural to ask whether $\mathsf{SR}_\kappa \restriction \mathrm{IA}$ gives any bound on 2^μ for an uncountable regular $\mu < \kappa$. In §6, we give a negative answer to this question (Corollary 6.2 (2)).

(III) For any regular uncountable cardinal $\mu < \kappa$, $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}$ does not give any bound on 2^μ.

In fact, we will prove that, under some mild assumption, $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}$ is preserved by $\mathrm{Add}(\mu, \rho)$ for any regular uncountable $\mu < \kappa$ and any ordinal ρ, where $\mathrm{Add}(\mu, \rho)$ denotes the forcing adding ρ-many subsets of μ (Theorem 6.1 (2)). (See §2 for $\mathrm{Add}(\mu, \rho)$.)

Another natural question arising from (II) is whether $\mathsf{SR}_\kappa \restriction \mathrm{IA}_{>\omega}$ has any consequences on cardinal arithmetic. Does it give any bound on 2^ω or imply SCH? In §6 and §7, we also give the following negative answer to this question. (Corollary 6.2 (1) and 7.2).

(IV) $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ does not give any bound on 2^ω.
(V) $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ does not imply SCH above κ.

For (IV), we prove that, under some mild assumption, $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ is preserved by $\mathrm{Add}(\omega, \rho)$ for any ordinal ρ (Theorem 6.1 (1)). For (V), we prove that, under another mild assumption, $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ is preserved by the Prikry forcing above κ (Theorem 7.1).

To prove Theorem 6.1, we show that the class of internally approachable sets have some rigidity under forcings (Proposition 6.3). This may be of independent interest.

Prior to prove the above mentioned results, we present our notation and basic facts in Set Theory in §2 and basics on internally approachable sets in §3.

2. Preliminaries

In this section, we present our notation and basic facts in Set Theory.

First, we give miscellaneous notation.

For a set A of ordinals, we let $\mathrm{cf}(A)$ denote the cofinality of $\langle A, < \rangle$, that is, the smallest order-type of a cofinal subset in $\langle A, < \rangle$. For a regular cardinal μ, $\mathrm{Cf}(\mu)$ denotes the class of all limit ordinals of cofinality μ.

Let A be a set of ordinals. For a regular cardinal μ, we say that A is *μ-closed* if $\sup(B) \in A$ for any $B \subseteq A$ of order-type μ. For a cardinal ν, we say that A is *$< \nu$-closed* if A is μ-closed for all regular $\mu < \nu$. For $\alpha \in A$ which is not the largest element of A, let $\mathrm{suc}_A(\alpha)$ denote the successor of α in A, i.e. $\mathrm{suc}_A(\alpha) = \min(A \setminus (\alpha+1))$.

Suppose $\mathcal{M}$ is a structure in which a well-ordering of its universe is definable. Then, for a subset A of the universe of $\mathcal{M}$, we let $\mathrm{Sk}^{\mathcal{M}}(A)$ denote the Skolem hull of A in $\mathcal{M}$, i.e. the smallest $M \prec \mathcal{M}$ with $A \subseteq M$.

Next, we give our notation and basic facts on $\mathcal{P}_\kappa(H)$.

Let κ be a regular uncountable cardinal and H be a set with $|H| \geq \kappa$. Then $\mathcal{P}_\kappa(H)$ is the set of all $x \subseteq H$ of cardinality $< \kappa$. In this paper, we adopt Jech's notion of club and stationary subsets of $\mathcal{P}_\kappa(H)$. That is, $Z \subseteq \mathcal{P}_\kappa(H)$ is *club* in $\mathcal{P}_\kappa(H)$ if Z is $\subseteq$-cofinal in $\mathcal{P}_\kappa(H)$, and $\bigcup_{\xi<\mu} x_\xi \in Z$ whenever $\langle x_\xi \mid \xi < \mu \rangle$ is a $\subseteq$-increasing sequence of elements of Z of length $\mu < \kappa$. $X \subseteq \mathcal{P}_\kappa(H)$ is said to be *stationary* if $X \cap Z \neq \emptyset$ for any club $Z \subseteq \mathcal{P}_\kappa(H)$.

We will use the following fact without any reference.

Fact 2.1 ((1) Kueker [12], (3) Menas [14])**.** *Let κ be a regular uncountable cardinal, H be a set with $\kappa \subseteq H$ and X be a subset of $\mathcal{P}_\kappa(H)$.*

(1) *X is stationary in $\mathcal{P}_\kappa(H)$ if and only if for any function $f : {}^{<\omega}H \to H$ there is $x \in X$ such that $x \cap \kappa \in \kappa$ and x is closed under f, i.e. $f(a) \in x$ for all $a \in {}^{<\omega}x$.*

(2) *Suppose $H' \supseteq H$. Then $X \subseteq \mathcal{P}_\kappa(H)$ is stationary in $\mathcal{P}_\kappa(H)$ if and only if the set $\{x' \in \mathcal{P}_\kappa(H') \mid x' \cap H \in X\}$ is stationary in $\mathcal{P}_\kappa(H')$.*

From (2) of the above fact, we have the following.

Lemma 2.2. *Suppose $\omega_1 \subseteq H \subseteq H'$ and $\mathsf{SR}_{\omega_1}(H')$ holds. Then $\mathsf{SR}_{\omega_1}(H)$ holds.*

Proof. Suppose X is a stationary subset of $\mathcal{P}_{\omega_1}(H)$. We find $R \subseteq H$ such that $|R| = \omega_1 \subseteq R$ and $X \cap \mathcal{P}_{\omega_1}(R)$ is stationary.

Let $X' := \{x' \in \mathcal{P}_{\omega_1}(H') \mid x' \cap H \in X\}$. Then X' is stationary in $\mathcal{P}_{\omega_1}(H')$ by Fact 2.1 (2). By $\mathsf{SR}_{\omega_1}(H')$, there is $R' \subseteq H'$ such that $|R'| \subseteq \omega_1 \subseteq R'$ and $X' \cap \mathcal{P}_{\omega_1}(R')$ is stationary. Let $R := R' \cap H$. Then, $|R| = \omega_1 \subseteq R$. Note also that

$$X' \cap \mathcal{P}_{\omega_1}(R') \;=\; \{x' \in \mathcal{P}_{\omega_1}(R') \mid x' \cap R \in X \cap \mathcal{P}_{\omega_1}(R)\}\,.$$

Then, since $X' \cap \mathcal{P}_{\omega_1}(R')$ is stationary, $X \cap \mathcal{P}_{\omega_1}(R)$ is stationary again by Fact 2.1 (2). Thus R witnesses $\mathsf{SR}_{\omega_1}(H)$ for X. □

Next, we give our notation and basic facts on forcing.

Let $\mathbb{P}$ be a poset, and suppose G is a $\mathbb{P}$-generic filter over V. For a $\mathbb{P}$-name $\dot{a}$, let $\dot{a}^G$ denote the evaluation of $\dot{a}$ by G. For a set $M \subseteq V$, we let $M[G]$ be the set $\{\dot{a}^G \mid \dot{a} \in V^{\mathbb{P}} \cap M\}$.

Let $\mathbb{P}$ be a poset and μ be a regular uncountable cardinal. We say that $\mathbb{P}$ has the *$<\mu$-c.c.* if $\mathbb{P}$ has no antichain of cardinality μ. $\mathbb{P}$ is said to be *$<\mu$-closed* if every descending sequence in $\mathbb{P}$ of length $<\mu$ has a lower bound in $\mathbb{P}$. We say that $\mathbb{P}$ is

$<\mu$-*directed closed* if every downward directed subset of $\mathbb{P}$ of cardinality $<\mu$ has a lower bound in $\mathbb{P}$. $\mathbb{P}$ is said to be $<\mu$-*Baire* if $\bigcap \mathcal{D}$ is dense in $\mathbb{P}$ for any family $\mathcal{D}$ of dense open subsets of $\mathbb{P}$ with $|\mathcal{D}|<\mu$. Note that if $\mathbb{P}$ is $<\mu$-Baire, then forcing extensions by $\mathbb{P}$ do not add new sequences of ordinals of length $<\mu$.

We will also use the approximation property introduced by Hamkins [8]. We say that $\mathbb{P}$ has the $<\mu$-*approximation property* if it satisfies the following:

> For any $\mathbb{P}$-generic filter G over V and any $A \in V[G]$ with $A \subseteq V$,
> if $A \cap B \in V$ for all $B \in V$ with $|B|^V < \mu$, then $A \in V$.

We will use the following standard lemma. Note that Y may not be club in a forcing extension of V by $\mathbb{P}$.

Lemma 2.3. *Suppose κ is a regular uncountable cardinal and $\kappa \subseteq H$. Let $\mathbb{P}$ be a poset with the $<\kappa$-c.c. and $\dot{Z}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\dot{Z}$ is club in $\mathcal{P}_\kappa(H)$". Then, in V, $Y := \{y \in \mathcal{P}_\kappa(H) \mid \Vdash_{\mathbb{P}}$ "$y \in \dot{Z}$"$\}$ is club in $\mathcal{P}_\kappa(H)$.*

Proof. Clearly, Y is closed, that is, $\bigcup_{\xi<\mu} x_\xi \in Y$ for any $\subseteq$-increasing sequence $\langle x_\xi \mid \xi < \mu\rangle$ in Y of length $\mu < \kappa$. To show that Y is $\subseteq$-cofinal, take an arbitrary $x \in \mathcal{P}_\kappa(H)$. We find $y \in Y$ with $x \subseteq y$.

By recursion on $n < \omega$, take $x_n \in \mathcal{P}_\kappa(H)$ and a $\mathbb{P}$-name $\dot{z}_n$ as follows. Let $x_0 := x$. Suppose that x_n has been taken. First, take a $\mathbb{P}$-name $\dot{z}_n$ so that $\Vdash_{\mathbb{P}}$ "$x_n \subseteq \dot{z}_n \in \dot{Z}$". Then, by the $<\kappa$-c.c. of $\mathbb{P}$, take $x_{n+1} \in \mathcal{P}_\kappa(H)$ so that $\Vdash_{\mathbb{P}}$ "$\dot{z}_n \subseteq x_{n+1}$".

Let $y := \bigcup_{n\in\omega} x_n$. Clearly $x \subseteq y$. Also, $\mathbb{P}$ forces that $\langle \dot{z}_n \mid n < \omega\rangle$ is a $\subseteq$-increasing sequence in $\dot{Z}$ and that $y = \bigcup_{n<\omega} \dot{z}_n$. Then, it follows from the closure of $\dot{Z}$ that $\Vdash_{\mathbb{P}}$ "$y \in \dot{Z}$". So $y \in Y$. □

Let μ be a regular cardinal and A be a set of ordinals. Then $\mathrm{Add}(\mu, A)$ denotes the poset of all partial functions $p : \mu \times A \to 2$ such that $|p| < \mu$. Also, $\mathrm{Col}(\mu, A)$ denotes the poset of all partial functions $p : \mu \times A \to \sup(A)$ such that $|p| < \mu$ and $p(\alpha, \beta) \in \beta$ for all $\langle \alpha, \beta\rangle \in \mu \times A$. Both $\mathrm{Add}(\mu, A)$ and $\mathrm{Col}(\mu, A)$ are ordered by reverse inclusions. Thus, both of them are $<\mu$-closed.

Recall that $\mathrm{Add}(\mu, \nu)$ has the $<(2^{<\mu})^+$-c.c. A forcing by $\mathrm{Add}(\mu, A)$ adds generic subsets of μ indexed by elements of A, and a forcing by $\mathrm{Col}(\mu, A)$ adds a surjection from μ to β for each $\beta \in A$. According to the custom, for a cardinal $\nu > \mu$, we denote $\mathrm{Col}(\mu, \nu)$ as $\mathrm{Col}(\mu, <\nu)$. Recall that if ν is inaccessible, then $\mathrm{Col}(\mu, <\nu)$ has the $<\nu$-c.c., and $\nu = \mu^+$ in its forcing extensions.

We will use the following fact by Mitchell [15]. In [15], it is proved in some general settings. Here we give a direct proof.

Fact 2.4 (Mitchell [15]). *Let μ be a regular cardinal and ρ be an ordinal. Then $\mathrm{Add}(\mu, \rho)$ has the $<(2^{<\mu})^+$-approximation property.*

Proof. We work in V. Let $\nu := 2^{<\mu}$. For a contradiction, assume $\mathrm{Add}(\mu,\rho)$ does not have the $<\nu^+$-approximation property. Then, we can take $p \in \mathrm{Add}(\mu,\rho)$ and a $\mathrm{Add}(\mu,\rho)$-name $\dot{A}$ with the following properties.

(i) $p \Vdash_{\mathrm{Add}(\mu,\rho)}$ " $\dot{A} \subseteq V \ \wedge \ \dot{A} \notin V$ ".
(ii) $p \Vdash_{\mathrm{Add}(\mu,\rho)}$ " $\dot{A} \cap B \in V$ for any $B \in V$ with $|B|^V \leq \nu$ ".

Let θ be a sufficiently large regular cardinal. Then we can take $M \prec \langle \mathcal{H}_\theta, \in \rangle$ such that $\mu, \rho, p, \dot{A} \in M$, $|M| = \nu$ and ${}^{<\mu}M \subseteq M$. By (ii) we can take $p^* \leq p$ and $A^* \subseteq M$ such that $p^* \Vdash_{\mathrm{Add}(\mu,\rho)}$ " $\dot{A} \cap M = A^*$ ". Let $q^* := p^* \cap M$. Then $q^* \in \mathrm{Add}(\mu,\rho) \cap M$ since ${}^{<\mu}M \subseteq M$. Note also that $q^* \leq p$.

By (i) and the elementarity of M, in M, we can take a and $q_0, q_1 \leq q^*$ such that $q_0 \Vdash_{\mathrm{Add}(\mu,\rho)}$ " $a \notin \dot{A}$ " and $q_1 \Vdash_{\mathrm{Add}(\mu,\rho)}$ " $a \in \dot{A}$ ". Note that both q_0 and q_1 are compatible with p^* since $q_0, q_1 \in M$ and $q_0, q_1 \leq q^* = p^* \cap M$. If $a \in A^*$, then this contradicts that q_0 and p^* are compatible, and if $a \notin A^*$, then this contradicts that q_1 and p^* are compatible. □

3. Internally approachable sets

In this section, we briefly review basics on internally approachable sets.

For a limit ordinal ζ, a set M is called *internally approachable* (i.a. for short) of length ζ if there is a $\subseteq$-increasing sequence $\langle M_\xi \mid \xi < \zeta \rangle$ such that

- $\bigcup_{\xi<\zeta} M_\xi = M$,
- $\langle M_\xi \mid \xi < \zeta' \rangle \in M$ for all $\zeta' < \zeta$.

A sequence $\langle M_\xi \mid \xi < \zeta \rangle$ as above is called an *i.a. sequence* to M.

Let M be a set. We say that M *i.a.* if M is i.a. of length ζ for some limit ordinal ζ. M is said to be *i.a. of regular length* if M is i.a. of length μ for some regular cardinal μ. For a regular cardinal μ, we say that M is *i.a. of cofinality* μ ($> \mu$, $< \mu$, respectively) if M is i.a. of length ζ for some limit ordinal ζ of cofinality μ ($> \mu$, $< \mu$, respectively).

Let IA denote the class of all i.a. sets. Let IA_μ ($\mathrm{IA}_{>\mu}$, $\mathrm{IA}_{<\mu}$, respectively) be the class of all sets which are i.a. of cofinality μ ($> \mu$, $< \mu$, respectively).

Note that if M is countable and $M \prec \langle \mathcal{H}_\lambda, \in \rangle$ for some regular uncountable cardinal λ, then M is i.a. of length ω. (Take an enumeration $\langle a_n \mid n < \omega \rangle$ of M, and let $M_n := \{a_m \mid m < n\}$ for $n < \omega$. Then $\langle M_n \mid n < \omega \rangle$ is an i.a. sequence to M.) Thus, for any regular uncountable cardinal λ, $\mathcal{P}_{\omega_1}(\mathcal{H}_\lambda) \cap \mathrm{IA}$ is club in $\mathcal{P}_{\omega_1}(\mathcal{H}_\lambda)$, and so $\mathsf{SR}_{\omega_1}(\mathcal{H}_\lambda)$ is equivalent to $\mathsf{SR}_{\omega_1}(\mathcal{H}_\lambda) \restriction \mathrm{IA}$. Thus SR_{ω_1} is equivalent to $\mathsf{SR}_{\omega_1} \restriction \mathrm{IA}$ by Lemma 2.2.

The following are basic facts on i.a. sets. The proofs of (1) and (2) are found in [7, Lemma 28], and those of (3) and (4) are found in [3, Lemma 2.3] and [6, Proposition 2.4], respectively.

Fact 3.1. *Let μ, κ, λ and λ' be regular cardinals with $\mu < \kappa \leq \lambda < \lambda'$.*

(1) $\mathcal{P}_\kappa(\mathcal{H}_\lambda) \cap \mathrm{IA}_\mu$ *is stationary in* $\mathcal{P}_\kappa(\mathcal{H}_\lambda)$.
(2) *Every stationary* $X \subseteq \mathcal{P}_\kappa(\mathcal{H}_\lambda) \cap \mathrm{IA}$ *remains stationary in any* $<\kappa$*-closed forcing extension.*
(3) *Suppose* $M \in \mathrm{IA}_\mu$ *and* $M \prec \langle \mathcal{H}_\lambda, \in \rangle$. *Then* $M \cap \lambda$ *is* ν*-closed for every regular cardinal* $\nu < \mu$*, and* $\mathrm{cf}(M \cap \nu) = \mu$ *for any regular cardinal* $\nu \in M$ *with* $\nu \geq \mu$ *and for* $\nu = \lambda$.
(4) *For any* $X \subseteq \mathcal{P}_\kappa(\mathcal{H}_\lambda) \cap \mathrm{IA}$*,* X *is stationary in* $\mathcal{P}_\kappa(\mathcal{H}_\lambda)$ *if and only if the set* $\{M' \in \mathcal{P}_\kappa(\mathcal{H}_{\lambda'}) \cap \mathrm{IA} \mid M' \cap \mathcal{H}_\lambda \in X\}$ *is stationary in* $\mathcal{P}_\kappa(\mathcal{H}_{\lambda'})$.

Using Fact 3.1, the same argument as Lemma 2.2 yields the following.

Lemma 3.2. *Let* μ *and* κ *be regular cardinals with* $\mu < \kappa$*, and suppose* C *is* IA, IA_μ, $\mathrm{IA}_{<\mu}$ *or* $\mathrm{IA}_{>\mu}$. *Assume that* λ *and* λ' *are regular cardinals with* $\kappa \leq \lambda \leq \lambda'$ *and that* $\mathsf{SR}_\kappa(\mathcal{H}_{\lambda'}) \restriction C$ *holds. Then,* $\mathsf{SR}_\kappa(\mathcal{H}_\lambda) \restriction C$ *holds.*

Proof. Take an arbitrary stationary $X \subseteq \mathcal{P}_\kappa(\mathcal{H}_\lambda) \cap C$. We find $R \subseteq \mathcal{H}_\lambda$ such that $|R| = \kappa \subseteq R$ and $X \cap \mathcal{P}_\kappa(R)$ is stationary.

Let X' be the set of all $M' \in \mathcal{P}_\kappa(\mathcal{H}_{\lambda'}) \cap \mathrm{IA}$ such that $\lambda \in M' \prec \langle \mathcal{H}_{\lambda'}, \in \rangle$ and $M' \cap \mathcal{H}_\lambda \in X$. Then X' is stationary in $\mathcal{P}_\kappa(\mathcal{H}_\lambda)$ by Fact 3.1 (4).

We claim that $X' \subseteq C$. This is clear if $C = \mathrm{IA}$. Suppose C is $\mathrm{IA}_{<\mu}$, IA_μ or $\mathrm{IA}_{>\mu}$. Take an arbitrary $M' \in X'$. Let $M := M' \cap \mathcal{H}_\lambda$. Then $M \in X \subseteq C$. Let ν be a regular cardinal with $M \in \mathrm{IA}_\nu$. Note that $\nu > \mu$, $\nu = \mu$, or $\nu < \mu$ if $C = \mathrm{IA}_{<\mu}$, $C = \mathrm{IA}_\mu$, or $C = \mathrm{IA}_{>\mu}$, respectively. Note also that $M \prec \langle \mathcal{H}_\lambda, \in \rangle$. So $\mathrm{cf}(M' \cap \lambda) = \mathrm{cf}(M \cap \lambda) = \nu$ by Fact 3.1 (3). Then, since $M' \in \mathrm{IA}$ and $\lambda \in M' \prec \langle \mathcal{H}_{\lambda'}, \in \rangle$, we have that $M' \in \mathrm{IA}_\nu$ again by Fact 3.1 (3). So $M' \in C$.

By $\mathsf{SR}_\kappa(\mathcal{H}_{\lambda'}) \restriction C$, there is $R' \subseteq \mathcal{H}_{\lambda'}$ such that $|R'| = \kappa \subseteq R'$ and $X' \cap \mathcal{P}_\kappa(R')$ is stationary. Let $R := R' \cap \mathcal{H}_\lambda$. Clearly, $|R| = \kappa \subseteq R$. Note also that the set $Y' := \{M' \in \mathcal{P}_\kappa(R') \mid M' \cap R \in X \cap \mathcal{P}_\kappa(R)\}$ includes $X' \cap \mathcal{P}_\kappa(R')$, and so Y' is stationary in $\mathcal{P}_\kappa(R')$. Then, $X \cap \mathcal{P}_\kappa(R)$ is stationary in $\mathcal{P}_\kappa(R)$ by Fact 2.1 (2). Therefore R is as desired. □

The notion of i.a. is closely related to Shelah's approachability ideal $I[\kappa]$. For a regular uncountable cardinal κ, let $I[\kappa]$ be the set of all $S \subseteq \kappa$ with the following property.

> There are a sequence $\langle b_\alpha \mid \alpha < \kappa \rangle$ of bounded subsets of κ and a club $C \subseteq \kappa$ such that for any limit ordinal $\gamma \in S \cap C$ there is $b \subseteq \gamma$ of order-type $\mathrm{cf}(\gamma)$ with $b \cap \beta \in \{b_\alpha \mid \alpha < \gamma\}$ for all $\beta < \gamma$.

We will use the following folklore.

Lemma 3.3. *Let* κ *and* λ *be regular cardinals with* $\omega_1 \leq \kappa \leq \lambda$*, and suppose* $\kappa \in I[\kappa]$. *Then, there is a club* $Z \subseteq \mathcal{P}_\kappa(\mathcal{H}_\lambda)$ *such that every* $M \in Z \cap \mathrm{IA}$ *is i.a. of regular length.*

Proof. Take a pair $\vec{b} = \langle b_\alpha \mid \alpha < \kappa \rangle$ and $C \subseteq \kappa$ witnessing that $\kappa \in I[\kappa]$. Let $\mathcal{M} := \langle \mathcal{H}_\lambda, \in, \kappa, \vec{b}, C \rangle$, and let Z be the set of all $M \in \mathcal{P}_\kappa(\mathcal{H}_\lambda)$ such that $M \prec \mathcal{M}$ and $M \cap \kappa \in \kappa$. Then, Z is club in $\mathcal{P}_\kappa(\mathcal{H}_\lambda)$. We claim that Z witnesses the lemma.

Suppose $M \in Z \cap \mathrm{IA}$. Take an i.a. sequence $\langle M_\xi \mid \xi < \zeta \rangle$ to M. Let $\mu := \mathrm{cf}(\zeta)$. We show that M is i.a. of length μ. Let $\gamma := M \cap \kappa$ and $\gamma_\xi := \sup(M_\xi \cap \kappa)$ for each $\xi < \zeta$. Then, $\langle \gamma_\xi \mid \xi < \zeta \rangle$ is an increasing cofinal sequence in γ, and so $\mathrm{cf}(\gamma) = \mathrm{cf}(\zeta) = \mu$. Note also that $\gamma \in C$ since $M \prec \mathcal{M}$.

Since $\vec{b}$ and C witnesses that $\kappa \in I[\kappa]$, we can take a cofinal $b \subseteq \gamma$ of order-type μ all of whose proper initial segments are in $\{b_\alpha \mid \alpha < \gamma\}$. Note that $\{b_\alpha \mid \alpha < \gamma\} \subseteq M$ since $\gamma \subseteq M \prec \mathcal{M}$. So all proper initial segments of b are in M. Let $\langle \beta_\eta \mid \eta < \mu \rangle$ be the increasing enumeration of b, and for each $\eta < \mu$ let ξ_η be the least $\xi < \zeta$ with $\beta_\eta < \gamma_\xi$. Then, $\langle \xi_\eta \mid \eta < \mu \rangle$ is increasing and cofinal in ζ. In particular, $\langle M_{\xi_\eta} \mid \eta < \mu \rangle$ is $\subseteq$-increasing, and $\bigcup_{\eta<\mu} M_{\xi_\eta} = M$. Moreover, all proper initial segments of $\langle M_{\xi_\eta} \mid \eta < \mu \rangle$ are in M, since they are definable from proper initial segments of b and $\langle M_\xi \mid \xi < \zeta \rangle$, which are in M. So $\langle M_{\xi_\eta} \mid \eta < \mu \rangle$ is an i.a. sequence to M, and M is i.a. of length μ. □

Suppose κ is a regular uncountable cardinal. The assertion that $\kappa \in I[\kappa]$ is known to be easily forced by the following poset $\mathbb{I}_\kappa$, which adds a witness of $\kappa \in I[\kappa]$ by initial segments. Let $\mathbb{I}_\kappa$ be the poset of all $\langle s, c \rangle$ such that

- (i) c is a closed bounded subset of κ.
- (ii) s is a sequence $\langle b_\alpha \mid \alpha < \max(c) \rangle$ of bounded subsets of κ.
- (iii) For any limit ordinal $\gamma \in c$, there is $b \subseteq \gamma$ of order-type $\mathrm{cf}(\gamma)$ with $b \cap \beta \in \{b_\alpha \mid \alpha < \gamma\}$ for all $\beta < \gamma$.

$\langle s, c \rangle \leq \langle t, d \rangle$ in $\mathbb{I}_\kappa$ if s and c are end-extensions of t and d, respectively.

Lemma 3.4. *Let κ be a regular uncountable cardinal. Then $\mathbb{I}_\kappa$ is $<\kappa$-Baire.*

Proof. Suppose $\langle D_\xi \mid \xi < \mu \rangle$ is a sequence of dense open subsets of $\mathbb{I}_\kappa$, where μ is a cardinal $< \kappa$. Take an arbitrary $p \in \mathbb{I}_\kappa$. We must find $p^* \leq p$ with $p^* \in \bigcap_{\xi<\mu} D_\xi$.

For each limit ordinal $\zeta \leq \mu$, take a cofinal $e_\zeta \subseteq \zeta$ of order-type $\mathrm{cf}(\zeta)$. By recursion on $\xi \leq \mu$, we will take $p_\xi = \langle c_\xi, s_\xi \rangle \in \mathbb{I}_\kappa$ so that $\langle p_\xi \mid \xi \leq \mu \rangle$ is descending and $p_{\xi+1} \in D_\xi$. When p_ξ has been taken, we let $\beta_\xi := \max(c_\xi)$ and let $s_\xi = \langle b_\alpha \mid \alpha \leq \beta_\xi \rangle$. First of all, let $p_0 := p$.

Suppose $\xi < \mu$ and p_ξ has been taken. We take $p_{\xi+1}$. First, take $p' = \langle c', s' \rangle \leq p_\xi$ with $p' \in D_\xi$. Let $\beta' := \max(c')$, $\beta_{\xi+1} := \beta' + \mu + 1$ and $c_{\xi+1} := c' \cup \{\beta_{\xi+1}\}$. Next, take a sequence $\langle b_\alpha \mid \beta' \leq \alpha < \beta_{\xi+1} \rangle$ of bounded subsets of κ such that

(∗) $\{\beta_\eta \mid \eta \in e_\zeta \cap (\xi + 1)\} \in \{b_\alpha \mid \beta' \leq \alpha < \beta_{\xi+1}\}$ for all limit $\zeta \leq \mu$.

Let $s_{\xi+1} := s' \cup \langle b_\alpha \mid \beta' \leq \alpha < \beta_{\xi+1} \rangle$. Finally, let $p_{\xi+1} := \langle c_{\xi+1}, s_{\xi+1} \rangle$. Note that $p_{\xi+1} \in \mathbb{I}_\kappa$ since $p' \in \mathbb{I}_\kappa$, and $\beta_{\xi+1}$ is a successor ordinal. Note also that $p_{\xi+1} \leq p_\xi$ and $p_{\xi+1} \in D_\xi$.

Next, suppose ξ is a limit ordinal $\leq \mu$ and $\langle p_\eta \mid \eta < \xi \rangle$ has been taken. Let $\beta_\xi := \sup_{\eta<\xi} \beta_\eta$, $c_\xi := (\bigcup_{\eta<\xi} c_\eta) \cup \{\beta_\xi\}$, $s_\xi := \bigcup_{\eta<\xi} s_\eta$ and $p_\xi := \langle c_\xi, s_\xi \rangle$. To see that $p_\xi \in \mathbb{I}_\kappa$, it suffices to check the property (iii) of conditions of $\mathbb{I}_\kappa$ for $\gamma = \beta_\xi$. But, by $(*)$ at successor steps, $b = \{\beta_\eta \mid \eta \in e_\xi\}$ witnesses this. Note also that $p_\xi \leq p_\eta$ for all $\eta < \xi$.

Now, we have defined $\langle p_\xi \mid \xi \leq \mu \rangle$. Clearly, $p^* := p_\mu$ is as desired. □

By the previous lemma, $\mathbb{I}_\kappa$ preserves all cofinalities $\leq \kappa$. Moreover, it is easy to check that for any $\gamma < \kappa$ the set $\{\langle c, s \rangle \in \mathbb{I}_\kappa \mid \gamma \leq \max(c)\}$ is dense. Then, for an $\mathbb{I}_\kappa$-generic filter G over V, $\bigcup\{s \mid \exists c,\ \langle c, s \rangle \in G\}$ and $\bigcup\{c \mid \exists s,\ \langle c, s \rangle \in G\}$ witness that $\kappa \in I[\kappa]$. So we have the following.

Lemma 3.5. *Let κ be a regular uncountable cardinal. Suppose G is an $\mathbb{I}_\kappa$-generic filter over V. Then $\kappa \in I[\kappa]$ in $V[G]$.*

Next, we turn our attention to scales in the PCF theory. It is well-known that i.a. sets have nice properties in connection with scales. See [1], [3] and [17] for example. Below, we briefly review a very basic one (Lemma 3.6). In this paper, we only use scales at singular cardinals of cofinality ω.

Let ν be a singular cardinal of cofinality ω and I be a set of regular cardinals with $\sup(I) = \nu$. Then, ΠI is the set of all functions $f : I \to \mathrm{On}$ such that $f(\iota) < \iota$ for all $\iota \in I$. For $f, g \in \Pi I$, we write $f <^* g$ ($f \leq^* g$, $f =^* g$, respectively) if there is $\delta < \sup I$ such that $f(\iota) < g(\iota)$ ($f(\iota) \leq g(\iota)$, $f(\iota) = g(\iota)$, respectively) for all $\iota \in I \setminus \delta$. A *scale* in ΠI is a $<^*$-increasing $<^*$-cofinal sequence of elements of ΠI of length ν^+.

To state the nice property of i.a. sets, we introduce the notion of μ-continuity of scales. Let ν and I be as above.

For $\mathcal{F} \subseteq \Pi I$ with $|\mathcal{F}| < \nu$, let $\sup(\mathcal{F})$ be $g \in \Pi I$ such that $g(\iota) = \sup_{f \in \mathcal{F}} f(\iota)$ for $\iota > |\mathcal{F}|$ and $g(\iota) = 0$ for all $\iota \leq |\mathcal{F}|$. Suppose γ is a limit ordinal $< \nu^+$ of uncountable cofinality. For a sequence $\langle \delta_\alpha \mid \alpha < \gamma \rangle$ of ordinals, let

$$\operatorname{csup}_{\alpha<\gamma} \delta_\alpha := \min\{\sup_{\alpha \in c} \delta_\alpha \mid c \text{ is a club subset of } \gamma \text{ of order-type } \operatorname{cf}(\gamma)\}\,.$$

For a sequence $\langle f_\alpha \mid \alpha < \gamma \rangle$ in ΠI, let $\operatorname{csup}_{\alpha<\gamma} f_\alpha$ be $g \in \Pi I$ such that $g(\iota) = \operatorname{csup}_{\alpha<\gamma} f_\alpha(\iota)$ for $\iota > \operatorname{cf}(\gamma)$ and $g(\iota) = 0$ for $\iota \leq \operatorname{cf}(\gamma)$.

Note that if $\langle f_\alpha \mid \alpha < \gamma \rangle$ is $<^*$-increasing, then $f_\beta <^* \operatorname{csup}_{\alpha<\gamma} f_\alpha$ for all $\beta < \gamma$: Assume not. Let $f := \operatorname{csup}_{\alpha<\gamma} f_\alpha$, and take $\beta < \gamma$ with $f_\beta \not<^* f$. Then we can take a countable unbounded $I' \subseteq I$ such that $\operatorname{cf}(\gamma) < \min(I')$ and $f_\beta(\iota) \geq f(\iota)$ for all $\iota \in I'$. For each $\iota \in I'$, take a club $c_\iota \subseteq \gamma$ with $f(\iota) = \sup_{\alpha \in c_\iota} f_\alpha(\iota)$. Then $c := \bigcap_{\iota \in I'} c_\iota$ is club in γ. Take $\alpha \in c$ with $\beta < \alpha$. Since $f_\beta <^* f_\alpha$, we can take $\iota \in I'$ with $f_\beta(\iota) < f_\alpha(\iota)$. But $f_\alpha(\iota) \leq f(\iota)$ since $\alpha \in c \subseteq c_\iota$. So $f_\beta(\iota) < f(\iota)$. This contradicts that $\iota \in I'$.

For a regular uncountable cardinal $\mu < \nu$, a scale $\langle f_\alpha \mid \alpha < \nu^+ \rangle$ in ΠI is said to be *μ-continuous* if $f_\gamma = \operatorname{csup}_{\alpha<\gamma} f_\alpha$ for all $\gamma \in \nu^+ \cap \operatorname{Cf}(\mu)$.

We also use characteristic functions of sets. For a set A with $|A| < \nu$, *the characteristic function* of A on I, which we denote as ch^I_A, is the function on I defined by $\mathrm{ch}^I_A(\iota) := \sup(A \cap \iota)$ for $\iota > |A|$ and $\mathrm{ch}^I_A(\iota) := 0$ for $\iota \leq |A|$. Note that $\mathrm{ch}^I_A \in \Pi I$.

I.a. sets have the following nice property in connection with scales. We give its proof for the convenience of the readers.

Lemma 3.6 (Shelah [17])**.** *Let ν be a singular cardinal of cofinality ω and μ be a regular uncountable cardinal $< \nu$. Suppose I is a countable set of regular cardinals with $\sup I = \nu$ and $\vec{f} = \langle f_\alpha \mid \alpha < \nu^+ \rangle$ is a μ-continuous scale in ΠI. Suppose also that M is an i.a. set of length μ, $|M| < \nu$ and $\vec{f} \in M \prec \langle \mathcal{H}_\lambda, \in \rangle$ for some regular cardinal $\lambda > \nu^+$. Then $\mathrm{ch}^I_M =^* f_{\sup(M \cap \nu^+)}$.*

Proof. We may assume $\min I > \mu$. Let $\gamma := \sup(M \cap \nu^+)$. Note that $\mathrm{cf}(\gamma) = \mu$. For each $\iota \in I$, let c_ι be a club subset of γ of order-type μ such that $f_\gamma(\iota) = \sup_{\alpha \in c_\iota} f_\alpha(\iota)$. Let $c := (\bigcap_{\iota \in I} c_\iota) \cap M$. Then c is club in γ, and $f_\gamma = \sup_{\alpha \in c} f_\alpha$. Let $\langle M_\xi \mid \xi < \mu \rangle$ be an i.a. sequence to M.

First, we prove $f_\gamma \leq^* \mathrm{ch}^I_M$. For each $\alpha \in c$, $\mathrm{ran}(f_\alpha) \subseteq M$ since $\alpha, \vec{f} \in M$, and $\mathrm{dom}(f_\alpha) = I$ is countable. Then, it follows that $f_\gamma = \sup_{\alpha \in c} f_\alpha \leq^* \mathrm{ch}^I_M$.

Next, we prove $\mathrm{ch}^I_M \leq^* f_\gamma$. For each $\xi < \mu$, let $g_\xi := \mathrm{ch}^I_{M_\xi}$. Note that $g_\xi \in M$. Then, for each $\xi < \mu$, there is $\alpha \in \nu^+ \cap M$ with $g_\xi <^* f_\alpha$ since $\vec{f} \in M$, and $\vec{f}$ is $<^*$-cofinal in ΠI. So $g_\xi <^* f_\gamma$ for all $\xi < \mu$. For each $\xi < \mu$, let $\iota_\xi \in I$ be such that $g_\xi(\iota) < f_\gamma(\iota)$ for all $\iota \in I$ with $\iota \geq \iota_\xi$. Then, we can take $\iota^* \in I$ such that $b := \{\xi < \mu \mid \iota_\xi = \iota^*\}$ is unbounded in μ, since μ is regular uncountable, and I is countable. Then, $\sup_{\xi \in b} g_\xi \leq^* f_\gamma$. On the other hand, $\mathrm{ch}^I_M = \sup_{\xi \in b} g_\xi$ since $M = \bigcup_{\xi \in b} M_\xi$. So $\mathrm{ch}^I_M \leq^* f_\gamma$. □

We will also use the following.

Lemma 3.7. *Let ν be a singular cardinal of cofinality ω, I be a set of regular cardinals with $\sup(I) = \nu$ and μ be a regular uncountable cardinal $< \nu$.*

(1) *If $2^\nu = \nu^+$, then there is a μ-continuous scale in ΠI.*
(2) *Suppose $\vec{f} = \langle f_\alpha \mid \alpha < \nu^+ \rangle$ is a μ-continuous scale in ΠI. Then $\vec{f}$ is a μ-continuous scale in ΠI in any $<\mu$-closed $<\mu^+$-c.c. forcing extension.*

Proof. (1) Suppose $2^\nu = \nu^+$. Then $|\Pi I| = \nu^+$. Let $\langle g_\alpha \mid \alpha < \nu^+ \rangle$ be an enumeration of ΠI. By induction on $\alpha < \nu^+$ take $f_\alpha \in \Pi I$ as follows: If α is a successor ordinal, then let $f_\alpha \in \Pi I$ be such that $f_{\alpha-1}, g_{\alpha-1} <^* f_\alpha$. Suppose α is a limit ordinal. If $\mathrm{cf}(\alpha) = \mu$, then let $f_\alpha := \mathrm{csup}_{\beta < \alpha} f_\beta$. Otherwise, take an unbounded $b \subseteq \alpha$ of order-type $\mathrm{cf}(\alpha) < \nu$, and let $f_\alpha := \sup\{f_\beta \mid \beta \in b\}$. Then $\langle f_\alpha \mid \alpha < \nu^+ \rangle$ is a μ-continuous scale in ΠI.

(2) Let V' be a $<\mu$-closed $<\mu^+$-c.c. forcing extension of V. Note that cofinalities and cardinalities are absolute between V and V'. We show that $\vec{f}$ is a μ-continuous scale in ΠI in V'. We work in V'.

First, we check that $\vec{f}$ is a scale. It suffices to show that $\vec{f}$ is $<^*$-cofinal in ΠI. Take an arbitrary $g \in \Pi I$. We find $\alpha < \nu^+$ with $g <^* f_\alpha$. Since V' is a $<\mu^+$-c.c. forcing extension, there is a function $H \in V$ on I such that $g(\iota) \in H(\iota)$ and $|H(\iota)| \leq \mu$ for all $\iota \in I$. In V, define $h \in \Pi I$ as $h(\iota) := \sup(H(\iota))$ for $\iota > \mu$ and $h(\iota) := 0$ for $\iota \leq \mu$. Then $g \leq^* h$. Since $\vec{f}$ is $<^*$-cofinal in ΠI in V, there is $\alpha < \nu^+$ with $h <^* f_\alpha$. Then $g <^* f_\alpha$.

Next, we show that $\vec{f}$ is μ-continous in V'. For this, it suffices to prove that if $\gamma \in \mathrm{Cf}(\mu)$, and $\langle \delta_\alpha \mid \alpha < \gamma \rangle$ is a sequence of ordinals which belongs to V, then $\mathrm{csup}_{\alpha<\gamma} \delta_\alpha$ is absolute between V and V'. Let γ and $\langle \delta_\alpha \mid \alpha < \gamma \rangle$ be as above, and let δ and δ' be $\mathrm{csup}_{\alpha<\gamma} \delta_\alpha$ in V and V', respectively. Clearly, $\delta' \leq \delta$. On the other hand, in V', there is a club $c' \subseteq \gamma$ with $\sup_{\alpha \in c'} \delta_\alpha = \delta'$. Then, using the fact that $\gamma \in \mathrm{Cf}(\mu)$ and V' is a $<\mu$-closed forcing extension of V, in V we can easily take a club $c \subseteq \gamma$ of order-type μ with $\sup_{\alpha \in c} \delta_\alpha \leq \delta'$. So $\delta \leq \delta'$. □

4. Generic supercompactness and SR^*

As we mentioned at the introduction, if a supercompact cardinal is Lévy collapsed to κ^+, then κ^+ is generically supercompact with respect to $<\kappa$-closed forcings. Moreover, this generic supercompactness of κ^+ implies $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}$. In this section, we review this and study consequences of generic supercompactness on cardinal arithmetic. For a systematic and detailed study of generic large cardinals, see Foreman [5].

First, we recall the notion of generic supercompactness. For classes M, N, a class function $j : M \to N$ and ordinals τ, ν with $\tau \leq \nu$, we say that $j : M \to N$ a (τ, ν)-*supercompact embedding* if

(i) M and N are transitive models of ZFC.
(ii) j is an elementary embedding with $\mathrm{crit}(j) = \tau$, where $\mathrm{crit}(j)$ denotes the critical point of j.
(iii) $j(\tau) > \nu$ and $j[\nu] \in N$.

Recall that τ is supercompact if there is a (τ, ν)-supercompact embedding $j : V \to M$ for any $\nu \geq \tau$.

For a regular uncountable cardinals κ and τ with $\kappa < \tau$, we say that τ is *generically supercompact* with respect to $<\kappa$-closed forcings if for any cardinal $\nu \geq \tau$, in some $<\kappa$-closed forcing extension of V, there is a (τ, ν)-supercompact embedding with its domain V.

The following proposition is standard. See Cummings [2, §10] for example. We will later prove more general fact in Proposition 4.3.

Proposition 4.1. *Let κ be a regular cardinal and τ be a supercompact cardinal $> \kappa$. Suppose G is a $\mathrm{Col}(\kappa, {<}\tau)$-generic filter over V. Then, in $V[G]$, $\tau = \kappa^+$ and τ is generically supercompact with respect to ${<}\kappa$-closed forcing.*

As we mentioned at the introduction, $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}$ follows from the generic supercompactness.

Proposition 4.2. *Assume κ^+ is a generic supercompact cardinal with respect to ${<}\kappa$-closed forcings. Then $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}$ holds.*

Proof. In V, let λ be a regular cardinal $> \kappa$ and X be a stationary subset of $\mathcal{P}_\kappa(\mathcal{H}^V_\lambda) \cap \mathrm{IA}$. We show that in V there is $R \subseteq \mathcal{H}^V_\lambda$ such that $|R| = \kappa \subseteq R$, R is i.a. of length κ and $X \cap \mathcal{P}_\kappa(R)$ is stationary.

Let $\nu := |\mathcal{H}^V_\lambda|$ in V. Then, in some ${<}\kappa$-closed forcing extension V' of V, we can take a $((\kappa^+)^V, \nu)$-supercompact embedding $j : V \to M$. Note that $j(\kappa) = \kappa$. By the elementarity of j, it suffices to show that in M there is $R^* \subseteq j(\mathcal{H}^V_\lambda)$ such that $|R^*| = \kappa \subseteq R^*$, R^* is i.a. of length κ and $j(X) \cap \mathcal{P}_\kappa(R^*)$ is stationary.

Let $R^* := j[\mathcal{H}^V_\lambda]$. Then it is easy to see that $R^* \in M$ using the fact that $|\mathcal{H}^V_\lambda| = \nu$ in V and $j[\nu] \in M$. We show that R^* is as desired. Clearly $\kappa \subseteq R^*$. Moreover, $|R^*| \leq \kappa$ in M since $|R^*|^M \leq \nu < j((\kappa^+)^V) = (\kappa^+)^M$.

Note that X remains stationary in $\mathcal{P}_\kappa(\mathcal{H}^V_\lambda)$ in V' by Fact 3.1 (2). Then, $\{j[x] \mid x \in X\}$ is stationary in $\mathcal{P}_\kappa(R^*)$ in V'. But $j[x] = j(x)$ for all $x \in X$ since $|x|^V < \kappa < \mathrm{crit}(j)$. So $\{j[x] \mid x \in X\} = j[X] \subseteq j(X) \cap \mathcal{P}_\kappa(R^*)$. Thus $j(X) \cap \mathcal{P}_\kappa(R^*)$ is stationary in V'. Then so is in M since $M \subseteq V'$.

It remains to show that R^* is i.a. of length κ in M. Recall that $|\mathcal{H}^V_\lambda| = \kappa$ in M. Take a bijection $f : \kappa \to \mathcal{H}^V_\lambda$ in M, and for each $\xi < \kappa$ let $N'_\xi := f[\xi]$. Note that if $\zeta < \kappa$, then $f \restriction \zeta \in V$ since V' is a ${<}\kappa$-closed forcing extension of V, and so $\langle N'_\xi \mid \xi < \zeta \rangle \in \mathcal{H}^V_\lambda$. For each $\xi < \kappa$, let $N_\xi := j[N'_\xi] = j(N_\xi) = j(f)[\xi]$. Note that $\langle N_\xi \mid \xi < \kappa \rangle \in M$ since $j(f) \in M$. Also, $\bigcup_{\xi<\kappa} N_\xi = j[\mathcal{H}^V_\lambda] = R^*$. Moreover, for all $\zeta < \kappa$, $\langle N_\xi \mid \xi < \zeta \rangle = j(\langle N'_\xi \mid \xi < \zeta \rangle) \in j[\mathcal{H}^V_\lambda] = R^*$. So $\langle N_\xi \mid \xi < \kappa \rangle$ is an i.a. sequence to R^* in M. □

In the rest of this section, we study consequences of generic supercompactness on cardinal arithemetic. The next proposition shows that the generic supercompactness of κ^+ with respect to ${<}\kappa$-closed forcings does not give any bound on 2^μ for a regular $\mu \geq \kappa$. In particular, by Proposition 4.2, $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}$ does not give any bound on 2^μ for a regular $\mu \geq \kappa$.

Proposition 4.3. *Let κ be a regular cardinal and τ be a supercompact cardinal $> \kappa$.*

(1) *For any $\rho \in \mathrm{On}$, $\mathrm{Col}(\kappa, {<}\tau) * \mathrm{Add}(\kappa, \rho)$, forces that τ is generically supercompact with respect to ${<}\kappa$-closed forcings.*

(2) *For any* $\mathrm{Col}(\kappa, {<}\tau)$*-name* $\dot{\mathbb{Q}}$ *for a* ${<}\tau$*-directed closed poset,* $\mathrm{Col}(\kappa, {<}\tau) * \dot{\mathbb{Q}}$ *forces that* τ *is generically supercompact with respect to* ${<}\kappa$*-closed forcings. In particular, for any regular* $\mu \geq \tau$ *and any* $\rho \in \mathrm{On}$*,* $\mathrm{Col}(\kappa, {<}\tau) * \mathrm{Add}(\mu, \rho)$ *forces that* τ *is generically supercompact with respect to* ${<}\kappa$*-closed forcings.*

Proof. In the proof below, we will deal with several transitive models, but ${}^{<\kappa}\mathrm{On}$ will be absolute among them. In particular, $\mathrm{Col}(\kappa, *)$ and $\mathrm{Add}(\kappa, *)$ will be absolute among them.

(1) Suppose $G \times H$ is a $\mathrm{Col}(\kappa, <\tau) \times \mathrm{Add}(\kappa, \rho)$-generic filter over V. Take an arbitrary ordinnal $\nu \geq \tau$. We will find a $<\kappa$-closed forcing extension of $V[G][H]$ in which there is a (τ, ν)-supercompact embedding with its domain $V[G][H]$. We may assume $\nu \geq |\mathrm{Add}(\kappa, \rho)|$ in $V[G]$.

In V, take a (τ, ν)-supercompact embedding $j : V \to M$. Then,

$$\begin{aligned} j(\mathrm{Col}(\kappa, {<}\tau) \times \mathrm{Add}(\kappa, \rho)) &= \mathrm{Col}(\kappa, {<}j(\tau)) \times \mathrm{Add}(\kappa, j(\rho)) \\ &\cong \mathrm{Col}(\kappa, {<}\tau) \times \mathrm{Col}(\kappa, j(\tau) \setminus \tau) \times \mathrm{Add}(\kappa, j[\rho]) \times \mathrm{Add}(\kappa, j(\rho) \setminus j[\rho]) \end{aligned}$$

Suppose $G' \times H'$ is a $\mathrm{Col}(\kappa, j(\tau) \setminus \tau) \times \mathrm{Add}(\kappa, j(\rho) \setminus j[\rho])$-generic filter over $V[G][H]$. Then $V[G][H][G'][H']$ is a $<\kappa$-closed forcing extension of $V[G][H]$. We show that in $V[G][H][G'][H']$, $j : V \to M$ can be extended to an elementary embedding from $V[G][H]$.

First, note that $j \restriction \mathrm{Add}(\kappa, \rho) : \mathrm{Add}(\kappa, \rho) \to \mathrm{Add}(\kappa, j[\rho])$ is isomorphic and $j \restriction \mathrm{Add}(\kappa, \rho) \in M$. Then, $G \times j[H]$ is $\mathrm{Col}(\kappa, <\tau) \times \mathrm{Add}(\kappa, j[\rho])$-generic over M, and $G' \times H'$ is $\mathrm{Col}(\kappa, j(\tau) \setminus \tau) \times \mathrm{Add}(\kappa, j(\rho) \setminus j[\rho])$-generic over $M[G][j[H]]$.

Let $\bar{G}$ be the $\mathrm{Col}(\kappa, {<}j(\tau))$-generic filter corresponding to $G \times G'$ and $\bar{H}$ be the $\mathrm{Add}(\kappa, j(\rho))$-generic filter corresponding to $j[H] \times H'$. Then, note that $\bar{G} \times \bar{H}$ is $j(\mathrm{Col}(\kappa, <\tau) \times \mathrm{Add}(\kappa, \rho))$-generic over M, and $j[G \times H] \subseteq \bar{G} \times \bar{H}$. So, by the standard argument, j can be extended to an elementary embedding from $V[G][H]$ to $M[\bar{G}][\bar{H}]$.

(2) Suppose G is a $\mathrm{Col}(\kappa, <\tau)$-generic filter over V. Let $\mathbb{Q} := \dot{\mathbb{Q}}^G$, and suppose H is a $\mathbb{Q}$-generic filter over $V[G]$. Take an arbitrary cardinal $\nu \geq \tau$. We will find a $<\kappa$-closed forcing extension of $V[G][H]$ in which there is a (τ, ν)- supercompact embedding with its domain $V[G][H]$. We may assume $\nu \geq |\mathbb{Q}|$ in $V[G][H]$.

In V, take a (τ, ν)-supercompact embedding $j : V \to M$. Then, in M,

$$\begin{aligned} j(\mathrm{Col}(\kappa, {<}\tau) * \dot{\mathbb{Q}}) &= \mathrm{Col}(\kappa, {<}j(\tau)) * j(\dot{\mathbb{Q}}) \\ &\cong (\mathrm{Col}(\kappa, <\tau) \times \mathrm{Col}(\kappa, j(\tau) \setminus \tau)) * j(\dot{\mathbb{Q}}) \,. \end{aligned}$$

Let I be a $\mathrm{Col}(\kappa, j(\tau) \setminus \tau)$-generic filter over $V[G][H]$. Note that in $V[G]$, $\mathbb{Q} \times \mathrm{Col}(\kappa, j(\tau) \setminus \tau)$ is forcing equivalent to $\mathrm{Col}(\kappa, j(\tau) \setminus \tau)$. Let G' be a $\mathrm{Col}(\kappa, j(\tau) \setminus \tau)$-generic filter over $V[G]$ which corresponds to $H \times I$. Moreover, let $\bar{G}$ be a $\mathrm{Col}(\kappa, <$

$j(\tau))$-generic filter naturally obtained from $G \times G'$. Note that $H \in M[\bar{G}]$ and $V[\bar{G}]$ is a $<\kappa$-closed forcing extension of $V[G][H]$.

SInce $j[G] \subseteq \bar{G}$, $j : V \to M$ can be extended to an elementary embedding $j : V[G] \to M[\bar{G}]$ by the standard argument. Here note that $j(\mathbb{Q})$ is $< j(\kappa)$-directed closed in $M[\bar{G}]$. Note also that $j[H] \in M[\bar{G}]$. Moreover, in $M[\bar{G}]$, $j[H]$ is a directed subset of $j(\mathbb{Q})$ of size $\leq \nu < j(\kappa)$. So we can take a lower bound q^* of $j[H]$. Let $\bar{H}$ be a $j(\mathbb{Q})$-generic filer over $V[\bar{G}]$ containing q^*. Then $j[H] \subseteq \bar{H}$. So, in $V[\bar{G}][\bar{H}]$, $j : V[G] \to M[\bar{G}]$ can be extended to an elementary embedding $j : V[G][H] \to M[\bar{G}][\bar{H}]$.

Note also that $j(\mathbb{Q})$ is $< \kappa$-closed in $V[\bar{G}]$, since so is in $M[\bar{G}]$ and ${}^{<\kappa}\mathrm{On}$ is absolute among $V[\bar{G}]$, V, M and $M[\bar{G}]$. Hence $V[\bar{G}][\bar{H}]$ is a $< \kappa$-closed forcing extension of $V[G][H]$. □

The following proposition is essenttially proved in Matsubara [13].

Proposition 4.4. *Let κ be a regular uncountable cardinal, and suppose κ^+ is generically supercompact with respect to $< \kappa$-closed forcings. Then $\lambda^{<\kappa} = \lambda$ for every regular cardinal $\lambda \geq \kappa$. In particular, $2^{<\kappa} = \kappa$, and* SCH *holds above κ.*

Proof. The latter statement clearly follows from the former, For the former, it suffices to prove the following.

(1) $2^{<\kappa} = \kappa$.

(2) For any regular $\lambda \geq \kappa$, there is a $\subseteq$-cofinal subset of $\mathcal{P}_\kappa(\lambda)$ of size λ.

Let $\tau := \kappa^+$.

(1) For a contradiction, assume $\nu := 2^{<\kappa} > \kappa$. Then we can take a bijection $f : \nu \to {}^{<\kappa}2$.

By the generic supercompactness of τ, in some $<\kappa$-closed forcing extension V' of V, we can define an elementary embedding j from V to a transitive M with critical point τ. Then $j(f)$ is a bijection from $j(\nu)$ to $({}^{<\kappa}2)^M$. Let $A := j(f)(\tau)$. Then $A \in ({}^{<\kappa}2)^M \subseteq ({}^{<\kappa}2)^{V'} = ({}^{<\kappa}2)^V$ since V' is a $<\kappa$-closed forcing extension of V. Then we can take $\alpha < \nu$ with $f(\alpha) = A$ since f is surjective. Then, $j(f)(j(\alpha)) = j(f(\alpha)) = j(A) = A = j(f)(\tau)$. But $j(\alpha) \neq \tau$ since τ is the critical point of j. This contradicts that $j(f)$ is injective.

(2) Suppose λ is a regular cardinal $\geq \kappa$. Take a partition $\langle S_\alpha \mid \alpha < \lambda \rangle$ of $\lambda \cap \mathrm{Cf}(\omega)$ into stationary sets. Let X be the set of all $x \in \mathcal{P}_\kappa(\lambda)$ such that $\mathrm{cf}(x) > \omega$ and

$$x \;=\; \{\alpha < \lambda \mid S_\alpha \cap \sup(x) \text{ is stationary in } \sup(x)\}\,.$$

Note that the mapping $x \mapsto \sup(x)$ is an injection from X to λ. So $|X| \leq \lambda$. We show that X is $\subseteq$-cofinal in $\mathcal{P}_\kappa(\lambda)$. Take an arbitrary $y \in \mathcal{P}_\kappa(\lambda)$. We show that there is $x \in X$ with $x \supseteq y$.

By the generic supercompactness of τ, suppose V' is a $<\kappa$-closed forcing extension of V, and $j : V \to M$ is a (τ, λ)-supercompact embedding definable in V'. By the elementarity of j, it suffices to show that in M there is $x \in j(X)$ with $x \supseteq j(y)$. Below, we work in V'.

Note that $x^* := j[\lambda] \in M$ and $j[\lambda] \in \mathcal{P}_{j(\kappa)}(j(\lambda))$ in M. Note also that $j(y) = j[y] \subseteq x^*$ since $y \in \mathcal{P}_\kappa(\lambda)$ in V, and $\kappa < \mathrm{crit}(j)$. So it suffices to show that $x^* \in j(X)$. Let $j(\langle S_\alpha \mid \alpha < \lambda\rangle) = \langle S'_{\alpha'} \mid \alpha' < j(\lambda)\rangle$. We must show that for $\alpha' < j(\lambda)$, $\alpha' \in j[\lambda]$ if and only if $S'_{\alpha'} \cap \sup(j[\lambda])$ is stationary in M.

First, suppose $\alpha' \in j[\lambda]$. Take $\alpha < \lambda$ with $j(\alpha) = \alpha'$. Note that S_α remains stationary in λ in V' since V' is a $<\kappa$-closed forcing extension of V. Note also that $j \restriction \mathrm{Cf}(\omega)^V$ is continuous, that is, $j(\beta) = \sup\{j(\gamma) \mid \gamma < \beta\}$ for any $\beta \in \mathrm{Cf}(\omega)^V$. So, $j[S_\alpha]$ is stationary in $\sup(j[\lambda])$. But

$$j[S_\alpha] \ \subseteq\ j(S_\alpha) \cap \sup(j[\lambda]) \ =\ S'_{\alpha'} \cap \sup(j[\lambda])\,.$$

Thus $S'_{\alpha'} \cap \sup(j[\lambda])$ is stationary in V', and so is in M since $M \subseteq V'$.

Next, suppose $\alpha' \in j(\lambda) \setminus j[\lambda]$. Let $T' := \bigcup_{\beta' \in j[\lambda]} S'_{\beta'}$. Then $S'_{\alpha'} \cap T' = \emptyset$ since $\langle S'_{\gamma'} \mid \gamma' < j(\lambda)\rangle$ is pairwise disjoint. Let $C' := j[\lambda \cap \mathrm{Cf}(\omega)^V] \in M$. Then $C' \subseteq T'$ since $\lambda \cap \mathrm{Cf}(\omega)^V = \bigcup_{\alpha<\lambda} S_\alpha$. So $S'_{\alpha'} \cap C' = \emptyset$. Note that C' is ω-club in $\sup(j[\lambda])$ in V' since $j \restriction \mathrm{Cf}(\omega)^V$ is continuous, and $({}^\omega\mathrm{On})^{V'} = ({}^\omega\mathrm{On})^V$. Then so is in M since $M \subseteq V'$. Since $S'_{\alpha'} \subseteq \mathrm{Cf}(\omega)^M$, it follows that $S'_{\alpha'} \cap \sup(j[\lambda])$ is non-stationary. □

5. $\mathsf{SR}_\kappa \restriction \mathrm{IA}_\omega$ and λ^ω

In this section, we prove the following:

Theorem 5.1. *Assume κ is a regular uncountable cardinal and $\mathsf{SR}_\kappa \restriction \mathrm{IA}_\omega$ holds. Then $\lambda^\omega = \lambda$ for all regular cardinal $\lambda > \kappa$.*

Theorem 5.1 can be proved by a straightforward generalization of the proof of the fact that SR_{ω_1} implies $\lambda^\omega = \lambda$ for all regular cardinal $\lambda > \omega_1$. Here we give a proof for the completeness of this paper. We will prove the following proposition.

Proposition 5.2. *Assume κ is a regular uncountable cardinal and $\mathsf{SR}_\kappa \restriction \mathrm{IA}_\omega$ holds. Then the following hold.*

1. $(\kappa^+)^\omega = \kappa^+$.
2. *For any singular cardinal ν of cofinality ω, if $\mu^\omega < \nu$ for all $\mu < \nu$, then $\nu^\omega = \nu^+$.*

Using this proposition, Theorem 5.1 can be easily proved by induction on λ. Below, we prove Proposition 5.2. Our proofs of (1) and (2) are based on Veličović [21] and Sakai [16], respectively.

We will use a game, which is a combination of games introduced in Shelah-Shioya [19] and Veličković [20]. Suppose κ is a regular uncountable cardinal, δ is an ordinal $< \kappa$, and $\mathcal{M}$ is a countable expansion of a structure $\langle \mathcal{H}_\lambda, \in, \Delta\rangle$, where λ is a regular cardinal $> \kappa$ and Δ is a well-ordering of $\mathcal{H}_\lambda$. Then let $\Game(\kappa, \mathcal{M}, \delta)$ be the following two players game of length ω:

I	α_0, β_0	α_1, β_1	α_2, β_2	$\cdots$	α_n, β_n	$\cdots$
II	γ_0	γ_1	γ_2	$\cdots$	γ_n	$\cdots$

At the n-th stage, first I chooses α_n, β_n with $\alpha_n < \beta_n < \lambda$, and then II chooses γ_n with $\beta_n < \gamma_n < \lambda$. If $n \geq 1$, then I must choose $\alpha_n > \gamma_n$. After ω stages, let $M := \mathrm{Sk}^{\mathcal{M}}(\delta \cup \{\alpha_n \mid n < \omega\})$. I wins if $M \cap \lambda \subseteq \bigcup_{n<\omega} \beta_n \setminus \alpha_n$ and $M \cap \kappa = \delta$.

Note that I wins if and only if $M_n \cap \gamma_n \subseteq \bigcup_{m \leq n} \beta_m \setminus \alpha_m$ and $M_n \cap \kappa = \delta$ for all $n < \omega$, where $M_n = \mathrm{Sk}^{\mathcal{M}}(\delta \cup \{\alpha_m \mid m \leq n\})$. Thus, $\Game(\kappa, \mathcal{M}, \delta)$ is a closed game for I, and so it is determined. The following lemma is a key.

Lemma 5.3. *Let κ be a regular uncountable cardinal. Suppose λ is a regular cardinal $> \kappa$, Δ is a well-ordering of $\mathcal{H}_\lambda$ and $\mathcal{M}$ is a countable expansion of $\langle \mathcal{H}_\lambda, \in, \Delta \rangle$. Then, there are club many $\delta < \kappa$ such that I has a winning strategy for $\Game(\kappa, \mathcal{M}, \delta)$.*

Proof. Let C be the set of all $\delta < \kappa$ such that I has a winning strategy for $\Game(\kappa, \mathcal{M}, \delta)$. For a contradiction, assume that C does not contain a club subset of κ, that is, $\kappa \setminus C$ is stationary.

For each $\delta \in \kappa \setminus C$, we can take a winning strategy σ_δ of II for $\Game(\kappa, \mathcal{M}, \delta)$, since $\Game(\kappa, \mathcal{M}, \delta)$ is determined. Take a sufficiently large regular cardinal θ, and let $\mathcal{N} := \langle \mathcal{H}_\theta, \in, \mathcal{M}, \langle \sigma_\delta \mid \delta \in \kappa \setminus C \rangle \rangle$. Then, we can take a $\subseteq$-increasing sequence $\langle N_n \mid n < \omega \rangle$ of elementary submodels of $\mathcal{N}$ such that $N_n \in N_{n+1}$ and $N_n \cap \lambda \in \lambda \cap \mathrm{Cf}(\kappa)$ for all $n < \omega$. Note that $\kappa \subseteq \bigcup_{n \in \omega} N_n \prec \mathcal{N}$. Then, since $\kappa \setminus C$ is stationary, we can take $N \prec \mathcal{N}$ of size $< \kappa$ such that

- $\{N_n \mid n \in \omega\} \subseteq N \subseteq \bigcup_{n \in \omega} N_n$
- $\delta^* := N \cap \kappa \in \kappa \setminus C$.

We will find a play $\langle \alpha_n, \beta_n, \gamma_n \mid n < \omega \rangle$ of $\Game(\kappa, \mathcal{M}, \delta^*)$ in which II has moved according to σ_{δ^*}, but I wins. This will contradict that σ_{δ^*} is a winning strategy of II. For $n \in \omega$, define α_n, β_n and γ_n as follows:

- $\alpha_0 := 0$, and $\alpha_n := N_{n-1} \cap \lambda \in \lambda \cap \mathrm{Cf}(\kappa)$ for $n \geq 1$.
- $\beta_n := \sup(N \cap \alpha_{n+1})$.
- $\gamma_n := \sigma_{\delta^*}(\langle \alpha_m, \beta_m \mid m \leq n \rangle)$.

First, we check that $\langle \alpha_n, \beta_n, \gamma_n \mid n < \omega \rangle$ is a legal play of $\Game(\kappa, \mathcal{M}, \delta^*)$ in which II has moved according to σ_{δ^*}. For this, it is enough to check that $\alpha_n < \beta_n < \gamma_n < \alpha_{n+1}$ for all $n < \omega$. First, $\alpha_n < \beta_n$ since $N_{n-1} \in N$. Next, $\beta_n < \gamma_n$ since σ_{δ^*} is a strategy for II. To see that $\gamma_n < \alpha_{n+1}$, note that $\beta_n < \alpha_{n+1}$ since $|N| < \kappa = \mathrm{cf}(\alpha_{n+1})$. So $\langle \alpha_m, \beta_m \mid m \leq n \rangle \in N_n$. Note also that $\sigma_{\delta^*} \in N_n$ since $\kappa \subseteq N_n \prec \mathcal{N}$. Thus $\gamma_n < N_n \cap \lambda = \alpha_{n+1}$.

Next, we check that I wins with the play $\langle \alpha_n, \beta_n, \gamma_n \mid n < \omega \rangle$. For this let M be $\mathrm{Sk}^{\mathcal{M}}(\delta^* \cup \{\alpha_n \mid n < \omega\})$. Then $M \subseteq N \cap \mathcal{H}_\lambda$ since $\delta^* \cup \{\alpha_n \mid n < \omega\} \subseteq N \cap \mathcal{H}_\lambda \prec \mathcal{M}$.

So $M \cap \kappa = \delta^*$ since $N \cap \kappa = \delta^* \subseteq M$. Note also that $N \cap \lambda \subseteq \bigcup_{n<\omega} \beta_n \setminus \alpha_n$ by the construction of $\langle \alpha_n, \beta_n \mid n < \omega \rangle$ and the fact that $N \subseteq \bigcup_{n\in\omega} N_n$. So $M \cap \lambda \subseteq \bigcup_{n<\omega} \beta_n \setminus \alpha_n$. Hence I wins with $\langle \alpha_n, \beta_n, \gamma_n \mid n < \omega \rangle$. □

We also note that if $\mathrm{cf}(\delta) = \omega$, then the resulting model M in $\eth(\kappa, \mathcal{M}, \delta)$ is i.a. of length ω.

Lemma 5.4. *Let κ be a regular uncountable cardinal. Suppose λ is a regular cardinal $> \kappa$, Δ is a well-ordering of $\mathcal{H}_\lambda$ and $\mathcal{M}$ is a countable expansion of $\langle \mathcal{H}_\lambda, \in, \Delta \rangle$. Then, for any $\delta \in \kappa \cap \mathrm{Cf}(\omega)$ and any $\{\alpha_n \mid n < \omega\} \subseteq \lambda$, it holds that $\mathrm{Sk}^{\mathcal{M}}(\delta \cup \{\alpha_n \mid n < \omega\})$ is i.a. of length ω.*

Proof. Take $\delta \in \kappa \cap \mathrm{Cf}(\omega)$ and $\{\alpha_n \mid n < \omega\} \subseteq \lambda$ arbitrarily. We prove that $M := \mathrm{Sk}^{\mathcal{M}}(\delta \cup \{\alpha_n \mid n < \omega\})$ is i.a. of length ω.

Take an increasing sequence $\langle \delta_n \mid n < \omega \rangle$ converging to δ and an enumeration $\langle \varphi_n(u, v_1, \ldots, v_{k_n}) \mid n < \omega \rangle$ of all formulas of the language of $\mathcal{M}$ in which u is a free variable. For each $n < \omega$, let $h_n : {}^{k_n}\mathcal{H}_\lambda \to \mathcal{H}_\lambda$ be the Skolem function of φ_n in $\mathcal{M}$, that is, $h_n(b_1, \ldots, b_{k_n})$ is the Δ-least a with $\mathcal{M} \models \varphi_n(a, b_1, \ldots, b_{k_n})$ if such a exists, and $h_n(b_1, \ldots, b_{k_n})$ is undefined otherwise. Note the following:

(i) Each h_n is definable over $\mathcal{M}$.
(ii) $M = \bigcup_{n<\omega} h_n \left[{}^{k_n}(\delta \cup \{\alpha_l \mid l < \omega\})\right]$.

For each $n < \omega$, let

$$M_n := \bigcup_{m \le n} h_m \left[{}^{k_m}(\delta_n \cup \{\alpha_l \mid l < n\})\right] .$$

Clearly, $\langle M_n \mid n < \omega \rangle$ is $\subseteq$-increasing. Moreover $\bigcup_{n<\omega} M_n = M$ by (ii). Note also that each M_n belongs to M since $M \prec \mathcal{M}$, and M_n is definable in $\mathcal{M}$ from parameters $\delta_n, \alpha_0, \ldots, \alpha_{n-1} \in M$. Hence $\langle M_n \mid n < \omega \rangle$ witnesses that M is i.a. of length ω. □

By Lemma 5.3 and 5.4, we obtain the next lemma.

Lemma 5.5. *Suppose κ and λ are regular uncountable cardinals with $\kappa < \lambda$, and Z is a club subset of $\mathcal{P}_\kappa(\mathcal{H}_\lambda)$. Then, there are a club $C \subseteq \lambda$ and $\delta \in \kappa \cap \mathrm{Cf}(\omega)$ such that for any strictly increasing sequence $\langle \gamma_n \mid n < \omega \rangle$ of elements of C there is $M \in Z \cap \mathrm{IA}_\omega$ with the following properties.*

(i) *$M \cap \kappa = \delta$.*
(ii) *$\{\gamma \in C \mid M \cap (\mathrm{succ}_C(\gamma) \setminus \gamma) \neq \emptyset\} = \{\gamma_n \mid n < \omega\}$.*

In particular, $|\{M \cap \lambda \mid M \in Z \cap \mathrm{IA}_\omega\}| \ge \lambda^\omega$.

Proof. First note that the latter statement follows from the former, since there are λ^ω many strictly increasing sequences $\langle \gamma_n \mid n < \omega \rangle$ of elements of C. Below, we prove the former statement.

First, we define C and δ. Take a well-ordering Δ of $\mathcal{H}_\lambda$ and a countable expansion $\mathcal{M}$ of $\langle \mathcal{H}_\lambda, \in, \Delta \rangle$ such that for any $M \in \mathcal{P}_\kappa(\mathcal{H}_\lambda)$, if $M \prec \mathcal{M}$ and $M \cap \kappa \in \kappa$, then $M \in Z$. By Lemma 5.3, we can take $\delta \in \kappa \cap \mathrm{Cf}(\omega)$ such that I has a winning strategy σ for $\Game(\kappa, \mathcal{M}, \delta)$. Let C be the set of all $\gamma < \lambda$ which is closed under σ, that is, for any $s \in {}^{<\omega}\gamma$, if $\langle \alpha, \beta \rangle = \sigma(s)$, then $\alpha, \beta < \gamma$. Note that C is club in λ.

To see that C and δ are as desired, suppose $\langle \gamma_n \mid n < \omega \rangle$ is a strictly increasing sequence of elements of C. Then, let $\langle \alpha_n, \beta_n \rangle := \sigma(\langle \gamma_m \mid m < n \rangle)$ for each $n < \omega$, and let $M := \mathrm{Sk}^{\mathcal{M}}(\delta \cup \{\alpha_n \mid n < \omega\})$. We check that M belongs to $Z \cap \mathrm{IA}_\omega$ and satisfies (i),(ii).

First, since each $\gamma \in C$ is closed under σ, we have the following.

(iii) $\alpha_0 < \beta_0 < \min(C) \leq \gamma_0$, and $\gamma_n < \alpha_{n+1} < \beta_{n+1} < \mathrm{succ}_C(\gamma_n) \leq \gamma_{n+1}$ for all $n < \omega$,

In particular, $\langle \alpha_n, \beta_n, \gamma_n \mid n < \omega \rangle$ is a play of $\Game(\kappa, \mathcal{M}, \delta)$ in which I has moved according to a winning strategy σ. So M satisfies (i) and the following.

(iv) $M \cap \lambda \subseteq \bigcup_{n<\omega} \beta_n \setminus \alpha_n$.

Then, $M \prec \mathcal{M}$, and $M \cap \kappa \in \kappa$ by (i). So $M \in Z$ by the choice of $\mathcal{M}$. Moreover, $M \in \mathrm{IA}_\omega$ by Lemma 5.4. Finally, (ii) follows from (iii) and (iv). □

Now, we can easily prove Proposition 5.2 (1).

Proof of Proposition 5.2 (1). Assume $\mathsf{SR}_\kappa \restriction \mathrm{IA}_\omega$. Let $\mathcal{R}$ be the set of all $R \subseteq \mathcal{H}_{\kappa^+}$ such that $|R| = \kappa \subseteq R$ and $R \cap \kappa^+ \in \kappa^+$.

First, we prove the following:

(i) For any stationary $X \subseteq \mathcal{P}_\kappa(\mathcal{H}_{\kappa^+}) \cap \mathrm{IA}_\omega$, there is $R \in \mathcal{R}$ with $X \cap \mathcal{P}_\kappa(R)$ is stationary.

Suppose X is a stationary subset of $\mathcal{P}_\kappa(\mathcal{H}_{\kappa^+}) \cap \mathrm{IA}_\omega$. By shrinking X if necessary, we may assume that $M \prec \langle \mathcal{H}_{\kappa^+}, \in \rangle$ for all $M \in X$. By $\mathsf{SR}_\kappa \restriction \mathrm{IA}_\omega$, we can take $R \subseteq \mathcal{H}_{\kappa^+}$ such that $|R| = \kappa \subseteq R$ and $X \cap \mathcal{P}_\kappa(R)$ is stationary. It suffices to prove that $R \cap \kappa^+ \in \kappa^+$. Since $X \cap \mathcal{P}_\kappa(R)$ is stationary, it follows that $R \prec \langle \mathcal{H}_{\kappa^+}, \in \rangle$. Then, from the fact that $\kappa \subseteq R$, it follows that $R \cap \kappa^+ \in \kappa^+$.

For each $\gamma \in \kappa^+ \setminus \kappa$, we can take a club $Y_\gamma \subseteq \mathcal{P}_\kappa(\gamma)$ of size κ since $|\gamma| = \kappa$. Note that for any $R \in \mathcal{R}$, there are non-stationary many $M \in \mathcal{P}_\kappa(R)$ with $M \cap \kappa^+ \notin Y_{R \cap \kappa^+}$. Let $Y := \bigcup_{\gamma \in \kappa^+ \setminus \kappa} Y_\gamma$. Then, we have the following.

(ii) $|Y| = \kappa^+$.

(iii) For any $R \in \mathcal{R}$, the set $\{M \in \mathcal{P}_\kappa(R) \mid M \cap \kappa^+ \notin Y\}$ is non-stationary.

By (i) and (iii), the set $\{M \in \mathcal{P}_\kappa(\mathcal{H}_{\kappa^+}) \cap \mathrm{IA}_\omega \mid M \cap \kappa^+ \notin Y\}$ is non-stationary, that is, there is a club $Z \subseteq \mathcal{P}_\kappa(\mathcal{H}_{\kappa^+})$ such that $\{M \cap \kappa^+ \mid M \in Z \cap \mathrm{IA}_\omega\} \subseteq Y$. Then, $(\kappa^+)^\omega \leq \kappa^+$ by (ii) and Lemma 5.5. □

Next, we prove Proposition 5.2 (2). We use the following technical lemma.

Lemma 5.6. *Let ν be a singular cardinal of cofinality ω such that $\mu^\omega < \nu$ for all $\mu < \nu$, and $\nu^\omega > \nu^+$. Suppose $\mathcal{A} \subseteq \mathcal{P}_\nu(\nu^+)$ and $|\mathcal{A}| \leq \nu^+$. Let C be a club subset of ν^+ and S be a stationary subset of $\nu^+ \cap \mathrm{Cf}(\omega)$. Then there is a strictly increasing sequence $\langle \gamma_n \mid n < \omega \rangle$ of elements of C with the following properties.*

(i) $\sup_{n<\omega} \gamma_n \in S$.
(ii) $A \cap \bigcup_{n<\omega} \mathrm{suc}_C(\gamma_n) \setminus \gamma_n$ *is bounded in* $\sup_{n<\omega} \gamma_n$ *for any* $A \in \mathcal{A}$.

Proof. Let D be the set of all $\delta \in C$ such that for any $\delta' < \delta$ there is $\delta'' < \delta$ with $|C \cap (\delta'' \setminus \delta')| = \nu$. Note that D is club in ν^+. Take $\delta \in D \cap S$. We will find $\langle \gamma_n \mid n < \omega \rangle$ satisfying (ii) such that $\sup_{n<\omega} \gamma_n = \delta$.

Take a strictly increasing sequence $\langle \delta_n \mid n < \omega \rangle$ in C converging to δ such that $\delta_0 = 0$ and $|C \cap (\delta_{n+1} \setminus \delta_n)| = \nu$ for all $n < \omega$. Moreover, take a bijection $\pi : C \cap \delta \to ({}^{<\omega}\nu \setminus \{\emptyset\})$ such that $\pi[C \cap (\delta_{n+1} \setminus \delta_n)] = {}^{n+1}\nu$.

For $\alpha < \delta$, let $\tau(\alpha) := \max(C \cap (\alpha + 1))$. Note that $\alpha \in \mathrm{suc}_C(\tau(\alpha)) \setminus \tau(\alpha)$. Then, for each $A \in \mathcal{A}$, let B_A be the set of all $b \in {}^\omega\nu$ such that there is a countable $x \subseteq A \cap \delta$ with $b = \bigcup_{\alpha \in x} \pi(\tau(\alpha))$. Note that $|B_A| \leq |A|^\omega < \nu$, since $|A| < \nu$, and $\mu^\omega < \nu$ for all $\mu < \nu$. Let $B := \bigcup_{A \in \mathcal{A}} B_A$. Then $|B| \leq \nu^+$ since $|\mathcal{A}| \leq \nu^+$.

Since $\nu^\omega > \nu^+$, we can take $b^* \in {}^\omega\nu \setminus B$. For $n < \omega$, take $\gamma_n \in C \cap (\delta_{n+1} \setminus \delta_n)$ with $\pi(\gamma_n) = b^* \restriction (n+1)$. Clearly, $\langle \gamma_n \mid n < \omega \rangle$ is strictly increasing, and $\sup_{n<\omega} \gamma_n = \delta \in S$. We check (ii). For a contradiction, $A \cap \bigcup_{n<\omega} \mathrm{suc}_C(\gamma_n) \setminus \gamma_n$ is unbounded in δ for some $A \in \mathcal{A}$. Take a countable cofinal $x \subseteq A \cap \bigcup_{n<\omega} \mathrm{suc}_C(\gamma_n)) \setminus \gamma_n$. Since $\tau(\alpha) = \gamma_n$ for $\alpha \in \mathrm{suc}_C(\gamma_n) \setminus \gamma_n$, we have that $b^* = \bigcup_{\alpha \in x} \pi(\tau(\alpha)) \in B_A \subseteq B$. This contradicts that $b^* \notin B$. □

Now, we prove Proposition 5.2 (2).

Proof of Proposition 5.2 (2). We prove the contraposition. Let κ be a regular uncountable cardinal and ν be a singular cardinal $> \kappa$ of cofinality ω. Suppose $\mu^\omega < \nu$ for all cardinals $\mu < \nu$, and $\nu^\omega > \nu^+$. We prove that $\mathsf{SR}_\kappa \restriction \mathrm{IA}_\omega$ fails. Below, let $\lambda := \nu^+$.

For each $\alpha \in \lambda \cap \mathrm{Cf}(> \omega)$, take a $\subseteq$-increasing sequence $\langle A_{\alpha,n} \mid n < \omega \rangle$ such that $\bigcup_{n<\omega} A_{\alpha,m} = \alpha$ and $|A_{\alpha,m}| < \nu$ for all $m < \omega$. Take a partition $\langle S_\delta \mid \delta < \kappa \rangle$ of $\lambda \cap \mathrm{Cf}(\omega)$ such that each S_δ is stationary in λ.

Then, let X be the set of all $M \in \mathcal{P}_\kappa(\mathcal{H}_{\lambda^+}) \cap \mathrm{IA}_\omega$ such that

(i) $M \cap \kappa \in \kappa$.
(ii) $\sup(M \cap \lambda) \in S_{M \cap \kappa} \setminus M$.
(iii) $M \cap A_{\alpha,m}$ is bounded in $\sup(M \cap \lambda)$ for any $\alpha \in \lambda \cap \mathrm{Cf}(> \omega)$ and any $m < \omega$.

It suffices to prove that X is stationary in $\mathcal{P}_\kappa(\mathcal{H}_\lambda)$ and that X is non-reflecting, i.e. $X \cap \mathcal{P}_\kappa(R)$ is non-stationary for any $R \subseteq \mathcal{H}_\lambda$ with $|R| = \kappa \subseteq R$.

First, we prove that X is stationary. Take an arbitrary club $Z \subseteq \mathcal{P}_\kappa(\mathcal{H}_\lambda)$. We find $M \in X \cap Z$.

Let C and δ be as in Lemma 5.5. By Lemma 5.6, take a strictly increasing sequence $\langle \gamma_n \mid n < \omega \rangle$ such that $\sup_{n<\omega} \gamma_n \in S_\delta$ and $A_{\alpha,m} \cap \bigcup_{n<\omega} \mathrm{suc}_C(\gamma_n) \setminus \gamma_n$ is bounded in $\sup_{n<\omega} \gamma_n$ for all $\alpha \in \lambda \cap \mathrm{Cf}(> \omega)$ and all $m < \omega$. Let M be the one as in Lemma 5.5 for this $\langle \gamma_n \mid n < \omega \rangle$. Then, it is easy to check that $M \in X \cap Z$.

Next, we prove that X is non-reflecting. Take an arbitrary $R \subseteq \mathcal{H}_\lambda$ with $|R| = \kappa \subseteq R$. We show that $X \cap \mathcal{P}_\kappa(R)$ is non-stationary.

Let $\alpha := \sup(R \cap \lambda)$. If $\alpha \in R$, then there is no $M \in X \cap \mathcal{P}_\kappa(R)$ with $\alpha \in M$, since $\sup(M \cap \lambda) \notin M$ for any $M \in X$ by (ii). So $X \cap \mathcal{P}_\kappa(R)$ is non-stationary in this case. Thus, we assume that $\alpha \notin R$. The rest of the proof splits into two cases according to $\mathrm{cf}(\alpha)$.

First, suppose $\mathrm{cf}(\alpha) = \omega$. Take $\delta < \kappa$ with $\alpha \in S_\delta$. Let Y_0 be the set of all $M \in \mathcal{P}_\kappa(R)$ with $\sup(M \cap \lambda) = \alpha$ and $\delta < M \cap \kappa \in \kappa$. Then it is easy to see that Y_0 is club in $\mathcal{P}_\kappa(R)$. But $X \cap Y_0 = \emptyset$ by the property (ii) of elements of X. So $X \cap \mathcal{P}_\kappa(R)$ is non-stationary.

Next, suppose $\mathrm{cf}(\alpha) > \omega$. Then, we can take $m < \omega$ with $\sup(R \cap A_{\alpha,m}) = \alpha$. Let Y_1 be the set of all $M \in \mathcal{P}_\kappa(R)$ such that $M \cap A_{\alpha,m}$ is unbounded in $\sup(M \cap \lambda)$. Then, it is easy to see that Y_1 is club in $\mathcal{P}_\kappa(R)$. But $X \cap Y_1 = \emptyset$ by the property (iii) of elements of X. Hence $X \cap \mathcal{P}_\kappa(R)$ is non-stationary. □

6. $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}$ and 2^μ for $\mu < \kappa$

In this section, we show that $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}$ does not give any bound on 2^μ for a regular uncountable cardinal $\mu < \kappa$. We also prove that $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ does not give any bound on 2^ω. More precisely, we prove the following.

Theorem 6.1. *Assume* GCH*. Let μ and κ be regular cardinals with $\mu < \kappa$ and $\kappa \in I[\kappa]$. Also, let $\mathbb{P}$ be a $< \mu$-closed poset with the $< \mu^+$-c.c. and the $< \mu^+$-approximation property. Suppose G is a $\mathbb{P}$-generic filter over V.*

(1) *If $\mu = \omega$, and $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ holds in V, then $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ holds also in $V[G]$.*
(2) *If $\mu > \omega$, and $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}$ holds in V, then $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}$ holds also in $V[G]$.*

Corollary 6.2. *Assume* GCH*. Let κ be a regular uncountable cardinal, τ be a supercompact cardinal $> \kappa$ and ρ be an ordinal.*

(1) *There is a forcing extension V^* of V in which the following hold.*
    (i) *If δ is a regular cardinal in V with $\delta \leq \kappa$ or $\delta \geq \tau$, then δ remains a regular cardinal.*
    (ii) $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ *holds, and $2^\omega \geq \rho$.*
(2) *Suppose μ is a regular uncountable cardinal $< \kappa$. Then there is a forcing extension V^{**} of V in which (i) above and the following holds.*
    (iii) $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}$ *holds, and $2^\mu \geq \rho$.*

Proof of Corollary 6.2 from Theorem 6.1. Let V' be a forcing extension of V by $\mathbb{I}_\kappa$ in §3. Then $\kappa \in I[\kappa]$ in V' by Lemma 3.5. Moreover, in V', GCH holds, all regular cardinals in V remain regular cardinals, and τ remains supercompact in V' by Lemma 3.4 and the fact that $|\mathbb{I}_\kappa| = \kappa$ in V.

Next, let V'' be a forcing extension of V' by $\mathrm{Col}(\kappa, {<}\tau)$. Then, (i) of the corollary holds in V''. Moreover, in V'', $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}$ holds in V'' by Proposition 4.1 and 4.2. Note also that GCH holds and $\kappa \in I[\kappa]$ in V''.

Now, let V^* be a forcing extension of V'' by $\mathrm{Add}(\omega, \rho)$. Then V^* witnesses (1) of the corollary by Theorem 6.1 (1) and Fact 2.4. For (2), let V^{**} be a forcing extension of V'' by $\mathrm{Add}(\mu, \rho)$. Then V^{**} witnesses (2) of the corollary by Theorem 6.1 (2) and Fact 2.4. □

Below, we prove Theorem 6.1. A difficulty to prove this lies in that a forcing by $\mathbb{P}$ adds many new sets of cardinality $< \kappa$, and $\mathcal{P}_\kappa(H)$ are not absolute between V and $V^{\mathbb{P}}$ for a set $H \in V$. In fact, Gitik [9] proved that if $\mu = \omega$, and $\mathbb{P}$ adds a real, then $X := \mathcal{P}_\kappa(\kappa^+)^{V^{\mathbb{P}}} \setminus \mathcal{P}_\kappa(\kappa^+)^V$ is stationary. Also, $X \cap \mathcal{P}_\kappa(\alpha)$ is non-stationary for any $\alpha < \kappa^+$ since $\mathcal{P}_\kappa(\alpha)^V$ contains a club set in $V^{\mathbb{P}}$.

We use the following proposition, which gets rid of this difficulty.

Proposition 6.3. *In V, assume GCH, let μ, κ and λ be regular cardinals with $\mu < \kappa < \lambda$, and let $\mathbb{P}$ be a $< \mu$-closed poset with the $< \mu^+$-c.c. and the $< \mu^+$-approximation property, which belongs to $\mathcal{H}^V_\lambda$. Suppose G is a $\mathbb{P}$-generic filter over V. Then, we have (1), (2) and (3) below in $V[G]$, where $\Phi(\zeta)$ is the following statement for a regular cardinal $\zeta < \kappa$.*

> *There is a club $Z \subseteq \mathcal{P}_\kappa(\mathcal{H}^{V[G]}_\lambda)$ such that if $M \in Z$, and M is i.a. of length ζ, then $N := M \cap \mathcal{H}^V_\lambda \in V$, N is i.a. of length ζ in V, and $M = N[G]$.*

(1) *$\Phi(\zeta)$ holds for every regular cardinal $\zeta < \mu$.*
(2) *$\Phi(\zeta)$ holds for every regular cardinal ζ with $\mu < \zeta < \kappa$.*
(3) *If $\mu > \omega$, then $\Phi(\mu)$ holds.*

First, we prove Theorem 6.1 using Proposition 6.3.

Proof of Theorem 6.1 from Proposition 6.3. We only prove (2). The proof of (1) is similar as (2) and left to the readers. Working in $V[G]$, we prove the contraposition of (2).

Assume $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}$ fails in $V[G]$. Then, we can take a regular cardinal $\lambda > \kappa$ and a stationary $X \subseteq \mathcal{H}^{V[G]}_\lambda \cap \mathrm{IA}$ such that for any $R \subseteq \mathcal{H}^{V[G]}_\lambda$, if $|R| = \kappa \subseteq R$ and R is i.a. of length κ, then $X \cap \mathcal{P}_\kappa(R)$ is non-stationary.

We may also assume $\mathbb{P} \in \mathcal{H}^V_\lambda$ by Lemma 3.2. Here note that $\kappa \in I[\kappa]$ in $V[G]$ since it holds in V, and cofinalities are absolute between V and $V[G]$. So we may assume that every $M \in X$ is i.a. of regular length by Lemma 3.3. Then, by Proposition 6.3, we may also assume that if $M \in X$, then $N := M \cap \mathcal{H}^V_\lambda \in V$, N is i.a. in V and $M = N[G]$.

In V, let $\mathcal{Q}$ be the set of all $Q \subseteq \mathcal{H}_\lambda^V$ such that $|Q| = \kappa \subseteq Q \prec \langle \mathcal{H}_\lambda^V, \in, \kappa, \mathbb{P}\rangle$ and Q is i.a. of length κ. We claim the following.

Claim. *There is $\langle Z_Q \mid Q \in \mathcal{Q}\rangle \in V$ such that the following hold for all $Q \in \mathcal{Q}$.*

(i) *Z_Q is club in $\mathcal{P}_\kappa(Q)$ in V.*

(ii) *$N[G] \notin X$ for any $N \in Z_Q$.*

Proof of Claim. First note that for any $Q \in \mathcal{Q}$, in $V[G]$, $|Q[G]| = \kappa \subseteq Q[G]$ and $Q[G]$ is i.a. of length κ. The latter is because if $\langle Q_\xi \mid \xi < \kappa\rangle$ is an i.a. sequence to Q in V, then $\langle Q_\xi[G] \mid \xi < \kappa\rangle$ is an i.a. sequence to $Q[G]$ in $V[G]$. So, for each $Q \in \mathcal{Q}$, $X \cap \mathcal{P}_\kappa(Q[G])$ is non-stationary in $V[G]$. Then, in $V[G]$, for each $Q \in \mathcal{Q}$, let Z''_Q be a club subset of $\mathcal{P}_\kappa(Q[G])$ with $Z''_Q \cap X = \emptyset$, and let Z'_Q be the set of all $N \in \mathcal{P}_\kappa(Q)$ with $N[G] \in Z''_Q$. Then it is easy to see that Z'_Q is club in $\mathcal{P}_\kappa(Q)$ in $V[G]$, and $N[G] \notin X$ for any $N \in Z'_Q$.

Take a sequence $\langle \dot{Z}'_Q \mid Q \in \mathcal{Q}\rangle \in V$ of $\mathbb{P}$-names such that $(\dot{Z}'_Q)^G = Z'_Q$ for all $Q \in \mathcal{Q}$. Also, take $p \in G$ which forces that $\dot{Z}'_Q$ is club in $\mathcal{P}_\kappa(Q)$ for all $Q \in \mathcal{Q}$. In V, for each $Q \in \mathcal{Q}$, let Z_Q be the set of all $N \in \mathcal{P}_\kappa(Q)$ such that $p \Vdash_{\mathbb{P}}$ " $N \in \dot{Z}'_Q$ ". Then $\langle Z_Q \mid Q \in \mathcal{Q}\rangle \in V$. Since $\mathbb{P}$ has the κ-c.c., each Z_Q is club in $\mathcal{P}_\kappa(Q)$ in V by Lemma 2.3. Moreover, $Z_Q \subseteq Z'_Q$ for all $Q \in \mathcal{Q}$. Therefore $\langle Z_Q \mid Q \in \mathcal{Q}\rangle$ is as desired. □ (Claim)

In V, let $Z := \bigcup_{Q \in \mathcal{Q}} Z_Q$ and $Y := (\mathcal{P}_\kappa(\mathcal{H}_\lambda^V) \cap \mathrm{IA}) \setminus Z$. If Y is stationary in $\mathcal{P}_\kappa(\mathcal{H}_\lambda^V)$ in V, then so is the set $Y' = \{N \in Y \mid N \prec \langle \mathcal{H}_\lambda^V, \in, \kappa, \mathbb{P}\rangle\}$. Moreover, $Y' \cap \mathcal{P}_\kappa(Q)$ is non-stationary for any i.a. set Q of length κ. (If $Q \prec \langle \mathcal{H}_\lambda^V, \in, \kappa, \mathbb{P}\rangle$, then $Q \in \mathcal{Q}$, and so $(Y' \cap \mathcal{P}_\kappa(Q)) \cap Z_Q = \emptyset$ by the definition of Y. Otherwise, $Y' \cap \mathcal{P}_\kappa(Q)$ is non-stationary since $N \prec \langle \mathcal{H}_\lambda^V, \in, \kappa, \mathbb{P}\rangle$ for all $N \in Y'$.) So Y' will be a counterexample of $\mathsf{SR}_\kappa^* \restriction \mathrm{IA}$ in V if Y is stationary. Thus it suffices to prove that Y is stationary in V.

In V, take an arbitrary $F : {}^{<\omega}\mathcal{H}_\lambda^V \to \mathcal{H}_\lambda^V$. It suffices to find $N \in Y$ such that $N \cap \kappa \in \kappa$ and N is closed under F. Since X is stationary in $V[G]$, we can take $M \in X$ such that $M \cap \kappa \in \kappa$ and M is closed under F. Let $N := M \cap \mathcal{H}_\lambda^V$. Note that $N \in V$, $N \in \mathrm{IA}$ in V and $M = N[G]$ by the assumption on X. Then, in V, $N \in \mathcal{P}_\kappa(\mathcal{H}_\lambda^V) \cap \mathrm{IA}$, and $N \notin Z$ by (ii) of Claim and the fact that $N[G] = M \in X$. So $N \in Y$. Moreover, Y is closed under F since so is M. Thus N is as desired. □

Below, we prove Proposition 6.3. First we prove (1).

Proof of Proposition 6.3 (1). We work in $V[G]$. Let ζ be a regular cardinal $< \mu$.

Take a well-ordering Δ of $\mathcal{H}_\lambda^{V[G]}$, and let $\mathcal{M} := \langle \mathcal{H}_\lambda^{V[G]}, \in, \Delta, \mathcal{H}_\lambda^V, \mu, \kappa, \mathbb{P}, G\rangle$. Then let Z be the set of all $M \in \mathcal{P}_\kappa(\mathcal{H}_\lambda^{V[G]})$ such that $M \prec \mathcal{M}$ and $M \cap \kappa \in \kappa$. Note that Z is club in $\mathcal{P}_\kappa(\mathcal{H}_\lambda^{V[G]})$. We show that Z witnesses $\Phi(\zeta)$.

Suppose $M \in Z$ and M is i.a. of length ζ. Let $N := M \cap \mathcal{H}_\lambda^V$. Note that for any $\mathbb{P}$-name $\dot{a} \in N$, we have $\dot{a}^G \in M$ since $M \prec \mathcal{M}$. Note also that for any $a \in M$, there is a $\mathbb{P}$-name $\dot{a} \in N$ with $\dot{a}^G = a$. So $M = N[G]$. We prove that $N \in V$ and N is i.a. of length ζ in V.

Let $\langle M_\xi \mid \xi < \zeta \rangle$ be an i.a. sequence to M. For each $\xi < \zeta$, by the $<\mu^+$-c.c. of $\mathbb{P}$, let N'_ξ be the Δ-least element of $\mathcal{P}_\kappa(\mathcal{H}_\lambda^V)^V$ such that $M_\xi \cap \mathcal{H}_\lambda^V \subseteq N'_\xi$. Note that $\langle N'_\xi \mid \xi < \zeta' \rangle \in M$ for all $\zeta' < \zeta$. Note also that $N'_\xi \subseteq M$ for all $\xi < \zeta$ since $N'_\xi \in M$ and $|N'_\xi| < \kappa$. So $N = \bigcup_{\xi<\zeta} N'_\xi$. But $\langle N'_\xi \mid \xi < \zeta \rangle \in V$ by the $<\mu$-closure of $\mathbb{P}$. So $N = \bigcup_{\xi<\zeta} N'_\xi \in V$.

For each $\xi < \zeta$, let $N_\xi := \bigcup_{\eta<\xi} N'_\eta$. Then $\langle N_\xi \mid \xi < \zeta \rangle \in V$. Moreover, $\langle N_\xi \mid \xi < \zeta \rangle$ is a $\subseteq$-increasing, $\bigcup_{\xi<\zeta} N_\xi = N$, and $\langle N_\xi \mid \xi < \zeta' \rangle \in M \cap \mathcal{H}_\lambda^V = N$ for all $\zeta' < \zeta$. Hence N is i.a. of length ζ in V. □

To prove Proposition 6.3 (2) and (3), first we reduce them to the following lemma.

Lemma 6.4. *In V, assume* GCH, *let μ and λ be regular cardinals with $\mu \leq \zeta < \lambda$, and let $\mathbb{P}$ be a $<\mu$-closed poset with the $<\mu^+$-c.c. and the $<\mu^+$-approximation property, which belongs to $\mathcal{H}_\lambda^V$. Suppose G is a $\mathbb{P}$-generic filter over V. Then, we have (1) and (2) below in $V[G]$, where $\Psi(\zeta)$ is the following statement for a regular cardinal ζ with $\mu \leq \zeta < \lambda$.*

> *There is a club $Z \subseteq \mathcal{P}_{\zeta^+}(\mathcal{H}_\lambda^{V[G]})$ such that if $M \in Z$, and M is i.a. of length ζ, then $N := M \cap \mathcal{H}_\lambda^V \in V$.*

(1) *$\Psi(\zeta)$ holds for every regular cardinal ζ with $\mu < \zeta < \lambda$.*
(2) *If $\mu > \omega$, then $\Psi(\mu)$ holds.*

First, we prove Proposition 6.3 (2) and (3) using Lemma 6.4. For this, we use the following lemma.

Lemma 6.5. *In V, assume* GCH. *Let μ, λ, $\mathbb{P}$ and G be as in Lemma 6.4, and let ζ be a regular cardinal with $\mu \leq \zeta < \lambda$. In $V[G]$, suppose $M \prec \langle \mathcal{H}_\lambda^{V[G]}, \in, \mathcal{H}_\lambda^V, \mu, \mathbb{P}, G \rangle$, $|M| = \zeta$ and M is i.a. of length ζ. Let $N := M \cap \mathcal{H}_\lambda^V$. Then, $({}^{<\zeta}N) \cap V \subseteq N$.*

Proof. Take an arbitrary $f \in ({}^{<\zeta}N) \cap V = ({}^{<\zeta}N) \cap \mathcal{H}_\lambda^V$. It suffices to prove $f \in M$. Let $\eta := \mathrm{dom}(f)$.

Let $\langle M_\xi \mid \xi < \zeta \rangle$ be an i.a. sequence to M. By the regularity of ζ, we can take $\xi < \zeta$ with $\mathrm{ran}(f) \subseteq M_\xi$. Since $\mathbb{P}$ has the $<\mu^+$-c.c., we can take $K \in \mathcal{H}_\lambda^V$ such that $M_\xi \subseteq K$ and $|K| = \max\{\mu, |M_\xi|\}$. By the elementarity of M, we can take such $K \in M$. Note that $\zeta \subseteq M$. So $({}^\eta K) \cap V = ({}^\eta K) \cap \mathcal{H}_\lambda^V \in M$. Note also that $|({}^\eta K) \cap V| \leq \zeta$ since $|K| \leq \zeta$, and GCH holds in V. Thus $({}^\eta K) \cap V \subseteq M$. Since $f \in ({}^\eta K) \cap V$, we have $f \in M$. □

Proof of Proposition 6.3 (2) and (3) from Lemma 6.4. Let μ, κ, λ, $\mathbb{P}$ and G be as in Proposition 6.3, and let ζ be a regular uncountable cardinal such that $\mu \leq \zeta < \kappa$. Working in $V[G]$, we prove $\Phi(\zeta)$.

Note that $\Psi(\zeta)$ holds by Lemma 6.4. Let Z' be a club subset of $\mathcal{P}_{\zeta^+}(\mathcal{H}_\lambda^{V[G]})$ witnessing $\Psi(\zeta)$. Take a well-ordering Δ of $\mathcal{H}_\lambda^{V[G]}$, and let $\mathcal{M}'$ be the structure

$\langle \mathcal{H}_\lambda^{V[G]}, \in, \Delta, \mathcal{H}_\lambda^V, \mu, \kappa, \mathbb{P}, G\rangle$. We may assume that if $M' \in Z'$, then $M' \prec \mathcal{M}'$, and $|M'| = \zeta \subseteq M'$. Let $\mathcal{M}$ be the structure obtained from $\mathcal{M}'$ by adding Z' as a predicate. Then, let Z be the set of all $M \in \mathcal{P}_\kappa(\mathcal{H}_\lambda^{V[G]})$ such that $M \prec \mathcal{M}$ and $M \cap \kappa \in \kappa$. We show that Z witnesses $\Phi(\zeta)$.

Suppose $M \in Z$ and M is i.a. of length ζ. Let $N := M \cap \mathcal{H}_\lambda^V$. Then $M = N[G]$ since $M \prec \mathcal{M}$. We show that $N \in V$ and N is i.a. of length ζ in V.

Let $\langle M_\xi \mid \xi < \zeta\rangle$ be an i.a. sequence to M. Using the $<\kappa$-c.c. of $\mathbb{P}$, for each $\xi < \zeta$, let K_ξ be the Δ-least element of $\mathcal{H}_\lambda^V$ such that $M_\xi \cap \mathcal{H}_\lambda^V \subseteq K_\xi$ and $|K_\xi| < \kappa$ in V. By recursion on $\xi < \zeta$, let M'_ξ be the Δ-least element of Z' such that $\bigcup_{\eta<\xi} M'_\eta \subseteq M'_\xi$ and $\langle M'_\eta \mid \eta < \xi\rangle, K_\xi \in M'_\xi$. Let $M' := \bigcup_{\xi<\zeta} M'_\xi$. Then $M' \in Z'$, and M' is i.a. of length ζ since $\langle M'_\xi \mid \xi < \zeta\rangle$ is an i.a. sequence to M'. Note also that $M'_\xi \in M$ for all $\xi < \zeta$ since M'_ξ is definable from $\langle M_\eta \mid \eta \leq \xi\rangle$ in $\mathcal{M}$. Then $M'_\xi \subseteq M$ since $M \prec \mathcal{M}$ and $M \cap \kappa \in \kappa$. So $M' \subseteq M$.

Let $N' := M' \cap \mathcal{H}_\lambda^V$. Then $N' \in V$ since Z' witnesses $\Psi(\zeta)$. Moreover, $N' \prec \langle \mathcal{H}_\lambda^V, \in, \kappa\rangle$ since $M' \prec \mathcal{M}'$.

Note also that $|N'| = \zeta$ in $V[G]$ since $|M'| = \zeta \subseteq N' \subseteq M'$. Then $|N'| = \zeta$ in V by the $<\zeta^+$-c.c. of $\mathbb{P}$. In V, take a bijection $f : \zeta \to N'$, and let $N'_\xi := f[\xi]$ for each $\xi < \zeta$. Then, $\bigcup_{\xi<\zeta} N'_\xi = N'$, and all initial segments of $\langle N'_\xi \mid \xi < \zeta\rangle$ belong to N by Lemma 6.5. So $\langle N'_\xi \mid \xi < \zeta\rangle$ is an i.a. sequence to N'.

In V, let $N^* := \bigcup\{K \in N' \mid |K| < \kappa\}$ and $N^*_\xi := \bigcup\{K \in N'_\xi \mid |K| < \kappa\}$ for $\xi < \zeta$. Then it is easy to see that $\langle N^*_\xi \mid \xi < \zeta\rangle$ is an i.a. sequence to N^* in V. So it suffices to show that $N^* = N$ $(= M \cap \mathcal{H}_\lambda^V)$.

Since $N' \subseteq M \prec \mathcal{M}$ and $M \cap \kappa \in \kappa$, we have $N^* \subseteq M$. Also, $N \subseteq \mathcal{H}_\lambda^V$ clearly. So $N^* \subseteq N$. On the other hand, $N \subseteq \bigcup_{\xi<\zeta} K_\xi$ by the choice of $\langle K_\xi \mid \xi < \zeta\rangle$. Moreover, $\bigcup_{\xi<\zeta} K_\xi \subseteq N^*$ since $K_\xi \in N'$ and $|K_\xi| < \kappa$ for all $\xi < \zeta$. Hence $N \subseteq N^*$. □

Below, we prove Lemma 6.4. First, we prove (1).

Proof of Lemma 6.4 (1). Let ζ be a regular cardinal with $\mu < \zeta < \lambda$. Working in $V[G]$, we prove $\Psi(\zeta)$.

Let $\mathcal{M}$ be the structure $\langle \mathcal{H}_\lambda^{V[G]}, \in, \mathcal{H}_\lambda^V, \mu, \zeta, \mathbb{P}, G\rangle$, and let Z be the set of all $M \in \mathcal{P}_{\zeta^+}(\mathcal{H}_\lambda^{V[G]})$ such that $M \prec \mathcal{M}$ and $\zeta \subseteq M$. Then Z is club in $\mathcal{P}_{\zeta^+}(\mathcal{H}_\lambda^{V[G]})$. We claim that Z witnesses $\Psi(\zeta)$.

Suppose $M \in Z$ and M is i.a. of length ζ. Let $N := M \cap \mathcal{H}_\lambda^V$. We must prove $N \in V$. Since $\mathbb{P}$ has the $< \mu^+$-approximation property, it suffices to prove that $N \cap A \in V$ for all $A \in V$ with $|A| \leq \mu$. Suppose $A \in V$ and $|A| \leq \mu$.

Let $\langle M_\xi \mid \xi < \zeta\rangle$ be an i.a. sequence to M. Since $\zeta > \mu$, we can take $\xi < \zeta$ with $N \cap A \subseteq M_\xi \cap \mathcal{H}_\lambda^V$. By the $<\mu^+$-c.c. of $\mathbb{P}$ and the elementarity of M, we can take $K \in M \cap \mathcal{H}_\lambda^V$ such that $M_\xi \cap \mathcal{H}_\lambda^V \subseteq K$ and $|K| \leq \zeta$. Then $N \cap A \subseteq K \subseteq M \cap \mathcal{H}_\lambda^V = N$, and $K, A \in V$. So $N \cap A = K \cap A \in V$. □

Before proving Lemma 6.4 (2), we make some preliminaries.

Lemma 6.6. *Let λ be a regular uncountable cardinal and Δ be a well-ordering of $\mathcal{H}_\lambda$. Suppose $N_0, N_1 \prec \langle \mathcal{H}_\lambda, \in, \Delta\rangle$, and both $N_0 \cap \lambda$ and $N_1 \cap \lambda$ are ω-closed. Suppose also that there is $\iota \in N_0 \cap N_1 \cap \lambda$ with $N_0 \cap \iota \neq N_1 \cap \iota$, and let ι^* be the least such ι. Then ι^* is a regular cardinal, and* $\sup(N_0 \cap \iota^*) \neq \sup(N_1 \cap \iota^*)$.

Proof. First, we prove that $\sup(N_0 \cap \iota^*) \neq \sup(N_1 \cap \iota^*)$. Assume not. Then, for any $\alpha \in (N_i \cap \iota^*) \setminus (N_{1-i} \cap \iota^*)$, we can take $\alpha' \in N_{i-1} \cap \iota^*$ with $\alpha' > \alpha$, and such α' is not in N_i since if $\alpha' \in N_i$, then $\alpha' \in N_i \cap N_{i-1} \cap \lambda$, $N_i \cap \alpha' \neq N_{i-1} \cap \alpha'$, and $\alpha' < \iota^*$, which contradicts to the choice of ι^*. Then we can recursively construct an increasing sequence $\langle \alpha_n \mid n < \omega\rangle$ such that $\alpha_n \in (N_0 \cap \iota^*) \setminus (N_1 \cap \iota^*)$ if n is even and $\alpha_n \in (N_1 \cap \iota^*) \setminus (N_0 \cap \iota^*)$ if n is odd. Note that $\alpha^* := \sup_{n<\omega} \alpha_n \leq \iota^*$ and $\alpha^* \in N_0 \cap N_1$ since both $N_0 \cap \lambda$ and $N_1 \cap \lambda$ are ω-closed. Let A be the Δ-least cofinal subset of α^* of order type ω. Then $A \in N_0 \cap N_1$, and so $A \subseteq N_0 \cap N_1$ since A is countable. Take $\alpha \in A$ with $\alpha_0 < \alpha$. Then $\alpha \in N_0 \cap N_1$, $N_0 \cap \alpha \neq N_1 \cap \alpha$, and $\alpha < \iota^*$. This contradicts to the choice of ι^*.

Next, we prove that ι^* is regular. Assume not. Let $\iota^{**} := \mathrm{cf}(\iota^*) < \iota^*$. Note that $\iota^{**} \in N_0 \cap N_1$ since $\iota^* \in N_0 \cap N_1$. Take the Δ-least increasing continuous cofinal function $f : \iota^{**} \to \iota^*$. Note that $f \in N_0 \cap N_1$. Then, for each $i = 0, 1$, it easily follows from the elementarity of N_i that $\sup(N_i \cap \iota^*) = f(\sup(N_i \cap \iota^{**}))$. Then, since $\sup(N_0 \cap \iota^*) \neq \sup(N_1 \cap \iota^*)$, we have $\sup(N_i \cap \iota^{**}) \neq \sup(N_i \cap \iota^{**})$. In particular, $N_0 \cap \iota^{**} \neq N_1 \cap \iota^{**}$. This contradicts the choice of ι^*. ☐

Lemma 6.7. *Let ν be a singular cardinal of cofinality ω, I be a set of regular cardinals with $\sup I = \nu$ and $\vec{f} = \langle f_\alpha \mid \alpha < \nu^+\rangle$ be a scale in ΠI. Suppose $\langle\langle \iota_s, \delta_s\rangle \mid s \in {}^{<\omega}2\rangle$ is a sequence with the following properties.*

(i) *$\delta_s < \iota_s \in I$ for all $s \in {}^{<\omega}2$.*
(ii) *For all $s \in {}^{<\omega}2$, $\iota_s < \iota_{s^\frown 0} = \iota_{s^\frown 1}$, and $\delta_{s^\frown 0} \neq \delta_{s^\frown 1}$.*
(iii) *For any $b \in {}^{\omega}2$, $I_b := \{\iota_{b\restriction n} \mid n < \omega\}$ is cofinal in ν.*

For each $b \in {}^{\omega}2$, let $g_b \in \Pi I_b$ be such that $g_b(\iota_{b\restriction n}) = \delta_{b\restriction n}$. Then, there is $b \in {}^{\omega}2$ such that $g_b \neq^* f_\alpha \restriction I_b$ for any $\alpha < \nu^+$.

Proof. For a contradiction, assume that for any $b \in {}^{<\omega}2$ there is $\alpha_b < \nu^+$ with $g_b =^* f_{\alpha_s} \restriction I_b$. For each $s \in {}^{<\omega}2$, let $B_s := \{b \in {}^{\omega}2 \mid s \subseteq b\}$ and $A_s := \{\alpha_b \mid b \in B_s\}$. Then let

$$\gamma := \min\{\sup{}^+ A_s \mid s \in {}^{<\omega}2\},$$

where $\sup^+ A$ denotes $\sup\{\alpha{+}1 \mid \alpha \in A\}$. Take $s_{-1} \in {}^{<\omega}2$ such that $\gamma = \sup^+ A_{s_{-1}}$. Note the following.

(iv) $\sup^+ A_s = \gamma$ for all $s \in {}^{<\omega}2$ with $s_{-1} \subseteq s$.

The rest of the proof splits into three cases according to γ.

First, suppose γ is a successor ordinal. We recursively construct a $\subseteq$-increasing sequence $\langle s_n \mid n < \omega \rangle$ of elements of ${}^{<\omega}2$ extending s_{-1} such that

(v) $\delta_{s_n} = f_{\gamma-1}(\iota_{s_n})$ if n is even and $\delta_{s_n} \neq f_{\gamma-1}(\iota_{s_n})$ if n is odd.

If s_{n-1} has been taken, then take s_n as follows. First, suppose n is even. By (iv), we can take $b \in B_{s_{n-1}}$ with $g_b =^* f_{\gamma-1} \restriction I_b$. Take $k < \omega$ such that $\mathrm{dom}(s_{n-1}) < k$ and $\delta_{b\restriction k} = g_b(\iota_{b\restriction k}) = f_{\gamma-1}(\iota_{b\restriction k})$. Then let $s_n := b \restriction k$. Next, suppose n is odd. Let $t_i := s_{n-1}{}^\frown i$ for $i = 0, 1$. By (ii), we can take $i < 2$ with $\delta_{t_i} \neq f_{\gamma-1}(\iota_{t_i})$. Let s_n be such t_i.

Let $b := \bigcup_{n<\omega} s_n$. By (v), both of the sets $\{\iota \in I_b \mid g_b(\iota) = f_{\gamma-1}(\iota)\}$ and $\{\iota \in I_b \mid g_b(\iota) \neq f_{\gamma-1}(\iota)\}$ are unbounded in I_b. So there is no $\alpha < \nu^+$ with $g_b =^* f_\alpha \restriction I_b$ since $\vec{f}$ is $<^*$-increasing. This contradicts that $g_b =^* f_{\alpha_b} \restriction I_b$.

Next, suppose γ is a limit ordinal of cofinality ω. Take a cofinal sequence $\langle \gamma_m \mid m < \omega \rangle$ in γ. We recursively take a $\subseteq$-increasing sequence $\langle s_n \mid n < \omega \rangle$ of elements of ${}^{<\omega}2$ extending s_{-1} such that

(vi) $\delta_{s_n} > f_{\gamma_m}(\iota_{s_n})$ for all $n < \omega$ and all $m \leq n$.

If s_{n-1} has been taken, then take s_n as follows. By (iv), take $b \in B_{s_{n-1}}$ with $\alpha_b > \gamma_m$ for all $m \leq n$. Then, we can take $k < \omega$ such that $\mathrm{dom}(s_{n-1}) < k$ and $\delta_{b\restriction k} = g_b(\iota_{b\restriction k}) = f_{\gamma_m}(\iota_{b\restriction k})$ for all $m \leq n$. Let $s_n := b \restriction k$.

Let $b := \bigcup_{n<\omega} s_n$. By (vi), $f_{\gamma_m} <^* g_b$ for all $m < \omega$. Then $\alpha_b \geq \sup_{m<\omega} \gamma_m = \gamma$. Since $b \in B_{s_{-1}}$, this contradicts that $\sup^+ A_{s_{-1}} = \gamma$.

Finally, suppose γ is a limit ordinal of cofinality $> \omega$. Then,

$$\beta := \sup{}^+\{\min A_s \mid s_{-1} \subseteq s \in {}^{<\omega}2\} < \gamma\,.$$

Note that $\min A_s < \beta < \sup A_s$ for all $s \in {}^{<\omega}2$ with $s_{-1} \subseteq s$. We recursively take a $\subseteq$-increasing sequence $\langle s_n \mid n < \omega \rangle$ of elements of ${}^{<\omega}2$ extending s_{-1} such that

(vii) $\delta_{s_n} < f_\beta(\iota_{s_n})$ if n is even, and $\delta_{s_n} > f_\beta(\iota_{s_n})$ if n is odd.

If s_{n-1} has been taken, then take s_n as follows. First, suppose n is even. Then, since $\min A_{s_{n-1}} < \beta$, we can take $b \in B_{s_{n-1}}$ with $g_b <^* f_\beta \restriction I_b$. Take $k < \omega$ such that $\mathrm{dom}(s_{n-1}) < k$ and $\delta_{b\restriction k} = g_b(\iota_{b\restriction k}) < f_\beta(\iota_{b\restriction k})$. Then let $s_n := b \restriction k$. Next, suppose n is odd. Then, since $\sup A_{s_{n-1}} > \beta$, we can take $b \in B_{s_{n-1}}$ with $f_\beta \restriction I_b <^* g_b$. Take $k < \omega$ such that $\mathrm{dom}(s_{n-1}) < k$ and $\delta_{b\restriction k} = g_b(\iota_{b\restriction k}) > f_\beta(\iota_{b\restriction k})$. Then let $s_n := b \restriction k$.

Let $b := \bigcup_{n\in\omega} s_n$. By (vii), there is no $\alpha < \nu^+$ with $g_b =^* f_\alpha \restriction I_b$. This contradicts that $g_b = f_{\alpha_b} \restriction I_b$. □

Now we prove Lemma 6.4 (2).

Proof of Lemma 6.4 (2). Suppose $\mu > \omega$. Working in $V[G]$, we prove $\Psi(\mu)$.

Take a bijection $F : \lambda \to \mathcal{H}_\lambda^V$ in V. Let $\mathcal{M} := \langle \mathcal{H}_\lambda^{V[G]}, \in, \mathcal{H}_\lambda^V, F, \mu, \mathbb{P}, G \rangle$, and let Z be the set of all $M \in \mathcal{P}_{\mu^+}(\mathcal{H}_\lambda^{V[G]})$ such that $M \prec \mathcal{M}$ and $\mu \subseteq M$. Then, Z is club in $\mathcal{P}_{\mu^+}(\mathcal{H}_\lambda^{V[G]})$. We show that Z witnesses $\Psi(\zeta)$.

Suppose $M \in Z$ and M is i.a. of length μ, and let $N := M \cap \mathcal{H}^V_\lambda$. We prove that $N \in V$. Note that $N = F[M \cap \lambda]$ since $M \prec \mathcal{M}$. So it suffices to prove that $M \cap \lambda \in V$. By induction on cardinals ν with $\mu \leq \nu \leq \lambda$, we prove that $M \cap \nu \in V$.

Note that if $\nu = \mu$, then $M \cap \nu = \mu \in V$. Suppose ν is a cardinal with $\mu < \nu \leq \lambda$, and $M \cap \rho \in V$ for all cardinals ρ with $\mu \leq \rho < \nu$. We show that $M \cap \nu \in V$. Our proof splits into four cases.

Case 1. ν is a successor cardinal.

Let ρ be a cardinal with $\nu = \rho^+$. If $\rho \notin M$, then $M \cap \nu = M \cap \rho \in V$ since $M \prec \mathcal{M}$. So we assume that $\rho \in M$.

Let $\delta := \sup(M \cap \nu) > \rho$. Since M is i.a. of length μ, there is a club $C \subseteq \delta$ of oder-type μ with $C \subseteq M$. (Take an i.a. sequence $\langle M_\xi \mid \xi < \zeta \rangle$ to M, and let $C = \{\sup((\bigcup_{\eta<\xi} M_\eta) \cap \nu) \mid \xi < \zeta\}$. Then $C \subseteq M$, and C is a club subset of δ of order-type μ.) Since $\delta > \rho$, we may assume $C \cap \rho = \emptyset$.

Let $A := M \cap \rho \in V$. For each α with $\rho \leq \alpha < \nu$, let $\pi_\alpha : \rho \to \alpha$ be the F-least bijection in V. Note that $M \cap \alpha = \pi_\alpha[A]$ since $M \prec \mathcal{M}$. Hence we have

(i) $\pi_\alpha[A] = \pi_\beta[A] \cap \alpha$ for all $\alpha, \beta \in C$ with $\alpha < \beta$.

Take a $\mathbb{P}$-name $\dot{C}$ of C and $p \in G$ which forces (i) and that $\dot{C}$ is club in δ. By the $<\mu$-closure of $\mathbb{P}$, in V, we can take a descending sequence $\langle p_\xi \mid \xi < \mu \rangle$ in $\mathbb{P}$ below p and an increasing continuous cofinal sequence $\langle \gamma_\xi \mid \xi < \mu \rangle$ in δ such that $p_\xi \Vdash_{\mathbb{P}}$ "$\gamma_\xi \in \dot{C}$". Let $D := \{\gamma_\xi \mid \xi < \mu\} \in V$. Then D is club in δ, and

(ii) $\pi_\alpha[A] = \pi_\beta[A] \cap \alpha$ for all $\alpha, \beta \in D$ with $\alpha < \beta$.

Since $\mu > \omega$, $C \cap D$ is unbounded in δ. Then, $M \cap \nu = \bigcup_{\alpha \in C \cap D} \pi_\alpha[A] = \bigcup_{\alpha \in D} \pi_\alpha[A]$ by (i) and (ii). Then, since $D, A, \langle \pi_\alpha \mid \alpha < \nu \rangle \in V$, we have that $M \cap \nu \in V$.

Case 2. ν is a limit cardinal with $\mathrm{cf}(\nu) < \mu$.

Take a sequence $\langle \rho_\xi \mid \xi < \mathrm{cf}(\nu) \rangle$ of cardinals which converges to ν. By the induction hypothesis, $M \cap \rho_\xi \in V$ for each $\xi < \mathrm{cf}(\nu)$. Then $\langle M \cap \rho_\xi \mid \xi < \mathrm{cf}(\mu) \rangle \in V$ since $\mathbb{P}$ is $<\mu$-closed. Then, $M \cap \nu = \bigcup_{\xi < \mathrm{cf}(\nu)} M \cap \rho_\xi \in V$.

Case 3. ν is a limit cardinal with $\mathrm{cf}(\nu) > \mu$.

Since $|M| = \mu < \mathrm{cf}(\nu)$, we can take a cardinal $\rho < \nu$ such that $M \cap \nu = M \cap \rho$. But $M \cap \rho \in V$ by the induction hypothesis.

Case 4. ν is a limit cardinal with $\mathrm{cf}(\nu) = \mu$.

For a contradiction, assume $M \cap \nu \notin V$. Then, note that $M \cap \nu$ is unbounded in ν. Let $\dot{M}$ be a $\mathbb{P}$-name of M and $\dot{G}$ be the canonical $\mathbb{P}$-name for a $\mathbb{P}$-generic filter. Then we can take $p \in G$ which forces the following.

(iii) $\dot{M} \prec \langle \mathcal{H}^{V[\dot{G}]}_\lambda, \in, \mathcal{H}^V_\lambda, F, \mu, \mathbb{P}, \dot{G} \rangle$, $|\dot{M}| = \mu$, and $\dot{M}$ is i.a. of length μ.

(iv) $\dot{M} \cap \rho \in V$ for all $\rho < \nu$.

(v) $\dot{M} \cap \nu \notin V$.

Below, we work in V.

Take a sufficiently large regular cardinal θ and a countable elementary submodel K of $\langle \mathcal{H}_\theta^V, \in, \lambda, F, \mu, \nu, \mathbb{P}, p, \dot{M} \rangle$. Let $\chi := \sup(K \cap \nu)$. Note that χ is a singular cardinal of cofinality ω. Let $\langle \chi_n \mid n < \omega \rangle$ be the F-least increasing cofinal sequence in χ. Let I be the set of all regular cardinals ι with $\mu < \iota < \chi$. By Lemma 3.7, let $\langle f_\alpha \mid \alpha < \chi^+ \rangle$ be the F-least μ-continuous scale in ΠI.

For each $s \in {}^{<\omega}2$, we will define ι_s, δ_s as in Lemma 6.7. For this, we will use the following claim.

Claim. *Suppose $q \in \mathbb{P} \cap K$ and $q \leq p$. Suppose also that $\rho \in K \cap \chi$, $A \in \mathcal{P}(\rho) \cap K$ and $q \Vdash_{\mathbb{P}}$ "$\dot{M} \cap \rho = A$". Then, $\chi_n \in A$ for all $n < \omega$ with $\chi_n < \rho$.*

Proof of Claim. First, p forces that $\dot{M} \cap \nu$ is unbounded in ν by (iv) and (v). Then, we can recursively take a descending sequence $\langle q_m \mid m < \omega \rangle$ in $\mathbb{P} \cap K$ below q and a sequence $\langle \alpha_m \mid m < \omega \rangle$ in $K \cap \chi$ so that $q_m \Vdash_{\mathbb{P}}$ "$\alpha_m \in \dot{M}$" and $\alpha_m \geq \chi_m$. By the $<\mu$-closure of $\mathbb{P}$, take $q^* \in \mathbb{P}$ with $q^* \leq q_m$ for all $m < \omega$. Then, by Fact 3.1 (3) and (iii) above, q^* forces that $\chi = \sup_{m<\omega} \alpha_m \in \dot{M}$ and so $\chi_n \in \dot{M}$ for all $n < \omega$. Also, q^* forces that $\dot{M} \cap \rho = A$. So $\chi_n \in A$ for all $n < \omega$ with $\chi_n < \rho$. □ (Claim)

By recursion on $s \in {}^{<\omega}2$, we will define $\iota_s \in I \cap \chi$, $\delta_s < \iota_s$, $p_s \in \mathbb{P} \cap K$, $\rho_s \in K \cap \chi$ and $A_s \in \mathcal{P}(\rho_s) \cap K$ so that the following hold for all $s \in {}^{<\omega}2$.

(vi) $p_\emptyset \leq p$, and $p_{s\frown i} \leq p_s$ for both $i = 0, 1$.

(vii) $p_s \Vdash_{\mathbb{P}}$ "$A_s = \dot{M} \cap \rho_s$".

(viii) $\iota_s \in A_s$, and $\delta_s = \sup(A_s \cap \iota_s)$.

(ix) $\iota_{s\frown 0} = \iota_{s\frown 1}$, and $\delta_{s\frown 0} \neq \delta_{s\frown 1}$.

(x) There is $n < \omega$ such that $\rho_s < \chi_n < \rho_{s\frown 0}, \rho_{s\frown 1}$.

Let $\rho_\emptyset := \mu^{++} \in K$. By (iv), we can take $p_\emptyset \leq p$ and $A_\emptyset \subseteq \mu^{++}$ in K such that $p_\emptyset \Vdash_{\mathbb{P}}$ "$\dot{M} \cap \mu^{++} = A_\emptyset$". Then, let $\iota_\emptyset := \mu^+ \in K$ and $\delta_\emptyset := \sup(A_\emptyset \cap \mu^+) \in K$. Note that $\iota_\emptyset \in A_\emptyset$ by (iii). So $\iota_\emptyset$, $\delta_\emptyset$, $p_\emptyset$, $\rho_\emptyset$ and $A_\emptyset$ satisfies (vi)–(x).

Suppose $s \in {}^{<\omega}2$, and ι_s, δ_s, p_s, ρ_s and A_s has been taken. We define $\iota_{s\frown i}$, $\delta_{s\frown i}$, $p_{s\frown i}$, $\rho_{s\frown i}$ and $A_{s\frown i}$ for $i = 0, 1$. By (iv) and (v), we can take $\rho' > \rho_s$, $p'_0, p'_1 \leq p_s$ and $A'_0, A'_1 \subseteq \rho'$ in K so that $p'_i \Vdash_{\mathbb{P}}$ "$\dot{M} \cap \rho' = A'_i$" and $A'_0 \neq A'_1$. Take $n < \omega$ with $\rho' \leq \chi_n$. For both $i = 0, 1$, let $\rho_{s\frown i}$ be the least $\rho \in K \cap \nu$ with $\chi_n < \rho$. Moreover, in K, by (iv), take $p_{s\frown i} \leq p'_i$ and $A_{s\frown i} \subseteq \rho_{s\frown i}$ such that $p_{s\frown i} \Vdash_{\mathbb{P}}$ "$\dot{M} \cap \rho_{s\frown i} = A_{s\frown i}$".

Note that $A_{s\frown 0} \cap \chi_n \neq A_{s\frown 1} \cap \chi_n$ and $\chi_n \in A_{s\frown 0} \cap A_{s\frown 1}$ by Claim. For both $i = 0, 1$, let $\iota_{s\frown i}$ be the least $\iota \in A_{s\frown 0} \cap A_{s\frown 1}$ with $A_{s\frown 0} \cap \iota \neq A_{s\frown 1} \cap \iota$. Moreover, let $\delta_{s\frown i} := \sup(A_{s\frown i} \cap \iota_{s\frown i})$. Here note that if we let N_i be the smallest ω-closed elementary submodel of $\langle \mathcal{H}_\lambda^V, \in, F \rangle$ such that $A_{s\frown i} \subseteq N_i$ and $N_i \cap \lambda$ is ω-closed, then $A_{s\frown i} = N_i \cap \rho_{s\frown i}$ by (iii) above, Fact 3.1 (3) and the fact that $p_{s\frown i} \Vdash_{\mathbb{P}}$ "$\dot{M} \cap \rho_{s\frown i} = A_{s\frown i}$". So $\iota_{s\frown i}$ is regular and $\delta_{s\frown 0} \neq \delta_{s\frown 1}$ by Lemma

6.6. Then, it is easy to check that $\iota_{s^\frown i}$, $\delta_{s^\frown i}$, $p_{s^\frown i}$, $\rho_{s^\frown i}$ and $A_{s^\frown i}$ satisfies all the requirements.

We have taken ι_s, δ_s, p_s, ρ_s and A_s for all $s \in {}^{<\omega}2$. Note that $\langle\langle \iota_s, \delta_s\rangle \mid s \in {}^{<\omega}2\rangle$ satisfies the assumption of Lemma 6.7. Thus there is $b \in {}^{\omega}2$ such that $g \neq^* f_\alpha \restriction J$ for any $\alpha < \chi^+$, where $J = \{\iota_{b\restriction n} \mid n < \omega\}$, and $g \in \Pi J$ is such that $g(\iota_{b\restriction n}) = \delta_{b\restriction n}$. Let $B := \bigcup_{n<\omega} A_{b\restriction n}$. Moreover, by the $<\mu$-closure of $\mathbb{P}$, take $q \in \mathbb{P}$ with $q \leq p_{b\restriction n}$ for all $n < \omega$.

Then, q forces that $\dot{M} \cap \chi = B$ by (vii). Moreover, $\mathrm{ch}^J_B = g$ by (viii). So q forces that there is no $\alpha < \chi^+$ with $\mathrm{ch}^J_{\dot{M}} =^* f_\alpha \restriction J$. On the other hand, q forces that $\chi = \sup B \in \dot{M}$ by Fact 3.1 (3) and (iii) above, and so $\langle f_\alpha \mid \alpha < \chi^+\rangle \in \dot{M}$. Moreover, since J is a countable subset of B, p forces that $J \in \dot{M}$ by Lemma 6.5. So p forces that $\langle f_\alpha \restriction J \mid \alpha < \chi^+\rangle \in \dot{M}$. Note also that $\langle f_\alpha \restriction J \mid \alpha < \chi^+\rangle$ remains to be a μ-continuous scale in $V^{\mathbb{P}}$ by Lemma 3.7 (2). So, by Lemma 3.6, q forces that there is $\alpha < \chi^+$ with $\mathrm{ch}^J_{\dot{M}} =^* f_\alpha \restriction J$. This is a contradiction. □

This completes the proof of Theorem 6.1.

7. $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ AND SCH

In this section, we show that $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ does not implies SCH above κ. For this, we prove that a Prikry forcing above κ preserves $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$.

For a measurable cardinal ν and a normal ultrafilter U over ν, let $\mathbb{P}(U)$ be the Prikry forcing with respect to U, that is,

- $\mathbb{P}(U)$ consists of all pairs (s, A) such that $s \in {}^{<\omega}\nu$, s is strictly increasing and $A \in U$,
- $(t, B) \leq (s, A)$ in $\mathbb{P}(U)$ if $t \supseteq s$, $B \subseteq A$, and $t(n) \in A$ for any $n \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$.

Recall that a forcing by $\mathbb{P}(U)$ adds no new bounded subsets of ν, preserves all cardinals and makes ν to be a singular strong limit cardinal of cofinality ω. See Jech [10, Chapter 21] for example.

We prove the following:

Theorem 7.1. *Suppose κ is a regular cardinal $> \omega_1$, $2^\mu < \kappa$ for all cardinal μ with $\mu^+ < \kappa$, and $\kappa \in I[\kappa]$. Let ν be a measurable cardinal $> \kappa$ and U be a normal measure over ν. Suppose G is a $\mathbb{P}(U)$-generic filter over V. If $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ holds in V, then so is in $V[G]$.*

Corollary 7.2. *Assume* GCH. *Let κ be a regular cardinal $\geq \omega_2$, and there are two supercompact cardinals $> \kappa$. Then there is a forcing extension V^* of V in which the following holds.*

(i) *All regular cardinals $\leq \kappa$ in V remain regular.*
(ii) $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ *holds but* SCH *fails above κ.*

Proof of Corollary 7.2 from Theorem 7.1. Let τ and ν be supercompact cardinals with $\kappa < \tau < \nu$.

We can take a $<\kappa$-Baire forcing extension V' of V in which the following hold.

(iii) $\kappa \in I[\kappa]$.
(iv) τ is supercompact.
(v) ν is measurable, and $2^\nu > \nu^+$.

For this, first make τ a Laver indestructible supercompact cardinal by a $<\kappa$-closed forcing of cardinality τ, next blow up 2^ν with preserving the measurability of ν by $<\tau$-directed closed forcing, and finally force $\kappa \in I[\kappa]$ by $\mathbb{I}_\kappa$.

Next, let V'' be a forcing extension of V' by $\mathrm{Col}(\kappa, <\tau)$. In V'', (iii), (v) and $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ holds. It also holds in V'' that $2^\mu < \kappa$ for all μ with $\mu^+ < \kappa$ since this holds in V, and V'' is a $<\kappa$-Baire forcing extension of V. Finally, let V^* be a forcing extension of V' by $\mathbb{P}(U)$ for some normal ultrafilter U over ν. Then $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ holds in V^* by Theorem 7.1. Moreover, in V^*, (i) holds, but SCH fails at ν. Thus V^* is as desired. □

Below, we prove Theorem 7.1. We use the following well-known fact.

Lemma 7.3. *In V, suppose U is a normal ultrafilter over a measurable cardinal ν and ζ is a regular uncountable cardinal $< \nu$. Suppose G is a $\mathbb{P}(U)$-generic filter over V and $f \in {}^\zeta V \cap V[G]$. Then, there is an unbounded $D \subseteq \zeta$ such that $f \restriction D \in V$.*

Proof. It suffices to prove that the following $(*)$ holds in V:

$(*)$ Suppose $\dot{f}$ is a $\mathbb{P}(U)$-name, and $(s, A) \in \mathbb{P}(U)$ forces $\dot{f} \in {}^\zeta V$. Then there are $(t, B) \leq (s, A)$ and an unbounded $D \subseteq \zeta$ such that (t, B) forces $\dot{f} \restriction D \in V$.

Take a $\mathbb{P}(U)$-generic filter G' over V which contains (s, A). In $V[G']$, for each $\xi < \zeta$, take $(t_\xi, C_\xi) \in G'$ below (s, A) which decides $\dot{f}(\xi)$. Then, since ζ is regular uncountable in $V[G']$, we can take t with $E := \{\xi < \omega_1 \mid t_\xi = t\}$ unbounded in ζ.

In V, let D be the set of all $\xi < \zeta$ for which there are a_ξ and $B_\xi \in U$ such that $B_\xi \subseteq A$ and $(t, B_\xi) \Vdash_{\mathbb{P}(U)}$ " $\dot{f}(\xi) = a_\xi$ ". Note that $E \subseteq D$. So D is unbounded in ζ. In V, let $B := \bigcap_{\xi \in D} B_\xi \in U$ and g be a function on D such that $g(\xi) = a_\xi$. Then $(t, B) \leq (s, A)$, and (t, B) forces that $\dot{f} \restriction D = g \in V$. □

The next lemma is a key for Theorem 7.1.

Lemma 7.4. *Let κ and λ be regular uncountable cardinals such that $\kappa < \lambda$, $\kappa \in I[\kappa]$ and $2^\mu < \kappa$ for all cardinals μ with $\mu^+ < \kappa$. Suppose ν is a measurable cardinal with $\kappa < \nu < 2^\nu < \lambda$ and U is a normal ultrafilter over ν. Suppose $(s, A) \in \mathbb{P}(U)$ and $\dot{X}$ is a $\mathbb{P}(U)$-name for a stationary subset of $\mathcal{P}_\kappa(\mathcal{H}_\lambda^{V^{\mathbb{P}(U)}}) \cap \mathrm{IA}_{>\omega}$. Let Y be the set of all $N \in \mathcal{P}_\kappa(\mathcal{H}_\lambda^V) \cap \mathrm{IA}_{>\omega}$ such that $(t, B) \Vdash_{\mathbb{P}(U)}$ "$N[\dot{G}] \in \dot{X}$ " for some $(t, B) \leq (s, A)$ with $t \in N$. Then Y is stationary in $\mathcal{P}_\kappa(\mathcal{H}_\lambda^V)$.*

Proof. In V, take an arbitrary $F : {}^{<\omega}\mathcal{H}_\lambda^V \to \mathcal{H}_\lambda^V$. We will find $N \in Y$ such that $N \cap \kappa \in \kappa$ and N is closed under F.

In V, take a well-ordering Δ of $\mathcal{H}_\lambda^V$, and let $\mathcal{N} := \langle \mathcal{H}_\lambda^V, \in, \Delta, \kappa, F \rangle$. Take a $\mathbb{P}(U)$-generic filter G over V with $(s, A) \in G$. Below, we basically work in $V[G]$.

Let θ be a regular cardinal $> 2^\lambda$, and let $\bar{\mathcal{M}} := \langle \mathcal{H}_\theta^{V[G]}, \in, \mathcal{N}, \lambda, \nu, U, G \rangle$. Let $X := \dot{X}^G$. Since X is stationary, we can take $\bar{M} \prec \bar{\mathcal{M}}$ such that $\bar{M} \cap \kappa \in \kappa$ and $M := \bar{M} \cap \mathcal{H}_\lambda^{V[G]} \in X$. Note that $\kappa \in I[\kappa]$ in $V[G]$. So M is i.a. of length ζ for some regular uncountable cardinal $\zeta < \kappa$ by Lemma 3.3. Let $\langle M_\xi \mid \xi < \zeta \rangle$ be an i.a. sequence to M.

Note that for each $\xi < \zeta$, we can take $K_\xi \in \mathcal{P}_\kappa(\mathcal{H}_\lambda^V)^V$ such that $K_\xi[G] = M_\xi$: Let $\mu := |M_\xi|^{V[G]} < \kappa$, and take a $\mathbb{P}(U)$-name $\dot{M}_\xi$ of M_ξ. Then, there is $p \in G$ which forces that $|\dot{M}_\xi| = \mu$ and $\dot{M}_\xi \subseteq \mathcal{H}_\lambda$. Then, in V, we can take a sequence $\langle \dot{a}_\alpha \mid \alpha < \mu \rangle$ of $\mathbb{P}(U)$-names such that $\dot{a}_\alpha \in \mathcal{H}_\lambda^V$ for all $\alpha < \mu$ and $p \Vdash$ " $\dot{M}_\xi = \{\dot{a}_\alpha \mid \alpha < \mu\}$ ". Let $K_\xi := \{\dot{a}_\alpha \mid \alpha < \mu\}$.

For each $\xi < \zeta$, let K_ξ be the Δ-least such one. Note that $\langle K_\xi \mid \xi < \zeta' \rangle$ belongs to $\bar{M}$ for each $\zeta' < \zeta$ since $\langle K_\xi \mid \xi < \zeta' \rangle$ is definable in $\bar{\mathcal{M}}$ from a parameter $\langle M_\xi \mid \xi < \zeta' \rangle \in \bar{M}$.

By Lemma 7.3, we can take an unbounded $D \subseteq \zeta$ such that $\langle K_\xi \mid \xi \in D \rangle \in V$. Here note also that $\mathcal{P}(\zeta') \subseteq \bar{M}$ since $\mathcal{P}(\zeta') \in \bar{M}$, $|\mathcal{P}(\zeta')| < \kappa$ by the assumption of the lemma, and $\bar{M} \cap \kappa \in \kappa$. So $\langle K_\xi \mid \xi \in D \cap \zeta' \rangle \in \bar{M}$ for all $\zeta' < \zeta$.

Note that $\langle \sup(M_\xi \cap \kappa) \mid \xi < \zeta \rangle \in V$ since $\mathcal{P}(\kappa)^V = \mathcal{P}(\kappa)^{V[G]}$. In V, by recursion on $\xi < \zeta$, let N'_ξ be the Δ-least element of $\mathcal{P}_\kappa(\mathcal{H}_\lambda^V)$ such that

(i) $N'_\xi \prec \mathcal{N}$,
(ii) $N'_\xi \cap \kappa \in \kappa$, and $\sup(M_\xi \cap \kappa) \leq N'_\xi \cap \kappa$,
(iii) $\bigcup_{\eta<\xi} N'_\eta \subseteq N'_\xi$, and $\langle N'_\eta \mid \eta < \xi \rangle \in N'_\xi$.
(iv) $K_\xi \subseteq N'_\xi$ if $\xi \in D$.

Let $N' := \bigcup_{\xi<\zeta} N'_\xi \in V$. Then $N' \in \mathcal{P}_\kappa(\mathcal{H}_\lambda^V)$ in V. Moreover, $\langle N'_\xi \mid \xi < \zeta \rangle$ is an i.a. sequence to N'. So $N' \in \mathrm{IA}_{>\omega}$ in V.

Note also that $N'[G] = M$: By (iv) and the unboundedness of D in ζ, we have $N'[G] \supseteq \bigcup_{\xi\in D} K_\xi[G] = \bigcup_{\xi\in D} M_\xi = M$. For the revserse inclusion, note that $N'_\xi \in \bar{M}$ for each $\xi < \zeta$ since N'_ξ is definable in $\bar{\mathcal{M}}$ from parameters $\langle M_\eta \mid \eta \leq \xi \rangle, \langle K_\eta \mid \eta \in D \cap \xi + 1 \rangle \in \bar{M}$. Then, $N'_\xi \subseteq \bar{M}$ for all $\xi < \zeta$ since $|N'_\xi| < \kappa$ and $\bar{M} \cap \kappa \in \kappa$. So $N' = \bigcup_{\xi<\zeta} N'_\xi \subseteq \bar{M}$. Then $N'[G] \subseteq \bar{M}$ since $\bar{M} \prec \bar{\mathcal{M}}$. Hence $N'[G] \subseteq \bar{M} \cap \mathcal{H}_\lambda^{V[G]} = M$.

Recall that $M \in X = \dot{X}^G$ and $M \prec \langle \mathcal{H}_\lambda^{V[G]}, \in, \mathcal{H}_\lambda^V, \Delta, \kappa, F, \nu, U, G \rangle$. Then, since $N'[G] = M$, we can take $(t, B) \in G$ such that

(v) $(t, B) \Vdash_{\mathbb{P}(U)}$ " $N'[\dot{G}] \in \dot{X}$ ",
(vi) $(t, B) \Vdash_{\mathbb{P}(U)}$ " $N'[\dot{G}] \prec \langle \mathcal{H}_\lambda^{V[\dot{G}]}, \in, \mathcal{H}_\lambda^V, \Delta, \kappa, F, \nu, U, G \rangle$ ".

We may assume $(t, B) \leq (s, A)$ since $(s, A) \in G$. In V, let $N := \mathrm{Sk}^{\mathcal{N}}(N' \cup \{t\}) \in \mathcal{P}_\kappa(\mathcal{H}_\lambda^V)$. We show that N is as desired.

First, N is closed under F since $N \prec \mathcal{N}$. Next, we check that $N \cap \kappa \in \kappa$. For this, note that $N' \subseteq N \subseteq M$, where the latter inclusion is because $N' \cup \{t\} \subseteq N'[G] = M$ since $(t, B) \in G$, and $M \cap \mathcal{H}^V_\lambda \prec \mathcal{N}$. Moreover, $N' \cap \kappa \supseteq M \cap \kappa$ by (ii). Hence $N \cap \kappa = M \cap \kappa \in \kappa$.

We must show that $N \in Y$. Below, we work in V. First, we show that $N \in \mathrm{IA}_{>\omega}$. Note that $N = \{f(t) \mid f : {}^{<\omega}\kappa \to \mathcal{H}^V_\lambda, f \in N'\}$ since $t \in {}^{<\omega}\kappa \in N \prec \mathcal{N}$. For each $\xi < \zeta$, let $N_\xi := \{f(t) \mid f : {}^{<\omega}\kappa \to \mathcal{H}^V_\lambda, f \in N'_\xi\}$. Here recall that $\langle N'_\xi \mid \xi < \zeta\rangle$ is an i.a. sequence to N'. Then it is easy to check that $\langle N_\xi \mid \xi < \zeta\rangle$ is an i.a. sequence to N. So $N \in \mathrm{IA}_{>\omega}$.

Next, we prove that $(t, B) \Vdash_{\mathbb{P}(U)}$ " $N[\dot{G}] \in \dot{X}$ ". This implies that $N \in Y$ since $t \in N$ and $(t, B) \le (s, A)$. First, note that $(t, B) \Vdash_{\mathbb{P}(U)}$ "$t \in N'[\dot{G}]$ ". Then, from (vi), it easily follows that $(t, B) \Vdash_{\mathbb{P}(U)}$ " $N[\dot{G}] = N'[\dot{G}]$ ". Hence (t, B) forces that $N[\dot{G}] \in \dot{X}$ by (v). □

Now, we prove Theorem 7.1.

Proof of Theorem 7.1. Assume $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ holds in V. We show that $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ holds also in $V^{\mathbb{P}(U)}$. Let $\dot{G}$ be the canonical $\mathbb{P}(U)$-name for a $\mathbb{P}(U)$-generic filter.

For a while, we work in V. Suppose λ is a regular cardinal $> 2^\nu$, $(s, A) \in \mathbb{P}(U)$ and $\dot{X}$ is a $\mathbb{P}(U)$-name for a stationary subset of $\mathcal{P}_\kappa(\mathcal{H}^{V[\dot{G}]}_\lambda) \cap \mathrm{IA}_{>\omega}$. It suffices to find $(t, B) \le (s, A)$ and a $\mathbb{P}(U)$-name $\dot{R}$ for a subset of $\mathcal{H}^{V[\dot{G}]}_\lambda$ such that (t, B) forces that $|\dot{R}| = \kappa \subseteq \dot{R}$, $\dot{R}$ is i.a. of length κ, and $\dot{X} \cap \mathcal{P}_\kappa(\dot{R})$ is stationary.

Let Y' be the set of all $N \in \mathcal{P}_\kappa(\mathcal{H}^V_\lambda) \cap \mathrm{IA}_{>\omega}$ such that $N \prec \langle \mathcal{H}^V_\lambda, \in, \kappa, \nu, U\rangle$ and $(t_N, B_N) \Vdash_{\mathbb{P}(U)}$ " $N[\dot{G}] \in \dot{X}$ " for some $(t_N, B_N) \le (s, A)$ with $t_N \in N$. Then Y' is stationary by Lemma 7.4. By the Pressing Down Lemma, we can take t such that $Y := \{N \in Y' \mid t_N = t\}$ is stationary.

By $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$, we can take $Q \subseteq \mathcal{H}^V_\lambda$ such that $|Q| = \kappa \subseteq Q$, Q is i.a. of length κ and $Y \cap \mathcal{P}_\kappa(Q)$ is stationary. Let $\dot{R}$ be a $\mathbb{P}(U)$-name for $Q[\dot{G}]$, and let

$$B := \bigcap \{B_N \mid N \in Y \cap \mathcal{P}_\kappa(Q)\}\,.$$

Note that $B \in U$ since $|Y \cap \mathcal{P}_\kappa(Q)| < \nu$. So $(t, B) \in \mathbb{P}(U)$.

We show that (t, B) and $\dot{R}$ are as desired. Take a $\mathbb{P}(U)$-generic filter G over V which contains (t, B). Let $R := \dot{R}^G = Q[G]$ and $X := \dot{X}^G$. Working in $V[G]$, we check that $|R| = \kappa \subseteq R$, R is i.a. of length κ and $X \cap \mathcal{P}_\kappa(R)$ is stationary.

First note that $Q \prec \langle \mathcal{H}^V_\lambda, \in, \kappa, \nu, U\rangle$ since $N \prec \langle \mathcal{H}^V_\lambda, \in, \kappa, \nu, U\rangle$ for all $N \in Y$, and $Y \cap \mathcal{P}_\kappa(Q)$ is stationary. So $Q \subseteq Q[G] = R \prec \langle \mathcal{H}^{V[G]}_\lambda, \in, \kappa, \nu, U, G\rangle$. Then, $|R| = \kappa \subseteq R$. Moreover, taking an i.a. sequence $\langle Q_\xi \mid \xi < \kappa\rangle$ to Q in V, it is easy to see that $\langle Q_\xi[G] \mid \xi < \kappa\rangle$ is an i.a. sequence to $R = Q[G]$. Finally, we prove that $X \cap \mathcal{P}_\kappa(R)$ is stationary. For this, note that $Y \cap \mathcal{P}_\kappa(Q)$ is staitonary in $V[G]$ since so is in V, and $|Q| = \kappa < \nu$. Take an $\subseteq$-increasing continuous cofinal sequence $\langle Q'_\xi \mid \xi < \kappa\rangle$ in $\mathcal{P}_\kappa(Q)$. Then $S := \{\xi < \kappa \mid Q'_\xi \in Y\}$ is stationary. Note that $\langle Q'_\xi[G] \mid \xi < \kappa \rangle$ is $\subseteq$-increasing continuous and cofinal in $\mathcal{P}_\kappa(R)$. Moreover, $Q'_\xi[G] \in X$ for all $\xi \in S$. So $X \cap \mathcal{P}_\kappa(R)$ is stationary. $\square$

We end this section with a question. Note that in our proof of Corollary 7.2, SCH fails at a singular cardinal of cofinality ω in V^*. So the following question naturally arises.

Question 7.5. *Does* $\mathsf{SR}^*_\kappa \restriction \mathrm{IA}_{>\omega}$ *imply that* SCH *holds at singular cardinals of cofinality* $> \omega$ *above* κ*?*

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