An extension of the Subcomplete Forcing Axiom which implies $\diamondsuit^+$

Hiroshi Sakai

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Abstract

We prove that the Subcomplete Forcing Axiom (SCFA), introduced by Jensen [6], is consistent with $\diamondsuit^+_{\omega_1}$. For this, we introduce a weaker variation of the subcompleteness of forcing notions. More precisely, for some kinds of $\diamondsuit_{\omega_1}$-like sequences $\vec{K}$, we define the notion of $\vec{K}$-subcompleteness, which is weaker than the subcompleteness. Then, the forcing axiom for $\vec{K}$-subcomplete forcing notions ($\vec{K}$-SCFA) implies SCFA. We show that $\vec{K}$-SCFA is consistent and implies $\diamondsuit^+_{\omega_1}$.

1 Introduction

Jensen [4, 7] introduced the notion of subcomplete forcing, which is weaker than the σ-closure. Besides σ-closed forcing notions, the class of all subcomplete forcing notions include Namba forcing (under the Continuum Hypothesis CH), Prikry forcing and forcing notions shooting clubs through stationary subsets of ordinals of countable cofinality. Subcomplete forcings add no new reals and preserve stationary subsets of ω_1. Also, all revised countable support iterations of subcomplete forcings are subcomplete. Jensen [5] used subcomplete forcing notions to settle the consistency strength of the Extended Namba Problem.

The original subcompleteness in [4, 5] has some technical condition which is used to prove that all revised countable support iterations of subcomplete forcings are subcomplete. Recently, Fuchs-Switzer [2] proved that this technical condition is not necessary for nice iterations developed by Miyamoto [9].

The forcing axiom for subcomplete forcing notions, which is called the Subcomplete Forcing Axiom and denoted as SCFA, is interesting as a fragment of Martin's Maximum (MM) consistent with CH. In fact, Jensen [6] proved that

SCFA is consistent with $\diamondsuit_{\omega_1}$. Jensen [6] also proved that SCFA implies several interesting consequences of MM such as the Singular Cardinal Hypothesis and the reflection of stationary sets consisting of ordinals of countable cofinality. Fuchs [1] and Fuchs-Switzer [2] investigated consequences of SCFA in more details.

Besides the above mentioned consequences of SCFA, MM has many interesting consequences which are consistent with $\diamondsuit_{\omega_1}$. For example, MM implies the reflection of stationary subsets of $\mathcal{P}_{\omega_1}(\lambda)$ for $\lambda \geq \omega_2$, which is often called the Weak Reflection Principle and denoted as WRP, Chang's Conjecture and the non-existence of ω_1-Kurepa trees. It was asked in [1] and [2] whether these consequences of MM also follows from SCFA. Here recall that

$$\mathsf{WRP} \ \Rightarrow \ \text{Chang's Conjecture} \ \Rightarrow \ \neg\exists\, \omega_1\text{-Kurepa trees} \ \Rightarrow \ \neg\diamondsuit^+_{\omega_1}\,.$$

In this paper, we answer this question negatively. Namely, we prove that if ZFC is consistent with the existence of a supercompact cardinal, then ZFC + SCFA + $\diamondsuit^+_{\omega_1}$ is consistent. (Corollary 5.8)

To prove this, we introduce the notion of $\vec{K}$-subcomplete forcing for a $\diamondsuit$-model sequence $\vec{K} = \langle K_\xi \mid \xi < \omega_1 \rangle$. It will follow from the definition that all subcomplete forcing notions are $\vec{K}$-subcomplete. So the forcing axiom for $\vec{K}$-subcomplete forcing notions, denoted as $\vec{K}$-SCFA, implies SCFA. We prove the following.

(I) For any $\diamondsuit$-model sequence $\vec{K}$, all nice iterations of $\vec{K}$-subcomplete forcings are $\vec{K}$-subcomplete. (Theorem 4.2)

(II) If ZFC is consistent with the existence of a supercompact cardinal, then ZFC is consistent with $\vec{K}$-SCFA for some $\diamondsuit$-model sequence $\vec{K}$. (Theorem 5.2)

(III) If $\vec{K}$-SCFA holds for some $\diamondsuit$-model sequence $\vec{K}$, then SCFA and $\diamondsuit^+_{\omega_1}$ holds. (Proposition 5.3 and 5.4)

This paper is organized as follows. In §2, we present our notation and basic facts used in this paper. In §3, we introduce the notion of $\vec{K}$-subcomplete forcing and study its basic properties. In §4, we discuss nice iterations of $\vec{K}$-subcomplete forcings to prove (I) above. Finally, in §5, we investigate $\vec{K}$-SCFA to prove (II) and (III).

2 Preliminaries

In this section, we present our notation and basic facts used in this paper. Consult Kunen [8] or Jech [3] for those which are not mentioned here.

2.1 $\diamondsuit$-principles

In this paper, we deal with several $\diamondsuit$-principles. Here we recall them.

- A $\diamondsuit_{\omega_1}$-*sequence* is a sequence $\langle b_\xi \mid \xi < \omega_1 \rangle$ such that for any $B \subseteq \omega_1$ there are stationary many $\xi < \omega_1$ with $B \cap \xi = b_\xi$.
- A $\diamondsuit^-_{\omega_1}$-*sequence* is a sequence $\langle \mathcal{B}_\xi \mid \xi < \omega_1 \rangle$ of countable sets such that for any $B \subseteq \omega_1$ there are stationary many $\xi < \omega_1$ with $B \cap \xi \in \mathcal{B}_\xi$.
- A $\diamondsuit^*_{\omega_1}$-*sequence* is a sequence $\langle \mathcal{B}_\xi \mid \xi < \omega_1 \rangle$ of countable sets such that for any $B \subseteq \omega_1$ there are club many $\xi < \omega_1$ with $B \cap \xi \in \mathcal{B}_\xi$.
- A $\diamondsuit^+_{\omega_1}$-*sequence* is a sequence $\langle \mathcal{B}_\xi \mid \xi < \omega_1 \rangle$ of countable sets such that for any $B \subseteq \omega_1$ there is a club $C \subseteq \omega_1$ with $B \cap \xi, C \cap \xi \in \mathcal{B}_\xi$ for all $\xi \in C$.

Let $\diamondsuit_{\omega_1}$ ($\diamondsuit^-_{\omega_1}$, $\diamondsuit^+_{\omega_1}$, respectively) be the assertion that a $\diamondsuit_{\omega_1}$-sequence (a $\diamondsuit^-_{\omega_1}$-sequence, a $\diamondsuit^+_{\omega_1}$-sequence, respectively) exists. Recall that $\diamondsuit^+_{\omega_1}$ implies $\diamondsuit^-_{\omega_1}$, and $\diamondsuit^-_{\omega_1}$ is equivalent to $\diamondsuit_{\omega_1}$. Recall also that $\diamondsuit^+_{\omega_1}$ implies the existence of an ω_1-Kurepa tree. See Kunen [8] for proofs of these facts.

2.2 Forcing and its iteration

In this paper, we follow Miyamoto [9] for notations on forcing. A *forcing notion* is a separative preorder with a largest element. Here recall that a preorder $\mathbb{P}$ is separative if for any $p, q \in \mathbb{P}$ with $p \not\leq_{\mathbb{P}} q$, there is $p' \leq_{\mathbb{P}} p$ which is incompatible with q. Recall also that a preorder $\mathbb{P}$ is separative if and only if for any $p, q \in \mathbb{P}$, $p \leq_{\mathbb{P}} q$ exactly when $p \Vdash_{\mathbb{P}}$ "$q \in \dot{G}$", where $\dot{G}$ is the canonical name for a $\mathbb{P}$-generic filter.

Let $\mathbb{P}$ be a forcing notion. The largest element of $\mathbb{P}$ is denoted as $1_{\mathbb{P}}$. For $p, q \in \mathbb{P}$, we let $p \equiv_{\mathbb{P}} q$ denote that $p \leq_{\mathbb{P}} q$ and $q \leq_{\mathbb{P}} p$. A subscripts $\mathbb{P}$ in $\leq_{\mathbb{P}}$, $\Vdash_{\mathbb{P}}$, $1_{\mathbb{P}}$ and $\equiv_{\mathbb{P}}$ is often omitted when it is clear from the context.

Next, we present notations on forcing iterations. A sequence $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$ of forcing notions, where $\delta \in \mathrm{On}$, is called an *iteration* if we have the following for all α, β with $\alpha \leq \beta < \delta$, where $\leq_\alpha$ and 1_α denote $\leq_{\mathbb{P}_\alpha}$ and $1_{\mathbb{P}_\alpha}$, respectively.

(i) $\mathbb{P}_\alpha$ consists of functions on α.

(ii) For all $p \in \mathbb{P}_\beta$ we have $p \restriction \alpha \in \mathbb{P}_\alpha$, and $1_\beta \restriction \alpha = 1_\alpha$.

(iii) For any $p \in \mathbb{P}_\alpha$ and any $q \in \mathbb{P}_\beta$, if $p \leq_\alpha q \restriction \alpha$, then $p ^\frown q \restriction [\alpha, \beta) \in \mathbb{P}_\beta$, and $p ^\frown q \restriction [\alpha, \beta) \leq_\beta q$.

(iv) For any $p, q \in \mathbb{P}_\beta$, if $p \leq_\beta q$, then $p \restriction \alpha \leq_\alpha q \restriction \alpha$, and $p \leq_\beta p \restriction \alpha ^\frown q \restriction [\alpha, \beta)$.

(v) If α is a limit ordinal, then for any $p, q \in \mathbb{P}_\alpha$, $p \leq_\alpha q$ if and only if $p \restriction \gamma \leq_\gamma q \restriction \gamma$ for all $\gamma < \alpha$.

For an iteration $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$, we let $\leq_\alpha$, 1_α and $\Vdash_\alpha$ denote $\leq_{\mathbb{P}_\alpha}$, $1_{\mathbb{P}_\alpha}$ and $\Vdash_{\mathbb{P}_\alpha}$, respectively. Also, let $\dot{G}_\alpha$ denote the canonical name for a $\mathbb{P}_\alpha$-generic filter. A subscript α in $\leq_\alpha$, 1_α and $\Vdash_\alpha$ is sometimes omitted if it is clear from the context. For $p \in \mathbb{P}_\alpha$, $\alpha = \mathrm{dom}(p)$ will be denoted as $l(p)$.

Suppose $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$ is an iteration, $\alpha \leq \beta < \delta$, and G_α is a $\mathbb{P}_\alpha$-generic filter over V. In $V[G_\alpha]$, let $\mathbb{P}_{\alpha,\beta}$ be the following forcing notion, where $\leq_{\alpha,\beta}$ and $1_{\alpha,\beta}$ are its order and largest element, respectively.

(i) $\mathbb{P}_{\alpha,\beta} := \{ p \restriction [\alpha, \beta) \mid p \in \mathbb{P}_\beta \wedge p \restriction \alpha \in G_\alpha \}$.

(ii) $p \leq_{\alpha,\beta} q$ if there are $p', q' \in G_\alpha$ such that $p' ^\frown p \leq_\beta q' ^\frown q$.

(iii) $1_{\alpha,\beta} := 1_\beta \restriction [\alpha, \beta)$.

Then, $\mathbb{P}_{\alpha,\beta}$ is a forcing notion in $V[G_\alpha]$. (See [9, Prop. 1.2].) Let $\dot{\mathbb{P}}_{\alpha,\beta}$ be a $\mathbb{P}_\alpha$-name for $\mathbb{P}_{\alpha,\beta}$.

If G_β is a $\mathbb{P}_\beta$-generic filter over V, then $G_\beta \restriction \alpha := \{ p \restriction \alpha \mid p \in G_\beta \}$ is a $\mathbb{P}_\alpha$-generic filter over V, and $G_\beta \restriction [\alpha, \beta) := \{ p \restriction [\alpha, \beta) \mid p \in G_\beta \}$ is a $(\dot{\mathbb{P}}_{\alpha,\beta})^{G_\beta \restriction \alpha}$-generic filter over $V[G_\beta \restriction \alpha]$. Also, if G_α is a $\mathbb{P}_\alpha$-generic filter over V, and H is a $(\dot{\mathbb{P}}_{\alpha,\beta})^{G_\alpha}$-generic filter over V, then

$$G_\alpha * H := \{ p \in \mathbb{P}_\beta \mid p \restriction \alpha \in G_\alpha \wedge p \restriction [\alpha, \beta) \in H \}$$

is a $\mathbb{P}_\beta$-generic filter over V. (See [9, Prop. 1.3].) So $\mathbb{P}_\beta$ is forcing equivalent to $\mathbb{P}_\alpha * \dot{\mathbb{P}}_{\alpha,\beta}$. In particular, $\mathbb{P}_{\alpha+1}$ is forcing equivalent to $\mathbb{P}_\alpha * \dot{\mathbb{P}}_{\alpha,\alpha+1}$.

2.3 Subcomplete forcing

We briefly review the notion of subcomplete forcing. Instead of the original one, we recall a simplified version, which is called ∞-subcompleteness, introduced in Fuchs-Switzer [2].

The notion of (∞-)subcomplete forcing involves models of set theory. First, we give our notation on them.

As is usual, for a first order structure M, we often identify M with its universe. For example, if we write $a \in M$, then it means that a belongs to the universe of M, and if we say that M is countable, then it means that the universe of M is countable.

For structures M and N of the same language and for a function $\sigma : M \to N$, let $\sigma : M \prec N$ denote that σ is an elementary embedding from M to N.

Let $\mathcal{L}_{\mathrm{ST}} = \{\tilde{\in}\}$ be the language of Set Theory, where $\tilde{\in}$ is a binary relation symbol for the $\in$-relation. Let $\mathcal{L}^+_{\mathrm{ST}} := \{\tilde{\in}, \tilde{P}\}$, where $\tilde{P}$ is a unary predicate symbol.

Suppose M is an $\mathcal{L}_{\mathrm{ST}}$-structure $\langle X, E\rangle$ or an $\mathcal{L}^+_{\mathrm{ST}}$-structure $\langle X, E, P\rangle$. We say that M is a model of ZF^- (ZFC^-) if M satisfies all axioms of ZF (ZFC) except for the Power Set Axiom. Here, if M is an $\mathcal{L}^+_{\mathrm{ST}}$-structure, then the Axiom Schemes of Replacement and Separation are applied to all $\mathcal{L}^+_{\mathrm{ST}}$-formulas. We say that M is *transitive* if X is a transitive set, and $E = \in \cap (X \times X)$.

For a set A and an ordinal χ, let L^A_χ denote the transitive $\mathcal{L}^+_{\mathrm{ST}}$-structure $\langle L_\chi[A], \in, A \cap L_\chi[A]\rangle$.

Suppose M is a transitive $\mathcal{L}^+_{\mathrm{ST}}$-model of ZFC^-. M is said to be *full* if there is a transitive $\mathcal{L}_{\mathrm{ST}}$-model N of ZF^- such that $M \in N$, and M is regular in N, where we say that M is *regular* in N if for any $x \in M$ and any function $f : x \to M$ with $f \in N$, we have $\mathrm{ran}(f) \in M$.

Suppose $M = \langle X, \in, P\rangle$ is a transitive $\mathcal{L}^+_{\mathrm{ST}}$-model of ZFC^-, and $\mathbb{P}$ is a forcing notion in M. For a $\mathbb{P}$-generic filter G over M, let $M[G]$ denote the $\mathcal{L}^+_{\mathrm{ST}}$-structure $\langle X[G], \in, P\rangle$, where $X[G] = \{\dot{a}^G \mid \dot{a} \text{ is a } \mathbb{P}\text{-name in } X\}$. Note that $M[G]$ is a transitive $\mathcal{L}^+_{\mathrm{ST}}$-model of ZFC^-. Note also that if M is full, then $M[G]$ is also full. In fact, if N witnesses the fullness of M, then G is $\mathbb{P}$-generic filter over N, and $N[G]$ witnesses the fullness of $M[G]$.

Now, we recall the notion of the subcompleteness. As we mentioned above, we recall a simplified version, which is called the ∞-subcompleteness, introduced by Fuchs-Switzer [2].

Definition 2.1 (Fuchs-Switzer [2]). *Suppose $\mathbb{P}$ is a forcing notion.*

For a regular cardinal θ, we say that θ verifies the ∞-subcompleteness *of $\mathbb{P}$ if $\mathbb{P} \in \mathcal{H}_\theta$, and the following hold: For any A, χ, $\bar{M}$, $\bar{\mathbb{P}}$, $\bar{b}$, σ, b and $\bar{G}$, if*

(i) A is a set, χ is an ordinal, and $\mathcal{H}_\theta \subseteq L^A_\chi \models \mathrm{ZFC}^-$,

(ii) $\bar{M}$ is a countable transitive full $\mathcal{L}^+_{\mathrm{ST}}$-model of ZFC^- with $\bar{\mathbb{P}}, \bar{b} \in \bar{M}$,

(iii) $\sigma : \bar{M} \prec L^A_\chi$, and $\sigma(\langle \bar{\mathbb{P}}, \bar{b}\rangle) = \langle \mathbb{P}, b\rangle$,

(iv) $\bar{G}$ is a $\bar{\mathbb{P}}$-generic filter over $\bar{M}$,

then there is $p^* \in \mathbb{P}$ which forces that there is a $\sigma^* : \bar{M} \prec L_\chi^A$ with $\sigma^*(\langle \bar{\mathbb{P}}, \bar{b}\rangle) = \langle \mathbb{P}, b\rangle$ and $\sigma^*[\bar{G}] \subseteq \dot{G}$, where $\dot{G}$ is the canonical name for a $\mathbb{P}$-generic filter.

We say that $\mathbb{P}$ is ∞-subcomplete *if there is a regular cardinal θ which verifies the ∞-subcompleteness of $\mathbb{P}$.*

In the above definition, note that σ^* exists in the forcing extension by $\mathbb{P}$, and it may not be in the ground model.

The original subcompleteness in Jensen [4, 5] has some additional condition which requests σ^* to have some similarity to σ. This condition is deleted in the ∞-subcompleteness. So the ∞-subcompleteness is weaker than the original subcompleteness. This additional condition is used to show that all revised countable support iterations of subcomplete forcings are subcomplete. See [4, 5] for details. Fuchs-Switzer [2] proved that this condition is not necessary for nice iterations developed by Miyamoto [9]. Namely, they proved the following.

Fact 2.2. *All nice iterations of ∞-subcomplete forcings are ∞-subcomplete.*

It should be noted here that a similar result for subproper forcings were obtained by Miyamoto [10] before.

Also, as far as we know, ∞-subcomplete forcings have all important properties of subcomplete forcings. For example, ∞-subcomplete forcings add no reals, preserve stationary subsets of ω_1 and preserve $\diamondsuit_{\omega_1}$.

Now, we turn our attention to the forcing axiom for ∞-subcomplete forcings. By Fact 2.2 and the standard argument for the consistency proof of forcing axioms, we can prove its consistency.

Definition 2.3. The (∞-)Subcomplete Forcing Axiom*, denoted as* (∞-)SCFA*, is the following assertion.*

> *For any (∞-)subcomplete forcing notion $\mathbb{P}$ and any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| \leq \omega_1$, there is a filter g on $\mathbb{P}$ such that $g \cap D \neq \emptyset$ for any $D \in \mathcal{D}$.*

Fact 2.4 (Switzer-Fuchs [2])**.** *Assume there is a supercompact cardinal. Then there is a forcing extension in which ∞-SCFA holds.*

In fact, since ∞-subcomplete forcings preserves $\diamondsuit_{\omega_1}$, if $\diamondsuit_{\omega_1}$ holds in the ground model, then so is in the extension. Note also ∞-SCFA implies SCFA since all subcomplete forcing notions are ∞-subcomplete.

3 $\vec{K}$-subcomplete forcing

In this section, we introduce the notion of $\vec{K}$-subcomplete forcing and study its basic properties. The $\vec{K}$-subcompleteness is defined for an adequate model sequence $\vec{K}$, which is a guessing sequence on ω_1. Roughly speaking, the $\vec{K}$-subcompleteness is obtained by restricting $\bar{M}$ and $\bar{G}$ in the ∞-subcompleteness to those captured by $\vec{K}$.

In §3.1, we introduce the notion of adequate model sequences and study its basic properties. In §3.2, we introduce the notion of $\vec{K}$-subcomplete forcings and study its basic properties.

3.1 Adequate model sequences

First, we introduce the notions of adequate model sequences and $\diamondsuit$-model sequences.

Definition 3.1. *A sequence $\vec{K} = \langle K_\xi \mid \xi < \omega_1 \rangle$ is called an* adequate model sequence *if*

(i) for each $\xi < \zeta$, K_ξ is a transitive $\mathcal{L}_{\mathrm{ST}}$-model ZFC$^-$*, $\xi \in K_\xi$, and ξ is countable in K_ξ,*

(ii) for any $B \subseteq \omega_1$, there are stationary many $\xi < \omega_1$ with $B \cap \xi \in K_\xi$.

An adequate model sequence $\vec{K} = \langle K_\xi \mid \xi < \omega_1 \rangle$ is called a $\diamondsuit$-model sequence *if*

(iii) K_ξ is countable for each $\xi < \omega_1$.

Note that if $\vec{K} = \langle K_\xi \mid \xi < \omega_1 \rangle$ satisfies (i) of the above definition, and $\mathcal{H}_{\omega_1} \subseteq K_\xi$ for all $\xi < \omega_1$, then $\vec{K}$ is an adequate model sequence.

The $\vec{K}$-subcompleteness is defined for an adequate model sequence $\vec{K}$. Before we give its definition, we study basic properties of adequate model sequences.

Lemma 3.2. *(1) A $\diamondsuit$-model sequence exists if and only if $\diamondsuit_{\omega_1}$ holds.*

(2) Suppose $\vec{K} = \langle K_\xi \mid \xi < \omega_1 \rangle$ is a sequence of transitive $\mathcal{L}_{\mathrm{ST}}$-models of ZFC^- *with $\xi \in K_\xi$. Then for any $R \subseteq \omega_1 \times \omega_1$, there are $B \subseteq \omega_1$ and a club $C \subseteq \omega_1$ such that for any $\xi \in C$ if $B \cap \xi \in K_\xi$, then $R \cap (\xi \times \xi) \in K_\xi$.*

(3) Suppose $\vec{K} = \langle K_\xi \mid \xi < \omega_1 \rangle$ is an adequate model sequence, and let F be the set of all $D \subseteq \omega_1$ such that $\{\xi < \omega_1 \mid B \cap \xi \in K_\xi\} \cap C \subseteq D$ for some $B \subseteq \omega_1$ and some club $C \subseteq \omega_1$. Then, F is a normal filter over ω_1.

Proof. (1) If there is a $\diamondsuit$-model sequence $\vec{K}$, then $\vec{K}$ witnesses $\diamondsuit^-_{\omega_1}$, which is equivalent to $\diamondsuit_{\omega_1}$. Conversely, suppose $\diamondsuit_{\omega_1}$ holds. Let $\langle b_\xi \mid \xi < \omega_1\rangle$ be a $\diamondsuit_{\omega_1}$-sequence, and for each $\xi < \omega_1$ take a countable K_ξ such that $b_\xi, \xi \in K_\xi \prec \langle \mathcal{H}_{\omega_1}, \in\rangle$. Then $\langle K_\xi \mid \xi < \omega_1\rangle$ is a $\diamondsuit$-model sequence.

(2) Let $\Gamma : \omega_1 \times \omega_1 \to \omega_1$ be the Gödel paring function. Then, let $B := \Gamma[R]$, and let C be the set of all $\xi < \omega_1$ such that $\Gamma[\xi \times \xi] = \xi$. Note that C is club in ω_1. If $\xi \in C$ and $B \cap \xi \in K_\xi$, then $R \cap (\xi \times \xi) = \Gamma^{-1}[B \cap \xi] \in K_\xi$ since $\xi, B \cap \xi \in K_\xi$. So B and C are as desired.

(3) We only prove the normality of F. The other properties are easily checked.

Suppose $\{D_\eta \mid \eta < \omega_1\} \subseteq F$. We show that $D := \Delta_{\eta<\omega_1} D_\eta \in F$. For each $\eta < \omega_1$, take $B_\eta \subseteq \omega_1$ and a club $C_\eta \subseteq \omega_1$ witnessing $D_\eta \in F$. Let $R := \{\langle \eta, \zeta\rangle \in \omega_1 \times \omega_1 \mid \zeta \in B_\eta\}$. By (3), we can take $B \subseteq \omega_1$ and a club $C' \subseteq \omega_1$ such that if $\xi \in C'$ and $B \cap \xi \in K_\xi$, then $R \cap (\xi \times \xi) \in K_\xi$. Let $C := C' \cap \Delta_{\eta<\omega_1} C_\eta$.

Then, A and C witnesses that $D \in F$: Assume $\xi \in C$ and $B \cap \xi \in K_\xi$. We must show that $\xi \in D_\eta$ for all $\eta < \xi$. Fix $\eta < \xi$. Since $\xi \in C'$ and $B \cap \xi \in K_\xi$, we have $R \cap (\xi \times \xi) \in K_\xi$. Then, $B_\eta \cap \xi = \{\zeta \mid \langle \eta, \zeta\rangle \in R \cap (\xi \times \xi)\} \in K_\xi$. Also, $\xi \in C_\eta$ since $\xi \in C$. So $\xi \in D_\eta$. $\square$

A definition and a lemma below are important in $\vec{K}$-subcomplete forcings.

Definition 3.3. *Suppose $\vec{K} = \langle K_\xi \mid \xi < \omega_1\rangle$ is an adequate model sequence. We call $\bar{M}$ a $\vec{K}$*-good model *if $\bar{M}$ is a transitive full $\mathcal{L}^+_{\mathrm{ST}}$-model of* ZFC$^-$ *such that $\bar{M} \in K_{\omega_1^{\bar{M}}}$, and $\bar{M}$ is countable in $K_{\omega_1^{\bar{M}}}$.*

Lemma 3.4. *Let $\vec{K} = \langle K_\xi \mid \xi < \omega_1\rangle$ be an adequate model sequence. Suppose $M = L^A_\chi$ for some set A and some regular cardinal χ, and $\mathcal{H}_{\omega_1} \in M$. Let $a \in M$. Then there are a $\vec{K}$-good model $\bar{M}$ and $\sigma : \bar{M} \prec M$ with $a \in \mathrm{ran}(\sigma)$.*

Proof. We may assume that $A \subseteq L_\chi[A]$. Take a regular cardinal $\lambda > \chi$, and let $N := \langle L_\lambda[A], \in\rangle$. Note that $M \in N \models \mathrm{ZFC}^-$. Moreover, M is regular in N:

Suppose $x \in M$, $f : x \to M$ and $f \in N$. We show that $f \in M$. Take $N' \prec N$ with $f, A, \chi \in N'$ and $\alpha := N' \cap \chi \in \chi$. Note that $f \subseteq X$ since $f \in N' \prec N$, and $|f|^N = |x|^N \in N' \cap \chi \subseteq N'$. Let $\pi : N' \to N''$ be the transitive collapse. Then $f = \pi(f) \in N''$. Note also that $N'' = \langle L_\beta[\pi(A)], \in\rangle$ for some $\beta < \chi$ and $\pi(A) = A \cap L_\alpha[A] \in M$. So $N'' \subseteq M$. Hence $f \in M$.

We will find $\bar{N}$ and τ such that

(i) $\bar{N}$ is a countable transitive $\mathcal{L}_{\mathrm{ST}}$-model of ZFC$^-$, $\bar{N} \in K_{\omega_1^{\bar{N}}}$, and $\bar{N}$ is countable in $K_{\omega_1^{\bar{N}}}$,

(ii) $\tau : \bar{N} \prec N$, and $a, M \in \mathrm{ran}(\sigma)$.

(We do not require that $\bar{N}$ is full.) If such $\bar{N}$ and τ are found, then $\bar{M} := \tau^{-1}(M)$ and $\sigma := \tau \restriction \bar{M}$ witnesses the lemma: $\bar{M}$ is full since $\bar{M} = \tau^{-1}(M) \in \bar{N} \models$ ZFC$^-$, and $\bar{M}$ is regular in $\bar{N}$ by the regularity of M in N. Note also that $\omega_1^M = \omega_1^N = \omega_1$ since $\mathcal{H}_{\omega_1} \in M$, and so $\omega_1^{\bar{M}} = \omega_1^{\bar{N}}$. Then $\bar{M}$ and σ are as desired clearly.

We construct $\bar{N}$ and τ satisfying (i) and (ii). Take $N' \prec N$ of cardinality ω_1 with $\omega_1 \cup \{a, M\} \subseteq N'$. Take a bijection $f : \omega_1 \to N'$ with $f(0) = \{a, M\}$, and let E be the pull-back of $\in \cap (N' \times N')$ by f. So f is an isomorphism from $J := \langle \omega_1, E \rangle$ to N'.

By Lemma 3.2 (3) and the adequateness of $\vec{K}$, there are stationary many $\xi < \omega_1$ such that $J \restriction \xi \in K_\xi$. Note also that there are club many $\xi < \omega_1$ such that $J \restriction \xi \prec J$ and $f[\xi] \cap \omega_1 = \xi$. So we can take $\xi > 0$ such that $J \restriction \xi \in K_\xi$, $J \restriction \xi \prec J$ and $f[\xi] \cap \omega_1 = \xi$.

Let $\pi : J \restriction \xi \to \bar{N}$ be the transitive collapse. Also, let $\tau := f \circ \pi^{-1} : \bar{N} \to N$. We claim that $\bar{N}$ and τ are as desired. Clearly, they satisfy (ii).

We check (i). Note that $\bar{N} \in K_\xi$ since $J \restriction \xi \in K_\xi$, and K_ξ is a transitive model of ZFC^-. Moreover $\bar{N}$ is countable in K_ξ since ξ is countable in K_ξ. So it suffices to prove that $\xi = \omega_1^{\bar{N}}$. For this, note that $\tau : \bar{N} \cong N \restriction f[\xi]$ is the inverse of the transitive collapse of $N \restriction f[\xi]$. Then, $\omega_1^{\bar{N}} = f[\xi] \cap \omega_1^N = f[\xi] \cap \omega_1 = \xi$. $\square$

3.2 $\vec{K}$-subcomplete forcings

First, we introduce the notion of $\vec{K}$-subcomplete forcings.

As we mentioned before, it is obtained from the ∞-subcompleteness by restricting $\bar{M}$ and $\bar{G}$ to those captured by $\vec{K}$. Note that if $\bar{M}$ is a $\vec{K}$-good model, and $\bar{\mathbb{P}}$ is a forcing notion in $\bar{M}$, then there is a $\bar{\mathbb{P}}$-generic filter $\bar{G} \in K_{\omega_1^{\bar{M}}}$ over $\bar{M}$.

In the definition below, we allow to use a parameter $a \in \mathcal{H}_\theta$ for the verification of the $\vec{K}$-subcompleteness. We will use this to prove that $\vec{K}$-subcompleteness is immutable with respect to the forcing equivalence (Lemma 3.6). Later, we will also make use of it to prove that some concrete forcing notion is $\vec{K}$-subcomplete in the proof of Lemma 5.7.

Definition 3.5. *Suppose $\mathbb{P}$ is a forcing notion and $\vec{K} = \langle K_\xi \mid \xi < \zeta \rangle$ is an adequate model sequence.*

For a regular cardinal θ with $\mathbb{P} \in \mathcal{H}_\theta$ and for $a \in \mathcal{H}_\theta$, we say that θ and a verify the $\vec{K}$-subcompleteness *of $\mathbb{P}$ if the following hold: For any A, χ, $\bar{M}$, $\bar{\mathbb{P}}$, $\bar{b}$, σ, b and $\bar{G}$, if* *(i)* *A is a set, χ is an ordinal, and $\mathcal{H}_\theta \subseteq L^A_\chi \models \mathrm{ZFC}^-$,*

(ii) *$\bar{M}$ is a $\vec{K}$-good model, and $\bar{\mathbb{P}}, \bar{b} \in \bar{M}$,*

(iii) *$\sigma : \bar{M} \prec L^A_\chi$, $a \in \mathrm{ran}(\sigma)$, and $\sigma(\langle \bar{\mathbb{P}}, \bar{b} \rangle) = \langle \mathbb{P}, b \rangle$,*

(iv) *$\bar{G}$ is a $\bar{\mathbb{P}}$-generic filter over $\bar{M}$ with $\bar{G} \in K_{\omega_1^{\bar{M}}}$,*

then there is $p^* \in \mathbb{P}$ which forces that there is $\sigma^* : \bar{M} \prec L^A_\chi$ with $\sigma^*(\langle \bar{\mathbb{P}}, \bar{b} \rangle) = \langle \mathbb{P}, b \rangle$ and $\sigma^*[\bar{G}] \subseteq \dot{G}$, where $\dot{G}$ is the canonical name for a $\mathbb{P}$-generic filter.

We say that $\mathbb{P}$ is $\vec{K}$-subcomplete *if there are a regular cardinal θ with $\mathbb{P} \in \mathcal{H}_\theta$ and $a \in \mathcal{H}_\theta$ which verify the $\vec{K}$-subcompleteness of $\mathbb{P}$.*

We make some remarks on the above definition.

(1) Every ∞-subcomplete forcing notion is $\vec{K}$-subcomplete for any adequate model sequence $\vec{K}$.

(2) Suppose $\vec{K} = \langle K_\xi \mid \xi < \omega_1 \rangle$ is an adequate model sequence with $\mathcal{H}_{\omega_1} \subseteq K_\xi$ for all $\xi < \omega_1$. Then, the $\vec{K}$-subcompleteness is the same as ∞-subcompleteness except for that we allow to use a parameter a in the $\vec{K}$-subcompleteness.

(3) If θ and a verify the $\vec{K}$-subcompleteness of $\mathbb{P}$, then any regular cardinal $\theta' \geq \theta$ together with a verify the $\vec{K}$-subcompleteness of $\mathbb{P}$.

(4) Suppose θ and a verify the $\vec{K}$-subcompleteness of $\mathbb{P}$, and A, χ, $\bar{M}$, $\bar{\mathbb{P}}$, σ, $\bar{G}$ satisfy (i)–(iv) of Definition 3.5. Suppose also that $\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_{n-1} \in \bar{M}$ and $\sigma(\bar{b}_i) = b_i$ for all $i < n$, where $n < \omega$. Then, by letting $\bar{b} := \langle \bar{b}_0, \bar{b}_1, \ldots, \bar{b}_{n-1} \rangle$, we can take p^* which forces the existence of $\sigma^* : \bar{M} \prec L^A_\chi$ with $\sigma^*(\bar{\mathbb{P}}) = \mathbb{P}$, $\sigma^*(\bar{b}_i) = b_i$ for all $i < n$ and $\sigma^*[\bar{G}] \subseteq \dot{G}$.

(5) Let $\mathbb{P}, A, \chi, \bar{M}, \sigma, \bar{G}, p^*$ be as in Definition 3.5. Suppose G is a $\mathbb{P}$-generic filter over V with $p^* \in G$, and in $V[G]$ let $\sigma^* : \bar{M} \prec L^A_\chi$ be such that $\sigma^*(\bar{\mathbb{P}}) = \mathbb{P}$ and $\sigma^*[\bar{G}] \subseteq G$. Then, in $V[G]$, σ^* can be naturally extended to $\sigma^{**} : \bar{M}[\bar{G}] \prec L^A_\chi[G]$ with $\sigma^{**}(\bar{G}) = G$ by letting $\sigma^{**}(\dot{\bar{x}}^{\bar{G}}) := \sigma^*(\dot{\bar{x}})^G$.

In the rest of this section, we observe basic properties of $\vec{K}$-subcomplete forcings. First, we prove that the $\vec{K}$-subcompleteness is preserved by forcing equivalence.

Lemma 3.6. *Let $\vec{K}$ be an adequate model sequence. If $\mathbb{P}$ is a $\vec{K}$-subcomplete forcing notion, and $\mathbb{P}'$ is a forcing notion which is forcing equivalent to $\mathbb{P}$, then $\mathbb{P}'$ is $\vec{K}$-subcomplete, too.*

Proof. It suffices to show that if there is a dense embedding between forcing notions $\mathbb{P}$ and $\mathbb{P}'$, then $\mathbb{P}$ is $\vec{K}$-subcomplete exactly when $\mathbb{P}'$ is $\vec{K}$-subcomplete. Suppose $\mathbb{P}$ and $\mathbb{P}'$ are forcing notions, and there is a dense embedding $d : \mathbb{P} \to \mathbb{P}'$. We only show that if $\mathbb{P}'$ is $\vec{K}$-subcomplete, then so is $\mathbb{P}$. The proof of the other direction is similar and is left to the readers. Let $\vec{K} = \langle K_\xi \mid \xi < \omega_1 \rangle$.

Take θ and a' which verify $\vec{K}$-subcompleteness of $\mathbb{P}'$. We may assume $a := \{a', \mathbb{P}', d\} \in \mathcal{H}_\theta$. We show that θ and a verify the $\vec{K}$-subcompleteness of $\mathbb{P}$. Suppose A, χ, $\bar{M}$, $\bar{\mathbb{P}}$, $\bar{b}$, σ, b and $\bar{G}$ satisfy (i)–(iv) of Definition 3.5. We find $p^* \in \mathbb{P}$ as in Definition 3.5. Let $M := L^A_\chi$.

Note that $\mathbb{P}', d \in \mathrm{ran}(\sigma)$ since $a \in \mathrm{ran}(\sigma)$. Let $\langle \bar{\mathbb{P}}', \bar{d} \rangle := \sigma^{-1}(\langle \mathbb{P}', d \rangle)$. Then $\bar{d} : \bar{\mathbb{P}} \to \bar{\mathbb{P}}'$ is a dense embedding in $\bar{M}$. Let $\bar{G}'$ be the filter on $\bar{\mathbb{P}}'$ generated by $\bar{d}[\bar{G}]$. Then $\bar{G}'$ is a $\bar{\mathbb{P}}'$-generic filter over $\bar{M}$ with $\bar{G}' \in K_{\omega_1^{\bar{M}}}$. Since θ and a' verify the $\vec{K}$-subcompleteness of $\mathbb{P}'$, we can take $p' \in \mathbb{P}'$ forcing the existence of $\sigma' : \bar{M} \to M$ with $\sigma'(\langle \bar{\mathbb{P}}, \bar{b}, \bar{\mathbb{P}}', \bar{d} \rangle) = \langle \mathbb{P}, b, \mathbb{P}', d \rangle$ and $\sigma'[\bar{G}'] \subseteq \dot{G}'$, where $\dot{G}'$ is the canonical name for a $\mathbb{P}'$-generic filter. Take $p^* \in \mathbb{P}$ with $d(p^*) \leq p'$.

We show that p^* is as desired. Suppose G is a $\mathbb{P}$-generic filter over V with $p^* \in G$. Working in $V[G]$, we show that there is $\sigma^* : \bar{M} \prec M$ with $\sigma^*(\langle \bar{\mathbb{P}}, \bar{b} \rangle) = \langle \mathbb{P}, b \rangle$ and $\sigma^*[\bar{G}] \subseteq G$. Let G' be the filter on $\mathbb{P}'$ generated by $d[G]$. Then G' is a $\mathbb{P}'$-generic filter over V with $p' \in G'$. By the choice of p', we can take $\sigma^* : \bar{M} \prec M$ with $\sigma^*(\langle \bar{\mathbb{P}}, \bar{b}, \bar{\mathbb{P}}', \bar{d} \rangle) = \langle \mathbb{P}, b, \mathbb{P}', d \rangle$ and $\sigma^*[\bar{G}'] \subseteq \dot{G}'$. Then,

$$\sigma^*[\bar{G}] = \sigma^*[\bar{d}^{-1}[\bar{G}']] \subseteq d^{-1}[G'] = G\,.$$

So σ^* is as desired. □

Next, recall that ∞-subcomplete forcings add no reals. We observe that this can be generalized to $\vec{K}$-subcomplete forcings.

Lemma 3.7. *Suppose that $\vec{K}$ is an adequate model sequence. Then any $\vec{K}$-subcomplete forcing adds no reals. In particular, every $\vec{K}$-subcomplete forcing preserves ω_1.*

Proof. Suppose $\mathbb{P}$ is a $\vec{K}$-subcomplete forcing notion, $p \in \mathbb{P}$ and $\dot{x}$ is a $\mathbb{P}$-name for a real. It suffices to find $p^* \leq p$ forcing that $\dot{x} \in V$.

Take a regular cardinal θ and $a \in \mathcal{H}_\theta$ verifying the $\vec{K}$-subcompleteness of $\mathbb{P}$. We may assume that $\omega_1 < \theta$ and $\dot{x} \in \mathcal{H}_\theta$. Take a set A and a regular cardinal χ such that $\mathcal{H}_\theta \subseteq L^A_\chi$. Let $M := L^A_\chi$.

By Lemma 3.4, we can take a $\vec{K}$-good model $\bar{M}$ and $\sigma : \bar{M} \prec M$ such that and $a, \mathbb{P}, p, \dot{x} \in \mathrm{ran}(\sigma)$. Let $\langle \bar{\mathbb{P}}, \bar{p}, \dot{\bar{x}} \rangle := \sigma^{-1}(\langle \mathbb{P}, p, \dot{x} \rangle)$. Since $\bar{M}$ is countable in $K_{\omega_1^{\bar{M}}}$, we can take a $\bar{\mathbb{P}}$-generic filter $\bar{G}$ over $\bar{M}$ with $\bar{p} \in \bar{G} \in K_{\omega_1^{\bar{M}}}$. Let $x := \dot{\bar{x}}^{\bar{G}}$.

Since θ and a verify the $\vec{K}$-subcompleteness of $\mathbb{P}$, we can take $p^* \in \mathbb{P}$ which forces the existence of $\sigma^* : \bar{M} \prec M$ with $\sigma^*(\langle \mathbb{P}, \dot{\bar{x}}, \bar{p} \rangle) = \langle \mathbb{P}, \dot{x}, p \rangle$ and $\sigma^*[\bar{G}] \subseteq \dot{G}$, where $\dot{G}$ is the canonical name for a $\mathbb{P}$-generic filter. p^* forces (v)–(vii) of Definition 3.5 for $\bar{b} = \langle \bar{p}, \dot{\bar{x}} \rangle$. We claim that p^* is as desired.

Note that $p^* \Vdash$ "$p = \sigma^*(\bar{p}) \in \sigma^*[\bar{G}] \subseteq \dot{G}$". So $p^* \Vdash$ "$p \in \dot{G}$". Thus $p^* \leq p$.

We prove that $p^* \Vdash$ "$\dot{x} \in V$". Suppose G is a $\mathbb{P}$-generic filter over V with $p^* \in G$. We show that $\dot{x}^G \in V$. In $V[G]$, take $\sigma^* : \bar{M} \prec M$ with $\sigma^*(\langle \mathbb{P}, \dot{\bar{x}}, \bar{p} \rangle) = \langle \mathbb{P}, \dot{x}, p \rangle$ and $\sigma^*[\bar{G}] \subseteq G$. Then, σ^* can be extended to $\sigma^{**} : \bar{M}[\bar{G}] \prec M[G]$ with $\sigma^{**}(\bar{G}) = G$. Then, $\dot{x}^G = \sigma^{**}(\dot{\bar{x}}^{\bar{G}}) = \sigma^{**}(x)$. But $\sigma^{**}(x) = x$ since x is a real. So, $\dot{x}^G = x \in V$. □

Next, we show that $\vec{K}$-subcomplete forcings preserve the adequateness of $\vec{K}$. The latter statement of the next lemma corresponds to the fact that ∞-subcomplete forcings preserve $\diamondsuit_{\omega_1}$.

Lemma 3.8. *If $\vec{K}$ is an adequate model sequence, then $\vec{K}$ remains to be an adequate model sequence in any $\vec{K}$-subcomplete forcing extensions. If $\vec{K}$ is a $\diamondsuit$-model sequence, then $\vec{K}$ remains to be a $\diamondsuit$-model sequence in any $\vec{K}$-subcomplete forcing extensions.*

Proof. The latter statement follows from the former. We prove the former.

Let $\vec{K} = \langle K_\xi \mid \xi < \omega_1 \rangle$ be an adequate model sequence and $\mathbb{P}$ be a $\vec{K}$-subcomplete forcing notion. Suppose $p \in \mathbb{P}$, $\dot{B}$ and $\dot{C}$ are $\mathbb{P}$-names, and p forces that $\dot{B} \subseteq \omega_1$ and $\dot{C}$ is club in ω_1. It suffices to find $p^* \leq p$ and $\xi < \omega_1$ such that p^* forces $\dot{B} \cap \xi \in K_\xi$ and $\xi \in \dot{C}$.

Take a regular cardinal θ and $a \in \mathcal{H}_\theta$ verifying the $\vec{K}$-subcompleteness of $\mathbb{P}$. We may assume $\dot{B}, \dot{C} \in \mathcal{H}_\theta$. Take a set A and a regular cardinal χ with $M := L^A_\chi \supseteq \mathcal{H}_\theta$.

By Lemma 3.4, we can take a $\vec{K}$-good model $\bar{M}$ and $\sigma : \bar{M} \prec M$ with $a, \mathbb{P}, p, \dot{B}, \dot{C} \in \operatorname{ran}(\sigma)$. Let $\xi := \omega_1^{\bar{M}}$, and let $\langle \bar{\mathbb{P}}, \bar{p}, \dot{\bar{B}}, \dot{\bar{C}} \rangle := \sigma^{-1}(\langle \mathbb{P}, p, \dot{B}, \dot{C} \rangle)$. Take a $\bar{\mathbb{P}}$-generic filter over $\bar{M}$ with $\bar{p} \in \bar{G} \in K_\xi$. Then, there is $p^* \in \mathbb{P}$ which forces the existence of $\sigma^* : \bar{M} \prec M$ with $\sigma^*(\langle \bar{\mathbb{P}}, \bar{p}, \dot{\bar{B}}, \dot{\bar{C}} \rangle) = \langle \mathbb{P}, p, \dot{B}, \dot{C} \rangle$ and $\sigma^*[\bar{G}] \subseteq \dot{G}$, where $\dot{G}$ is the canonical name for a $\mathbb{P}$-generic filter. We claim that p^* and ξ are as desired. We can prove that $p^* \leq p$ by the same argument as in the proof of Lemma 3.7.

Suppose G is a $\mathbb{P}$-generic filter over V with $p^* \in G$. Let $B := \dot{B}^G$ and $C := \dot{C}^G$. In $V[G]$, we show that $B \cap \xi \in K_\xi$ and $\xi \in C$. Take $\sigma^{**} : \bar{M}[\bar{G}] \prec M[G]$ with $\sigma^{**}(\langle \bar{\mathbb{P}}, \bar{p}, \dot{\bar{B}}, \dot{\bar{C}} \rangle) = \langle \mathbb{P}, p, \dot{B}, \dot{C} \rangle$ and $\sigma^{**}(\bar{G}) = G$. Let $\bar{B} := \dot{\bar{B}}^{\bar{G}}$ and $\bar{C} := \dot{\bar{C}}^{\bar{G}}$. Then, $\sigma^{**}(\bar{B}) = B$, $\sigma^{**}(\bar{C}) = C$, and $\sigma^{**}(\xi) = \omega_1$. Then, $\bar{B} = B \cap \xi$.

But $\bar{B} \in K_\xi$ since $\bar{B} \in \bar{M}[\bar{G}] \in K_\xi$. So $B \cap \xi \in K_\xi$. Also, $\bar{C} = C \cap \xi$. Moreover $\bar{C}$ is unbounded in ξ by the elementarity of σ^*. Then $\xi \in C$ since C is closed. $\square$

Recall that all ∞-subcomplete forcings preserve stationary subsets of ω_1. We prove that $\vec{K}$-subcomplete forcings preserve stationary subsets of ω_1 if $\vec{K}$ is strongly adequate in the sense below.

Definition 3.9. *An adequate model sequence $\vec{K} = \langle K_\xi \mid \xi < \omega_1 \rangle$ is called a* strongly adequate model sequence *if for any $B \subseteq \omega_1$, there are club many $\xi < \omega_1$ with $B \cap \xi \in K_\xi$.*

Lemma 3.10. *Suppose $\vec{K}$ is a strongly adequate model sequence, and $\mathbb{P}$ is a $\vec{K}$-subcomplete forcing notion. Then $\mathbb{P}$ preserves stationary subsets of ω_1.*

To prove this, we use the following modification of Lemma 3.4.

Lemma 3.11. *Let $\vec{K} = \langle K_\xi \mid \xi < \omega_1 \rangle$ be a strongly adequate model sequence. Suppose $M = L^A_\chi$ for some set A and some regular cardinal χ, $\mathcal{H}_{\omega_1} \in M$, and $a \in M$. Let S be a stationary subset of ω_1. Then, there are a $\vec{K}$-good model $\bar{M}$ with $\omega_1^{\bar{M}} \in S$ and $\sigma : \bar{M} \prec M$ with $a \in \mathrm{ran}(\sigma)$.*

Proof. The proof is almost the same as Lemma 3.4. In the fifth paragraph of the proof of Lemma 3.4, we took $\xi < \omega_1$ such that $J \restriction \xi \in K_\xi$, $J \restriction \xi \prec J$ and $f[\xi] \cap \omega_1 = \xi$. Note that we can take such $\xi \in S$ under the assumption of this lemma that $\vec{K}$ is strongly adequate and S is stationary. Then, the rest of the proof is exactly the same as that of Lemma 3.4. $\square$

Proof of Lemma 3.10. Let S be a stationary subset of ω_1. Suppose $p \in \mathbb{P}$, $\dot{C}$ is a $\mathbb{P}$-name and p forces $\dot{C}$ to be a club subset of ω_1. We find $p^* \leq p$ and $\xi \in S$ such that $p \Vdash$ "$\xi \in \dot{C}$".

Take a regular cardinal θ and $a \in \mathcal{H}_\theta$ verifying the $\vec{K}$-subcompleteness of $\mathbb{P}$. We may assume $\dot{C} \in \mathcal{H}_\theta$. Take a set A and a regular cardinal χ with $M := L^A_\chi \supseteq \mathcal{H}_\theta$.

By Lemma 3.11, we can take a $\vec{K}$-good model $\bar{M}$ with $\xi := \omega_1^{\bar{M}} \in S$ and $\sigma : \bar{M} \prec M$ with $a, \mathbb{P}, p, \dot{C} \in \mathrm{ran}(\sigma)$. Let $\langle \bar{\mathbb{P}}, \bar{p}, \dot{\bar{C}} \rangle := \sigma^{-1}(\langle \mathbb{P}, p, \dot{C} \rangle)$. Take a $\bar{\mathbb{P}}$-generic filter over $\bar{M}$ with $\bar{p} \in \bar{G} \in K_\xi$. Then, there is $p^* \in \mathbb{P}$ which forces the existence of $\sigma^* : \bar{M} \prec M$ with $\sigma^*(\langle \bar{\mathbb{P}}, \bar{p}, \dot{\bar{C}} \rangle) = \langle \mathbb{P}, p, \dot{C} \rangle$ and $\sigma^*[\bar{G}] \subseteq \dot{G}$, where $\dot{G}$ is the canonical name for a $\mathbb{P}$-generic filter.

We can prove that $p^* \leq p$ by the same argument as in the proof of Lemma 3.7. Recall also that $\xi \in S$. Moreover, we can prove that $p^* \Vdash$ "$\xi \in \dot{C}$" by the same argument as in the proof of Lemma 3.8. So p^* and ξ are as desired. $\square$

We have proved that $\vec{K}$-subcomplete forcings preserve stationary subsets of ω_1 if $\vec{K}$ is strongly adequate. But this is not true for an adequate $\vec{K}$ in general. In fact, a $\vec{K}$-subcomplete forcing $\mathbb{C}_{\vec{K},B}$ in §5 does not preserve stationary subsets of ω_1 if $\vec{K}$ is not strongly adequate, and B is its witness. See a remark after the proof of Proposition 5.4.

4 Nice iterations of $\vec{K}$-subcomplete forcings

In this section, we discuss nice iterations of $\vec{K}$-subcomplete forcings. Here an iteration of $\vec{K}$-subcomplete forcings means the following.

Definition 4.1. *Suppose $\vec{K}$ is an adequate model sequence. An iteration $\langle \mathbb{P}_\alpha \mid \alpha \leq \delta \rangle$ is called an* iteration of $\vec{K}$-subcomplete forcings *if*

$$\Vdash_\alpha \text{“}\vec{K} \text{ is an adequate model sequence, and } \dot{\mathbb{P}}_{\alpha,\alpha+1} \text{ is } \vec{K}\text{-subcomplete”}$$

for all $\alpha < \delta$.

We prove the following.

Theorem 4.2. *Suppose $\vec{K}$ is an adequate model sequence, and $\langle \mathbb{P}_\alpha \mid \alpha \leq \delta \rangle$ is a nice iteration of $\vec{K}$-subcomplete forcings. Then $\mathbb{P}_\delta$ is $\vec{K}$-subcomplete.*

The proof of this theorem is almost the same as the analogous theorem for ∞-subcomplete forcings (Fact 2.2). But, we give the proof for the completeness of this paper.

In §4.1, we briefly review nice iterations developed by Miyamoto [9]. In §4.2, we prove Theorem 4.2.

4.1 Nice iterations

Here we review nice iterations introduced by Miyamoto [9]. Key notions in nice iterations are those of nested antichains and their mixture.

We begin with the notion of nested antichains. In the definition below, S is essentially a tree of height ω consisting of conditions in $\bigcup\{\mathbb{P}_\alpha \mid \alpha < \delta\}$: S_n is the n-th level of S, and $\mathrm{suc}^n_S(s)$ is the set of immediate successors of $s \in S_n$.

Definition 4.3. *Let $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$ be an iteration. A* nested antichain *in $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$ is a pair $S = \langle \langle S_n \mid n < \omega \rangle, \langle \mathrm{suc}^n_S \mid n < \omega \rangle \rangle$ such that*

(i) $S_0 = \{s_0\}$ *for some* $s_0 \in \bigcup_{\alpha<\delta} \mathbb{P}_\alpha$,

(ii) $S_n \subseteq \bigcup_{\alpha<\delta} \mathbb{P}_\alpha$ *for all* $n < \omega$,

(iii) $\mathrm{suc}^n_S : S_n \to \mathcal{P}(S_{n+1})$ *and* $S_{n+1} = \bigcup\{\mathrm{suc}_n(s) \mid s \in S_n\}$ *for each* $n < \omega$,

(iv) if $s \in S_n$, *and* $s' \in \mathrm{suc}^n_S(s)$, *then* $l(s) \leq l(s')$, *and* $s' \restriction l(s) \leq s$,

(v) if $s \in S_n$, *then* $\langle s' \restriction l(s) \mid s' \in \mathrm{suc}^n_S(s)\rangle$ *is a maximal antichain below* s *in* $\mathbb{P}_{l(s)}$.

For a nested antichain S, we let $\langle S_n \mid n < \omega\rangle$ and $\langle \mathrm{suc}^n_S \mid n < \omega\rangle$ denote those such that $S = \langle\langle S_n \mid n < \omega\rangle, \langle \mathrm{suc}^n_S \mid n < \omega\rangle\rangle$. We write $s \in S$ for $s \in \bigcup_{n<\omega} S_n$. Scripts n and S in suc^n_S will sometimes be omitted if they are clear from the context.

Suppose S is a nested antichain in some iteration. A unique element of S_0 is called a *root* of S and denoted as $\mathrm{rt}(S)$. Suppose $n \leq n' < \omega$, $s \in S_n$ and $s' \in T_{n'}$. We write $(s, n) \leq_S (s', n')$ if there is a sequence $\langle s_m \mid n \leq m \leq n'\rangle$ such that $s_n = s$, $s_{n'} = s'$ and $s_{m+1} \in \mathrm{suc}^m_S(s_m)$ for all m.

Next, we recall mixtures of nested antichains. A nested antichain S can be identified with some condition in an iteration, and such condition is called a mixture of S. Here we adopt the following definition of mixtures, which [9, Proposition 2.5] proved to be equivalent to the original definition. The author believes that the following can be more intuitively understood.

Definition 4.4. *Let* S *be a nested antichain in an iteration* $\langle \mathbb{P}_\alpha \mid \alpha < \delta\rangle$ *with* $\mathrm{rt}(S) = s_0$. *For* $\beta < \delta$ *and* $p \in \mathbb{P}_\beta$, *we say that* p *is a* mixture *of* S *up to* β *if the following holds, where* $\dot{G}_\alpha$ *is the canonical name for a* $\mathbb{P}_\alpha$*-generic filter.*

(i) $p \equiv s_0 \restriction \beta$ *if* $\beta < l(s_0)$, *and* $p \restriction l(s_0) \equiv s_0$ *if* $\beta \geq l(s_0)$.

(ii) For any $s \in S$, $s \restriction \beta \leq p$ *if* $\beta < l(s)$, *and* $s \leq p \restriction l(s)$ *if* $\beta \geq l(s)$.

(iii) For any $s \in S$ *with* $l(s) \leq \beta$ *and any* $s' \in \mathrm{suc}(s)$,

- $s' \restriction l(s)^\frown p \restriction [l(s), \beta) \equiv s' \restriction \beta$ *if* $\beta < l(s')$,
- $s' \restriction l(s)^\frown p \restriction [l(s), l(s')) \equiv s'$ *if* $\beta \geq l(s')$.

(iv) For any $\alpha < \beta$ *and any* $u \in \mathbb{P}_\alpha$, *if* u *forces the following* $(*)$, *then we have* $u^\frown 1_\beta \restriction [\alpha, \beta) \equiv u^\frown p \restriction [\alpha, \beta)$.

  $(*)$ There is a sequence $\langle s_n \mid n < \omega\rangle \in \prod_{n<\omega} S_n$ *such that* $l(s_n) < \alpha$, $s_{n+1} \in \mathrm{suc}(s_n)$, *and* $s_n \in \dot{G}_{l(s_n)}$ *for all* $n < \omega$.

For a limit ordinal $\beta \leq \delta$ *and a sequence* p *on* β, *we say that* p *is* (S, β)-nice *if* $p \restriction \beta' \in \mathbb{P}_{\beta'}$, *and* $p \restriction \beta'$ *is a mixture of* S *up to* β' *for all* $\beta' < \beta$.

Note that if $p \in \mathbb{P}_\beta$ is a mixture of a nested antichain S up to β, then $q \in \mathbb{P}_\beta$ is a mixuture of S up to β if and only if $p \equiv q$. For p being (S, β)-nice, we do not request p to be in $\mathbb{P}_\beta$.

Now, we recall the notion of nice iterations.

Definition 4.5. *An iteration $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$, where $\delta \in \mathrm{On}$, is called a* nice iteration *if it satisfies the following.*

(i) For any $\alpha < \delta$ with $\alpha + 1 \leq \delta$, if $p \in \mathbb{P}_\alpha$, and $\dot{q}$ is a $\mathbb{P}_\alpha$-name such that $p \Vdash_\alpha$ "$\dot{q} \in \dot{\mathbb{P}}_{\alpha,\alpha+1}$ ", then there is $r \in \mathbb{P}_{\alpha+1}$ such that $r \restriction \alpha \equiv p$ and $p \Vdash_\alpha$ "$r \restriction [\alpha, \alpha+1) \equiv \dot{q}$ ".

(ii) For any limit ordinal $\beta < \delta$, $\mathbb{P}_\beta$ consists of all sequences p on β such that p is (S, β)-nice for some nested antichain S in $\langle \mathbb{P}_\alpha \mid \alpha < \beta \rangle$.

[9, Lemma 2.9] proved that if $\langle \mathbb{P}_\alpha \mid \alpha < \beta \rangle$ is an iteration for a limit ordinal β, then we can extend it to an iteration $\langle \mathbb{P}_\alpha \mid \alpha \leq \beta \rangle$ so that $\mathbb{P}_\beta$ satisfies (ii) of the above definition. So we can recursively construct a nice iteration as usual: Suppose Q is a class function, and $Q(\mathbb{P})$ is a $\mathbb{P}$-name of a forcing notion for all forcing notion $\mathbb{P}$. Then for any ordinal δ, we can construct a nice iteration $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$ such that $\Vdash_\alpha$ " $\dot{\mathbb{P}}_{\alpha,\alpha+1}$ is forcing equivalent to $Q(\mathbb{P}_\alpha)$ " for all $\alpha < \delta$.

In the rest of this subsection, we recall technical notions developed in [9], which will be used to prove Theorem 4.2.

First, we recall the notion of hooking.

Definition 4.6. *Suppose S and T are nested antichains in some iteration. We say that S* hooks *T and write $S \angle T$ if for any $n < \omega$ and any $s \in S_n$ there is $t \in T_{n+1}$ such that $l(t) \leq l(s)$ and $s \restriction l(t) \leq t$.*

It is easy to see that if S and T are nested antichains with $S \angle T$, and p, q are mixtures of S, T up to some ordinal, respectively, then $p \leq q$. We will use the following lemma. See [9] for the proof.

Lemma 4.7 ([9, Lemma 2.11])**.** *Let $\langle \mathbb{P}_\alpha \mid \alpha \leq \delta \rangle$ is a nice iteration for a limit ordinal δ. Suppose $p, q \in \mathbb{P}_\delta$, and S, s_1 satisfies the following.*

(i) S is a nested antichain in $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$, and $s \in S_1$.

(ii) p is a mixture of S up to δ.

(iii) $q \restriction l(s_1) \leq s_1$, and $q \leq p$.

Then there is a nested antichain T in $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$ such that q is a mixture of T up to δ and $T \angle S$.

Next, we recall the notion of fusion structures.

Definition 4.8. *Let $\langle \mathbb{P}_\alpha \mid \alpha \leq \delta \rangle$ be an iteration for a limit ordinal δ, and let S be a nested antichain in $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$. A* fusion structure *of S is a sequence $\langle q^{(s,n)}, T^{(s,n)} \mid n < \omega,\ s \in S_n \rangle$ which satisfies the following properties for all $n < \omega$ and $s \in S_n$.*

(i) $T^{(s,n)}$ is a nested antichain in $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$.

(ii) $q^{(s,n)}$ is a mixture of $T^{(s,n)}$ up to δ.

(iii) $l(\mathrm{rt}(T^{(s,n)})) = l(s)$, and $s \leq \mathrm{rt}(T^{(s,n)})$.

(iv) If $s' \in \mathrm{suc}^n_S(s)$, then $T^{(s',n+1)} \angle T^{(s,n)}$, and so $q^{(s',n+1)} \leq q^{(s,n)}$.

The following is a key lemma on a fusion structure. See [9] for the proof.

Lemma 4.9 ([9, Proposition 3.5])**.** *Let $\langle \mathbb{P}_\alpha \mid \alpha \leq \delta \rangle$ be an iteration for a limit ordinal δ. Suppose S is a nested antichain in $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$, and $\langle q^{(s,n)}, T^{(s,n)} \mid n < \omega,\ s \in S_n \rangle$ is a fusion structure of S. Assume G_δ is a $\mathbb{P}_\delta$-generic filter over V containing a mixture of S up to δ. Then, in $V[G]$, there is $\langle s_n \mid n < \omega \rangle \in \prod_{n<\omega} S_n$ such that $s_{n+1} \in \mathrm{suc}^n_S(s_n)$ and $q^{(s_n,n)} \in G_\delta$ for all $n < \omega$.*

4.2 Nice iterations of $\vec{K}$-subcomplete forcings

Here we prove Theorem 4.2. It will be proved by induction the length δ of the iteration $\langle \mathbb{P}_\alpha \mid \alpha \leq \delta \rangle$. In fact, as is usual, we prove something stronger by induction. We use the following notation.

Definition 4.10. *Let $\vec{K} = \langle K_\xi \mid \xi < \omega_1 \rangle$ be an adequate model sequence and $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha \mid \alpha \leq \delta \rangle$ be an iteration. Suppose θ is a regular cardinal with $\mathbb{P}_\delta \in \mathcal{H}_\theta$, $a \in \mathcal{H}_\theta$, $\beta < \delta$. We say that θ and a* verify $\vec{K}$-subcompleteness *of $\mathbb{P}_\delta$* relative to *β if the following holds: For any A, χ, b, $\bar{M}$, $\bar{b}$, $\bar{\vec{\mathbb{P}}} = \langle \bar{\mathbb{P}}_{\bar{\alpha}} \mid \bar{\alpha} \leq \bar{\delta} \rangle$, $\bar{\beta}$, $\bar{G}_{\bar{\beta}}$, $\bar{G}_{\bar{\delta}}$ and p if*

(i) A is a set, χ is an ordinal, $\mathcal{H}_\theta \subseteq L^A_\chi \models \mathrm{ZFC}^-$, and $b \in L^A_\chi$,

(ii) $\bar{M}$ is a $\vec{K}$-good model,

(iii) $\bar{b}, \bar{\vec{\mathbb{P}}}, \bar{\beta} \in \bar{M}$, $\bar{\vec{\mathbb{P}}}$ is an iteration in $\bar{M}$, and $\bar{\beta} < \bar{\delta}$,

(iv) $\bar{G}_{\bar{\delta}}$ is a $\bar{\mathbb{P}}_{\bar{\delta}}$-generic filter over $\bar{M}$ with $\bar{G}_{\bar{\delta}} \in K_\xi$, and $\bar{G}_{\bar{\beta}} = \bar{G}_{\bar{\delta}} \restriction \bar{\beta}$,

(v) $p \in \mathbb{P}_\beta$, and p forces the existence of $\sigma : \bar{M} \prec L^A_\chi$ with $a \in \mathrm{ran}(\sigma)$, $\sigma(\langle \vec{\bar{\mathbb{P}}}, \bar{\beta}, \bar{b} \rangle) = \langle \vec{\mathbb{P}}, \beta, b \rangle$ and $\sigma[\bar{G}_{\bar{\beta}}] \subseteq \dot{G}_\beta$,

then there is $p^* \in \mathbb{P}_\delta$ such that $p^* \restriction \beta = p$, and p^* forces the existence of $\sigma^* : \bar{M} \prec L^A_\chi$ with $\sigma^*(\langle \vec{\bar{\mathbb{P}}}, \bar{b} \rangle) = \langle \vec{\mathbb{P}}, b \rangle$ and $\sigma^*[\bar{G}_{\bar{\delta}}] \subseteq \dot{G}_\delta$.

We prove the following proposition by induction δ. Note that Theorem 4.2 follows from this proposition for $\alpha = 0$.

Proposition 4.11. *Let $\vec{K}$ be a model sequence and $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha \mid \alpha \leq \delta \rangle$ be a nice iteration of $\vec{K}$-subcomplete forcings. Suppose $\beta < \delta$. Then there are a regular cardinal θ and $a \in \mathcal{H}_\theta$ verifying the $\vec{K}$-subcompleteness of $\mathbb{P}_\delta$ relative to β.*

Proof. We prove the proposition by induction on the length δ of an iteration $\vec{\mathbb{P}}$. We have nothing to do for $\delta = 0$. Suppose $\delta > 0$, and the proposition holds for all $\delta' < \delta$. We prove the proposition for δ.

Case 1. δ is a limit ordinal.

By the induction hypothesis, for each α, γ with $\alpha < \gamma < \delta$, take $\theta_{\alpha,\gamma}$ and $a_{\alpha,\gamma}$ which verify the $\vec{K}$-subcompleteness of $\mathbb{P}_\gamma$ relative to α. Let θ be a regular cardinal such that $\theta > \theta_{\alpha,\gamma}$ for all α, γ. Also, let $a := \langle a_{\alpha,\gamma} \mid \alpha < \gamma < \delta \rangle$. We show that θ and a verify the $\vec{K}$-subcompleteness of $\mathbb{P}_\delta$ relative to β.

Suppose A, χ, b, $\bar{M}$, $\bar{b}$, $\vec{\bar{\mathbb{P}}} = \langle \bar{\mathbb{P}}_{\bar{\alpha}} \mid \bar{\alpha} \leq \bar{\delta} \rangle$, $\bar{\beta}$, $\bar{G}_{\bar{\beta}}$, $\bar{G}_{\bar{\delta}}$ and p satisfies (i)–(v) of Definition 4.10. We will show that there is $p^* \in \mathbb{P}_\delta$ as in Definition 4.10. Let $M := L^A_\chi$ and $\xi := \omega_1^{\bar{M}}$. For each $\bar{\alpha} < \bar{\delta}$, let $\bar{G}_{\bar{\alpha}} := \bar{G}_{\bar{\delta}} \restriction \bar{\alpha}$. Take an enumeration $\langle \bar{c}_n \mid n < \omega \rangle$ of $\bar{M}$ with $\bar{c}_0 = \emptyset$ and an enumeration $\langle \bar{r}_n \mid n < \omega \rangle$ of $\bar{G}_{\bar{\delta}}$ such that $\bar{r}_0$ is the largest element $\bar{1}_{\bar{\delta}}$ in $\bar{\mathbb{P}}_{\bar{\delta}}$.

We will construct a nested antichain S in $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$ with $\mathrm{rt}(S) = p$ and a fusion structure $\langle q^{(s,n)}, T^{(s,n)} \mid n < \omega,\ s \in S_n \rangle$ of S together with $\bar{q}^{(s,n)}$, $\bar{T}^{(s,n)}$, $\bar{\beta}^{(s,n)}$, $\dot{\sigma}^{(s,n)}$, $c^{(s,n)}$ for $n < \omega$ and $s \in S_n$ so that

(i) $\bar{q}^{(s,n)} \in \bar{G}_{\bar{\delta}}$, $\bar{q}^{(s,n)} \leq \bar{r}_n$, and $\bar{T}^{(s,n)}, \bar{\beta}^{(s,n)} \in \bar{M}$,

(ii) $\dot{\sigma}^{(s,n)}$ is a $\mathbb{P}_{l(s)}$-name, and s forces that

- $\dot{\sigma}^{(s,n)} : \bar{M} \prec M$, $a \in \mathrm{ran}(\dot{\sigma}^{(s,n)})$, and $\dot{\sigma}^{(s,n)}(\langle \vec{\bar{\mathbb{P}}}, \bar{b} \rangle) = \langle \vec{\mathbb{P}}, b \rangle$,
- $\dot{\sigma}^{(s,n)}(\langle \bar{q}^{(u,m)}, \bar{T}^{(u,m)}, \bar{c}_m \rangle) = \langle q^{(u,m)}, T^{(u,m)}, c^{(u,m)} \rangle$ for all $(u,m) \leq_S (s,n)$,
- $\dot{\sigma}^{(s,n)}(\bar{\beta}^{(s,n)}) = l(s)$, and $\dot{\sigma}^{(s,n)}[\bar{G}_{\bar{\beta}^{(s,n)}}] \subseteq \dot{G}_{l(s)}$.

First, assuming the above objects are constructed, we show that there is p^* as desired. Since $\vec{\mathbb{P}}$ is a nice iteration, we can take a mixture $p^* \in \mathbb{P}_\delta$ of S. Note that $p^* \restriction \beta \equiv \mathrm{rt}(S) = p$. So we may assume $p^* \restriction \beta = p$. Suppose G_δ is a $\mathbb{P}_\delta$-deneric filter over V with $p^* \in G_\delta$. Working in $V[G_\delta]$, we show that there is $\sigma^* : \bar{M} \prec M$ with $\sigma^*(\langle \bar{\vec{\mathbb{P}}}, b\rangle) = \langle \vec{\mathbb{P}}, b\rangle$, and $\sigma^*[\bar{G}_{\bar{\delta}}] \subseteq G_\delta$.

By Lemma 4.9, we can take $\langle s_n \mid n < \omega\rangle \in \prod_{n<\omega} S_n$ such that $s_{n+1} \in \mathrm{suc}^n_S(s_n)$ and $q^{(s_n,n)} \in G_\delta$ for all $n < \omega$. Then, define $\sigma^* : \bar{M} \to M$ by $\sigma^*(\bar{c}_n) := c^{(s_n,n)}$. We show that σ^* is as desired.

Let $\sigma_n := (\dot{\sigma}^{(s_n,n)})^{G_\delta \restriction l(s_n)}$ for each $n < \omega$. Note that, σ^* and σ_n coincides on $\{\bar{c}_m \mid m \leq n\}$ for all $n < \omega$. Then it follows that $\sigma^* : \bar{M} \prec M$, since $\sigma_n : \bar{M} \prec M$ for all $n < \omega$, and $\bar{M} = \{\bar{c}_m \mid m < \omega\}$. Moreover, since $\sigma_n(\langle \bar{\vec{\mathbb{P}}}, \bar{b}\rangle) = \langle \vec{\mathbb{P}}, b\rangle$, we also have that $\sigma^*(\langle \bar{\vec{\mathbb{P}}}, \bar{b}\rangle) = \langle \vec{\mathbb{P}}, b\rangle$. Finally, note that $\sigma^*(\bar{q}^{(s_n,n)}) = q^{(s_n,n)}$ for all $n < \omega$. Then $\sigma^*(\bar{r}_n) \geq \sigma^*(\bar{q}^{(s_n,n)}) = q^{(s_n,n)} \in G_\delta$ for all $n < \omega$. So $\sigma^*[\bar{G}_{\bar{\delta}}] \subseteq G_\delta$ since $\bar{G}_{\bar{\delta}} = \{\bar{r}_n \mid n < \omega\}$.

We start to construct the above objects. By recursion on n, we construct S_n and $q^{(s,n)}$, $T^{(s,n)}$, $\bar{q}^{(s,n)}$, $\bar{T}^{(s,n)}$, $\bar{\beta}^{(s,n)}$, $\dot{\sigma}^{(s,n)}$, $c^{(s,n)}$ for all $s \in S_n$. (suc^n_S will be constructed when we construct S_{n+1}.) We must construct them so that they satisfy (i),(ii) above and (iii)–(vi) below. (iii)–(vi) are properties assuring that S will be a nested antichain with $\mathrm{rt}(S) = p$ and that $\langle q^{(s,n)}, T^{(s,n)} \mid n < \omega,\ s \in S_n\rangle$ will be a fusion structure of S.

(iii) $S_0 = \{p\}$, and $S_{n+1} = \bigcup_{s \in S_n} \mathrm{suc}^n_S(s) \subseteq \bigcup_{\alpha<\delta} \mathbb{P}_\alpha$.

(iv) If $s' \in \mathrm{suc}^n_S(s)$, then $l(s') \geq l(s)$, and $s' \restriction l(s) \leq s$. Also, for all $s \in S_n$, $\langle s' \restriction l(s) \mid s' \in \mathrm{suc}^n_S(s)\rangle$ is a maximal antichain below s in $\mathbb{P}_{l(s)}$.

(v) $T^{(s,n)}$ is a nested antichain in $\langle \mathbb{P}_\alpha \mid \alpha < \delta\rangle$ with $l(\mathrm{rt}(T^{(s,n)})) = l(s)$ and $s \leq \mathrm{rt}(T^{(s,n)})$, and $q^{(s,n)}$ is a mixture of $T^{(s,n)}$ up to δ.

(vi) If $s' \in \mathrm{suc}^n_S(s)$, then $T^{(s',n+1)} \angle T^{(s,n)}$.

First, suppose $n = 0$. Let $S_0 := \{p\}$ and $q^{(p,0)} := 1_\beta$. Define $T^{(p,0)}$ by $T^{(p,0)}_k := \{1_\beta\}$ and $\mathrm{suc}^k_{T^{(p,0)}}(1_\beta) = \{1_\beta\}$ for all $k < \omega$. Note that (iii) and (v) hold for $n = 0$ and $s = p$. Next, let $\dot{\sigma}^{(p,0)}$ be a $\mathbb{P}_\beta$-name of σ in (v) of Definition 4.10. Also, let $\bar{q}^{(p,0)} := \bar{1}_{\bar{\delta}}$, and define $\bar{T}^{(p,0)}$ by $\bar{T}^{(p,0)}_k := \{\bar{1}_{\bar{\beta}}\}$ and $\mathrm{suc}^k_{\bar{T}^{(p,0)}}(\bar{1}_{\bar{\beta}}) := \{\bar{1}_{\bar{\beta}}\}$ for all $k < \omega$, where $\bar{1}_{\bar{\beta}}$ is the largest element of $\bar{\mathbb{P}}_{\bar{\beta}}$. Let $\bar{\beta}^{(p,0)} := \bar{\beta}$ and $c^{(s,0)} := \emptyset$. Clearly, (i) and (ii) hold for $n = 0$ and $s = p$. (iv) and (vi) are irrelevant for $n = 0$. This completes the construction for $n = 0$.

Next, suppose S_n and $q^{(s,n)}$, $T^{(s,n)}$, $\bar{q}^{(s,n)}$, $\bar{T}^{(s,n)}$, $\bar{\beta}^{(s,n)}$, $\dot{\sigma}^{(s,n)}$, $c^{(s,n)}$ for all $s \in S_n$ have been constructed. We construct suc^n_S, S_{n+1} and $q^{(s',n+1)}$, $T^{(s',n+1)}$, $\bar{q}^{(s',n+1)}$, $\bar{T}^{(s',n+1)}$, $\bar{\beta}^{(s',n+1)}$, $\dot{\sigma}^{(s',n+1)}$, $c^{(s',n+1)}$ for all $s' \in S_{n+1}$.

For $s \in S_n$, let E_s be the collection of all candidates of elements of $\mathrm{suc}^n_S(s)$. That is, let E_s be the set all $s' \in \bigcup_{\alpha<\delta} \mathbb{P}_\alpha$ such that we have the following for some q', T', $\bar{q}'$, $\bar{T}'$, $\bar{\beta}'$, $\dot{\sigma}'$ and c'.

(vii) $l(s') \geq l(s)$, and $s' \restriction l(s) \leq s$.

(viii) T' is a nested antichain in $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$ with $l(\mathrm{rt}(T')) = l(s')$ and $s' \leq \mathrm{rt}(T')$, and q' is a mixuture of T' up to δ.

(ix) $T' \angle T^{(s,n)}$,

(x) $\bar{q}' \in \bar{G}_{\bar{\delta}}$, $\bar{q}' \leq \bar{r}_{n+1}$, and $\bar{T}', \bar{\beta}' \in \bar{M}$.

(xi) $\dot{\sigma}'$ is a $\mathbb{P}_{l(s')}$-name, and s' forces that

- $\dot{\sigma}' : \bar{M} \prec M$, $a \in \mathrm{ran}(\dot{\sigma}')$, and $\dot{\sigma}'(\langle \vec{\bar{\mathbb{P}}}, \bar{b} \rangle) = \langle \vec{\mathbb{P}}, b \rangle$,
- $\dot{\sigma}'(\langle \bar{q}^{(u,m)}, \bar{T}^{(u,m)}, \bar{c}_m \rangle) = \langle q^{(u,m)}, T^{(u,m)}, c^{(u,m)} \rangle$ for all $(u,m) \leq_S (s,n)$, and $\dot{\sigma}'(\langle \bar{q}', \bar{T}', \bar{c}_{n+1} \rangle) = \langle q', T', c' \rangle$,
- $\dot{\sigma}'(\bar{\beta}') = l(s')$, and $\dot{\sigma}'[\bar{G}_{\bar{\beta}'}] \subseteq \dot{G}_{l(s')}$.

For $s \in S_n$, let $D_s := \{ s' \restriction l(s) \mid s' \in E_s \}$. We claim the following.

Claim 1. *For any $s \in S_n$, D_s is dense below s in $\mathbb{P}_{l(s)}$.*

Proof of Claim 1. Suppose $s \in S_n$. Take an arbitrary $u \leq s$ in $\mathbb{P}_{l(s)}$. We find $s' \in E_s$ with $s' \restriction l(s) \leq u$.

By (i),(ii),(v) for n and s, in $\bar{M}$, $\bar{T}^{(s,n)}$ is a nested antichain in $\langle \bar{\mathbb{P}}_{\bar{\alpha}} \mid \bar{\alpha} < \bar{\delta} \rangle$, and $\bar{q}^{(s,n)}$ is a mixture of $\bar{T}^{(s,n)}$. Since $\bar{q}^{(s,n)} \in \bar{G}_{\bar{\delta}}$, we can take $\bar{t} \in \bar{T}^{(s,n)}_1$ with $\bar{t} \in \bar{G}_{l(\bar{t})}$. Then, we can take $\bar{q}' \in \bar{G}_{\bar{\delta}}$ such that $\bar{q}' \leq \bar{q}^{(s,n)}, \bar{r}_{n+1}$ and $\bar{q}' \restriction l(\bar{t}) \leq \bar{t}$, since $\bar{q}^{(s,n)}, \bar{r}_{n+1} \in \bar{G}_{\bar{\delta}}$ and $\bar{t} \in \bar{G}_{l(\bar{t})}$. By Lemma 4.7, in $\bar{M}$, we can take a nested antichain $\bar{T}'$ in $\langle \bar{\mathbb{P}}_{\bar{\alpha}} \mid \bar{\alpha} < \bar{\delta} \rangle$ such that $\bar{T}' \angle \bar{T}^{(s,n)}$, and $\bar{q}'$ is a mixture of $\bar{T}'$ up to $\bar{\delta}$. In $\bar{M}$, let $\bar{t}' := \mathrm{rt}(\bar{T}')$ and $\bar{\beta}' := l(\bar{t}')$. Note that $\bar{t}' \in \bar{G}_{\bar{\beta}'}$ since a mixture $\bar{q}'$ of $\bar{T}'$ belongs to $\bar{G}_{\bar{\delta}}$.

Take $v \leq u$ in $\mathbb{P}_{l(s)}$ and $q', T', c', \beta' \in M$ such that

(xii) $v \Vdash_{l(s)}$ "$\dot{\sigma}^{(s,n)}(\langle \bar{q}', \bar{T}', \bar{c}_{n+1}, \bar{\beta}' \rangle) = \langle q', T', c', \beta' \rangle$".

Note that

(xiii) $v \Vdash_{l(s)}$ "$a_{l(s),\beta'} \in \dot{\sigma}^{(s,n)}$"

since v forces that $l(s), \beta', a = \langle a_{\alpha,\gamma} \mid \alpha < \gamma < \delta \rangle \in \mathrm{ran}(\dot{\sigma}^{(s,n)})$.

Recall that θ and $a_{l(s),\beta'}$ verify the $\vec{K}$-subcompleteness of $\mathbb{P}_{\beta'}$ relative to $l(s)$. Then, by (ii),(xii),(xiii), we can take $s' \in \mathbb{P}_{\beta'}$ with $s' \restriction l(s) = v$ and $\mathbb{P}_{\beta'}$-name $\dot{\sigma}'$ satisfying (xi). Note that $s' \leq \mathrm{rt}(T')$ since $\mathrm{rt}(\bar{T}') = \bar{t}' \in \bar{G}_{\bar{\beta}'}$, and

s' forces that $\mathrm{rt}(T) = \dot{\sigma}'(\mathrm{rt}(\bar{T}')) \in \dot{\sigma}'[\bar{G}_{\bar{\beta}'}] \subseteq \dot{G}_{l(s')}$. Then, $s' \restriction l(s) \leq u$, and $q', T', \bar{q}', \bar{T}', \bar{\beta}', \dot{\sigma}', c'$ witnesses that $s' \in E_s$. So s' is as desired. $\square$ (Claim 1)

For each $s \in S_n$, we construct $\mathrm{suc}^n_S(s)$ and $q^{(s',n+1)}$, $T^{(s',n+1)}$, $\bar{q}^{(s',n+1)}$, $\bar{T}^{(s',n+1)}$, $\bar{\beta}^{(s',n+1)}$, $\dot{\sigma}^{(s',n+1)}$, $c^{(s',n+1)}$ for all $s' \in \mathrm{suc}^n_S(s)$. Fix $s \in S_n$. By the claim above, we can take $A_s \subseteq D_s$ which is a maximal antichain below s in $\mathbb{P}_{l(s)}$. For each $u \in A_s$, choose $s'_u \in E_s$ with $s'_u \restriction l(s) = u$. Then, let $\mathrm{suc}^n_S(s) := \{s'_u \mid u \in A_s\}$. Moreover, for each $s' \in \mathrm{suc}^n_S(s)$, let $q^{(s',n+1)}$, $T^{(s',n+1)}$, $\bar{q}^{(s',n+1)}$, $\bar{T}^{(s',n+1)}$, $\bar{\beta}^{(s',n+1)}$, $\dot{\sigma}^{(s',n+1)}$, $c^{(s',n+1)}$ be q', T', $\bar{q}'$, $\bar{T}'$, $\bar{\beta}'$, $\dot{\sigma}'$ and c' witnessing $s' \in E_s$.

Finally, let $S_{n+1} := \bigcup_{s \in S_n} \mathrm{suc}^n_S(s)$. We have constructed suc^n_S, S_{n+1} and $q^{(s',n+1)}$, $T^{(s',n+1)}$, $\bar{q}^{(s',n+1)}$, $\bar{T}^{(s',n+1)}$, $\bar{\beta}^{(s',n+1)}$, $\dot{\sigma}^{(s',n+1)}$, $c^{(s',n+1)}$ for all $s' \in S_{n+1}$. Clearly, they are as desired.

This completes the proof for Case 1. $\square$ (Case 1)

Case 2. δ is a successor ordinal.

Let $\gamma := \delta - 1$. By the induction hypothesis, we can take a regular cardinal θ' and $a' \in \mathcal{H}_{\theta'}$ which verify the $\vec{K}$-subcompleteness of $\mathbb{P}_\gamma$ relative to β. Since 1_γ forces that $\dot{\mathbb{P}}_{\gamma,\delta}$ is $\vec{K}$-subcomplete, we can also take a regular cardinal θ'' and a $\mathbb{P}_\gamma$-name $\dot{a}''$ such that 1_γ forces θ'' and $\dot{a}''$ to verify the $\vec{K}$-subcompleteness of $\dot{\mathbb{P}}_{\gamma,\delta}$. Let $\theta := \max\{\theta', \theta''\}$ and $a := \langle a', \dot{a}'' \rangle$. We show that θ and a verify the $\vec{K}$-subcompleteness of $\mathbb{P}_\delta$ relative to β.

Suppose A, χ, b, $\bar{M}$, $\bar{b}$, $\vec{\bar{\mathbb{P}}} = \langle \bar{\mathbb{P}}_{\bar{\alpha}} \mid \bar{\alpha} \leq \bar{\delta} \rangle$, $\bar{\beta}$, $\bar{G}_{\bar{\beta}}$, $\bar{G}_{\bar{\delta}}$, p satisfies (i)–(v) of Definition 4.10. We will show that there is p^* as in Definition 4.10. Let $M := L^A_\chi$, $\bar{\gamma} := \bar{\delta} - 1$ and $\bar{G}_{\bar{\gamma}} := \bar{G}_{\bar{\delta}} \restriction \bar{\gamma}$.

Note that for $a' \in \mathrm{ran}(\sigma)$ for σ as in (v) of Definition 4.10. Since θ and a' verify the $\vec{K}$-subcompleteness of $\mathbb{P}_\gamma$ relative to β, we can take $p' \leq \mathbb{P}_\gamma$ such that $p' \restriction \beta = p$ and p' forces the existence of $\sigma' : \bar{M} \prec M$ with $\sigma'(\langle \vec{\bar{\mathbb{P}}}, \bar{b} \rangle) = \langle \vec{\mathbb{P}}, b \rangle$, $a \in \mathrm{ran}(\sigma')$ and $\sigma'[\bar{G}_{\bar{\gamma}}] \subseteq \dot{G}_\gamma$.

Let $\bar{\mathbb{P}}_{\bar{\gamma},\bar{\delta}}$ be the evaluation of $\dot{\bar{\mathbb{P}}}_{\bar{\gamma},\bar{\delta}}$ by $\bar{G}_{\bar{\gamma}}$, and let $\bar{H}_{\bar{\gamma}} := \bar{G}_{\bar{\delta}} \restriction [\bar{\gamma}, \bar{\delta})$. Note that $\bar{H}_{\bar{\gamma}}$ is a $\bar{\mathbb{P}}_{\bar{\gamma},\bar{\delta}}$-generic filter over $\bar{M}[\bar{G}_{\bar{\gamma}}]$. Also, let $\dot{H}_\gamma$ be a $\mathbb{P}_\gamma$-name of the canonical name for a $\dot{\mathbb{P}}_{\gamma,\delta}$-generic filter. We claim the following.

Claim 2. *There is $p^* \in \mathbb{P}_\delta$ such that $p^* \restriction \gamma = p'$ and*

$$p^* \restriction \gamma \Vdash_\gamma \text{“} p^* \restriction [\gamma, \delta) \Vdash_{\gamma,\delta} \Phi \text{”},$$

where Φ is the statement that there is $\tau : \bar{M}[\bar{G}_{\bar{\gamma}}] \prec M[\dot{G}_\gamma]$ with $\tau(\langle \vec{\bar{\mathbb{P}}}, \bar{b}, \bar{G}_{\bar{\gamma}} \rangle) = \langle \vec{\mathbb{P}}, b, \dot{G}_\gamma \rangle$ and $\tau[\bar{H}_{\bar{\gamma}}] \subseteq \dot{H}_\gamma$.

Proof of Claim 2. By (i) of Definition 4.5, it suffices to prove that

$$p' \Vdash_\gamma \text{“} \exists q \in \dot{\mathbb{P}}_{\gamma,\delta} \ (\, q \Vdash_{\gamma,\delta} \ \Phi \,) \text{”}.$$

Suppose G_γ is a $\mathbb{P}_\gamma$-generic filter over V with $p' \in G_\gamma$. Let $\mathbb{P}_{\gamma,\delta} := (\dot{\mathbb{P}}_{\gamma,\delta})^{G_\gamma}$. Working in $V[G_\gamma]$, we find $q \in \mathbb{P}_{\gamma,\delta}$ which forces Φ.

Let $a'' := (\dot{a}'')^{G_\gamma}$. Recall that θ and a'' verify the $\vec{K}$-subcompleteness of $\mathbb{P}_{\gamma,\delta}$. We want to use this to find q. For this, we make some preliminaries.

Since $p' \in G_\gamma$, there is $\sigma' : \bar{M} \prec M$ with $a \in \mathrm{ran}(\sigma')$, $\sigma'(\langle \vec{\bar{\mathbb{P}}}, \bar{b} \rangle) = \langle \vec{\mathbb{P}}, b \rangle$ and $\sigma'[\bar{G}_{\bar{\gamma}}] \subseteq G_\gamma$. Then, σ' can be naturally extended to $\sigma'' : \bar{M}[\bar{G}_{\bar{\gamma}}] \prec M[G_\gamma]$. with $\sigma''(\bar{G}_{\bar{\gamma}}) = G_\gamma$. Note that

(i) $a'' \in \mathrm{ran}(\sigma'')$, and $\sigma''(\bar{\mathbb{P}}_{\bar{\gamma},\bar{\delta}}) = \mathbb{P}_{\gamma,\delta}$.

Let $B := (\{0\} \times A) \cup (\{1\} \times G_\gamma)$ and $N := L^B_\chi$. Then it is easy to check that $M[G_\gamma] = \langle L_\chi[A][G_\gamma], \in, A \cap L_\chi[A] \rangle$ and $N = \langle L_\chi[B], \in, B \cap L_\chi[B] \rangle$ are equivalent in the sense that $L_\chi[A][G_\gamma] = L_\chi[B]$, $A \cap L_\chi[A]$ is definable in N, and $B \cap L_\chi[B]$ is definable in $M[G_\gamma]$. Note also that

(ii) $\mathcal{H}^{V[G_\gamma]}_\theta \subseteq L^B_\chi \models \mathrm{ZFC}^-$.

By the elementarity of σ', $\bar{M} = L^{\bar{A}}_{\bar{\chi}}$ for some set $\bar{A}$ and some ordinal $\bar{\chi}$. Let $\bar{B} := (\{0\} \times \bar{A}) \cup (\{1\} \times \bar{G})$ and $\bar{N} := L^{\bar{B}}_{\bar{\chi}}$. Then $\bar{M}[\bar{G}_{\bar{\gamma}}]$ and $\bar{N}$ are equivalent in the same sense as above. Then, we have that

(iii) $\sigma'' : \bar{N} \prec N$.

Next, we note that

(iv) $\bar{N}$ is a $\vec{K}$-good model,

(v) $\bar{H}_{\bar{\gamma}}$ is a $\bar{\mathbb{P}}_{\bar{\gamma},\bar{\delta}}$-generic filter over $\bar{N}$ with $\bar{H}_{\bar{\gamma}} \in K_{\omega_1^{\bar{N}}}$.

We only check that $\bar{N}, \bar{H}_{\bar{\gamma}} \in K_{\omega_1^{\bar{N}}}$, and $\bar{N}$ is countable in $K_{\omega_1^{\bar{N}}}$. The other properties are easily checked. Note that $\bar{N}, \bar{H}_{\bar{\gamma}} \in K_{\omega_1^{\bar{M}}}$ and $\bar{N}$ is countable in $K_{\omega_1^{\bar{M}}}$, since $\bar{M}, \bar{G}_{\bar{\gamma}}, \bar{G}_{\bar{\delta}} \in K_{\omega_1^{\bar{M}}}$ and $\bar{M}$ is countable in $K_{\omega_1^{\bar{M}}}$. So it suffices to check that $\omega_1^{\bar{N}} = \omega_1^{\bar{M}}$. For this, note that $\mathbb{P}_\gamma$ preserves the adequateness of $\vec{K}$ by the definition of an iteration of $\vec{K}$-subcomplete forcings. So $\mathbb{P}_\gamma$ preserves ω_1. Then, by the elementarity of σ', we have $\omega_1^{\bar{N}} = \omega_1^{\bar{M}}$.

Since θ and a'' verify the $\vec{K}$-subcompleteness of $\mathbb{P}_{\gamma,\delta}$, by (i)–(v) above, there is $q \in \mathbb{P}_{\gamma,\delta}$ which forces that there is $\tau : \bar{N} \prec N$ with $\tau(\langle \vec{\bar{\mathbb{P}}}, \bar{b} \rangle) = \langle \vec{\mathbb{P}}, b \rangle$ and $\tau[\bar{H}_{\bar{\gamma}}] \subseteq \dot{H}_\gamma$. Then q is as desired. □ (Claim 2)

Let p^* be as in Claim 2. We show that p^* is as desired. First, note that $p^* \restriction \beta = p' \restriction \beta = p$. Suppose G_δ is a $\mathbb{P}_\delta$-generic filter over V with $p^* \in G_\delta$. In $V[G_\delta]$, we find $\sigma^* : \bar{M} \prec M$ with $\sigma^*(\langle \vec{\bar{\mathbb{P}}}, \bar{b} \rangle) = \langle \vec{\mathbb{P}}, b \rangle$ and $\sigma^*[\bar{G}_{\bar{\delta}}] \subseteq \dot{G}_\delta$.

Note that $G_\gamma := G_\delta \restriction \gamma$ is $\mathbb{P}_\gamma$-generic filter over V, and $H_\gamma := G_\delta \restriction [\gamma, \delta)$ is a $(\dot{\mathbb{P}}_{\gamma,\delta})^{G_\gamma}$-generic filter over $V[G_\gamma]$. Moreover, $p^* \restriction \gamma \in G_\gamma$, and $p^* \restriction [\gamma, \delta) \in H_\gamma$. Then, by the choice of p^*, in $V[G_\delta] = V[G_\gamma][H_\gamma]$, there is $\tau : \bar{M}[\bar{G}_{\bar{\gamma}}] \prec M[G_\gamma]$ with $\tau(\langle \vec{\bar{\mathbb{P}}}, \bar{b}, \bar{G}_{\bar{\gamma}} \rangle) = \langle \vec{\mathbb{P}}, b, G_\gamma \rangle$ and $\tau[\bar{H}_{\bar{\gamma}}] \subseteq H_\gamma$. Let $\sigma^* := \tau \restriction \bar{M}$.

Clearly, $\sigma^* : \bar{M} \prec M$ and $\sigma^*(\langle \vec{\bar{\mathbb{P}}}, \bar{b} \rangle) = \langle \vec{\mathbb{P}}, b \rangle$. Note that $\tau[\bar{G}_{\bar{\gamma}}] \subseteq G_\gamma$ and $\tau[\bar{H}_{\bar{\gamma}}] \subseteq H_\gamma$. Note also that $\bar{G}_{\bar{\delta}} = \bar{G}_{\bar{\gamma}} * \bar{H}_{\bar{\gamma}}$ and $G_\delta = G_\gamma * H_\gamma$. So $\sigma^*[\bar{G}_{\bar{\delta}}] \subseteq G_\delta$. Thus σ^* is as desired. □ (Case 2)

This completes the proof of Proposition 4.11. □

5 $\vec{K}$-SCFA

In this section, we study the forcing axiom for $\vec{K}$-subcomplete forcing notions.

Definition 5.1. *Suppose $\vec{K}$ is an adequate model sequence. Then the $\vec{K}$-Subcomplete Forcing Axiom, $\vec{K}$-SCFA, is the following assertion:*

> *For any $\vec{K}$-subcomplete forcing notion $\mathbb{P}$ and any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| \leq \omega_1$, there is a filter g on $\mathbb{P}$ with $g \cap D \neq \emptyset$ for any $D \in \mathcal{D}$.*

First, we show the consistency of $\vec{K}$-SCFA for some $\diamondsuit$-model sequence $\vec{K}$. Using Lemma 3.8 and Theorem 4.2, this can be proved by the standard argument.

Theorem 5.2. *Assume there is a supercompact cardinal. Then there is a forcing extension in which $\vec{K}$-SCFA holds for some $\diamondsuit$-model sequence $\vec{K}$.*

Proof. By replacing V with its forcing extension by ${}^{<\omega_1}2$, we may assume that $\diamondsuit_{\omega_1}$ holds in V. Take a $\diamondsuit$-model sequence $\vec{K}$ in V.

In V, let κ be a supercompact cardinal, and take a Laver function $F : \kappa \to V_\kappa$. Then, we can construct a nice iteration $\langle \mathbb{P}_\alpha \mid \alpha \leq \kappa \rangle$ so that we have the following for all $\alpha < \kappa$.

- If 1_α forces that $\vec{K}$ is a $\diamondsuit$-model sequence, and $F(\alpha)$ is a $\mathbb{P}_\alpha$-name for a $\vec{K}$-subcomplete forcing axiom, then 1_α forces that $\dot{\mathbb{P}}_{\alpha,\alpha+1}$ is forcing equivalent to $F(\alpha)$.
- Otherwise, 1_α forces that $\dot{\mathbb{P}}_{\alpha,\alpha+1}$ is a trivial forcing notion.

Using Lemma 3.8 and Theorem 4.2, by induction on $\alpha \leq \kappa$, we can prove that $\mathbb{P}_\alpha$ is $\vec{K}$-subcomplete, and $\Vdash_\alpha$ " $\vec{K}$ is a $\diamondsuit$-model sequence ".

Let G_κ be a $\mathbb{P}_\kappa$-generic filter over V. Then $\vec{K}$ is a $\diamondsuit$-model sequence in $V[G_\kappa]$. Moreover, by the same argument as the proof of the consistency of PFA, we can prove that $\vec{K}$-SCFA holds in $V[G_\kappa]$. This proof is left to the readers. □

We turn our attention to consequences of $\vec{K}$-SCFA. First, recall that all subcomplete forcing notions are $\vec{K}$-subcomplete. So we have the following.

Proposition 5.3. *Suppose $\vec{K}$-SCFA holds for some adequate model sequence $\vec{K}$. Then SCFA holds.*

Next, we prove that $\vec{K}$-SCFA for a $\diamondsuit$-model sequence $\vec{K}$ implies $\diamondsuit^+_{\omega_1}$.

Proposition 5.4. *Suppose $\vec{K}$ is an adequate model sequence, and $\vec{K}$-SCFA holds. Then $\vec{K}$ is strongly adequate. If $\vec{K}$ is a $\diamondsuit$-model sequence in addition, then $\vec{K}$ is a $\diamondsuit^+_{\omega_1}$-sequence.*

For this, we use the following forcing notion.

Definition 5.5. *For an adequate model sequence $\vec{K} = \langle K_\xi \mid \xi < \omega_1 \rangle$ and $B \subseteq \omega_1$, let $\mathbb{C}_{\vec{K},B}$ be the following forcing notion.*

(i) $\mathbb{C}_{\vec{K},B}$ consists of all closed bounded $p \subseteq \omega_1$ such that $B \cap \xi, p \cap \xi \in K_\xi$ for all $\xi \in p$.

(ii) $p \leq q$ in $\mathbb{C}_{\vec{K},B}$ if p is an end-extension of q.

In the following two lemmata, we observe basic properties of $\mathbb{C}_{\vec{K},B}$.

Lemma 5.6. *Suppose $\vec{K} = \langle K_\xi \mid \xi < \omega_1 \rangle$ is an adequate model sequence, and $B \subseteq \omega_1$. Then, $D_\xi := \{p \in \mathbb{C}_{\vec{K},B} \mid \max(p) \geq \xi\}$ is dense in $\mathbb{P}$ for any $\xi < \omega_1$.*

Proof. Suppose $p \in \mathbb{C}_{\vec{K},B}$ and $\xi < \omega_1$. We must find $q \leq p$ with $q \in D_\xi$. By Lemma 3.2 (4), there is $\zeta < \omega_1$ such that $\xi, \max(p) < \zeta$ and $B \cap \zeta, p \in K_\zeta$. Let $q := p \cup \{\zeta\}$. Then, $q \in \mathbb{C}_{\vec{K},B}$ by the choice of ζ and the fact that $p \in \mathbb{C}_{\vec{K},B}$. Moreover $q \leq p$ clearly, and $q \in D_\xi$ since $\max(q) = \zeta > \xi$. □

Lemma 5.7. *Suppose $\vec{K} = \langle K_\xi \mid \xi < \omega_1 \rangle$ is an adequate model sequence, and $B \subseteq \omega_1$. Then $\mathbb{C}_{\vec{K},B}$ is $\vec{K}$-subcomplete.*

Proof. Let $\mathbb{P} := \mathbb{C}_{\vec{K},B}$. Let θ be a sufficiently large regular cardinal, and let $a := B$. We show that θ and a verify the $\vec{K}$-subcompleteness of $\mathbb{P}$. Suppose A, χ, $\bar{M}$, $\bar{\mathbb{P}}$, $\bar{b}$, σ, b and $\bar{G}$ satisfies (i)–(iv) of Definition 3.5. It suffices to find $p^* \in \mathbb{P}$

which forces that $\sigma[\bar{G}] \subseteq \dot{G}$, where $\dot{G}$ is the canonical name for a $\mathbb{P}$-generic filter. (Then p^* forces that $\sigma^* := \sigma$ is as in Definition 3.5.)

Let $\xi := \omega_1^{\bar{M}}$ and $\bar{B} := \sigma^{-1}(B)$. Note that $\bar{B} = B \cap \xi$ since the critical point of σ is ξ, and $\sigma(\xi) = \omega_1$. Note also that $\sigma \restriction \bar{\mathbb{P}}$ is an identity since $\mathbb{P} \subseteq \mathcal{H}_{\omega_1}$. In particular, $\sigma[\bar{G}] = \bar{G}$.

Let $p' := \bigcup \bar{G}$. Then p' is club in ξ by Lemma 5.6. Moreover, $B \cap \eta, p' \cap \eta \in K_\eta$ for all $\eta \in p'$ since $\bar{G} = \sigma[\bar{G}] \subseteq \mathbb{P}$. Note also that $B \cap \xi = \bar{B} \in \bar{M} \subseteq K_\xi$ and $p' \in K_\xi$. Then $p^* := p' \cup \{\xi\} \in \mathbb{P}$, and p^* is a lower bound of $\bar{G} = \sigma[\bar{G}]$. So p^* forces that $\sigma[\bar{G}] \subseteq \dot{G}$. □

Now, we prove Proposition 5.4.

Proof of Proposition 5.4. We only prove the latter statement. The proof of the former is the same. Let $\vec{K} = \langle K_\xi \mid \xi < \omega_1 \rangle$ be a $\diamondsuit$-model sequence, and suppose $\vec{K}$-SCFA holds. We show that $\vec{K}$ is a $\diamondsuit^+_{\omega_1}$-sequence.

Take an arbitrary $B \subseteq \omega_1$. We find a club $C \subseteq \omega_1$ with $B \cap \xi, C \cap \xi \in K_\xi$ for all $\xi \in C$. For each $\xi < \omega_1$, let $D_\xi := \{p \in \mathbb{C}_{\vec{K},B} \mid \max(p) \geq \xi\}$. By Lemma 5.6, each D_ξ is dense in $\mathbb{C}_{\vec{K},B}$. By $\vec{K}$-SCFA, we can take a filter g on $\mathbb{C}_{\vec{K},B}$ with $g \cap D_\xi \neq \emptyset$ for any $\xi < \omega_1$. Let $C := \bigcup g$.

Then C is a club subset of ω_1. Moreover $B \cap \xi, C \cap \xi \in K_\xi$ for all $\xi \in C$ since $g \subseteq \mathbb{C}_{\vec{K},B}$. So C is as desired. □

Here we make a remark on the preservation of stationary subsets of ω_1 by $\vec{K}$-subcomplete forcings. If $\vec{K}$ is an adequate model sequence, and $\vec{K}$-SCFA holds, then $\vec{K}$-subcomplete forcings preserve stationary subsets of ω_1 by Lemma 3.11 and Proposition 5.4. But, in general, $\vec{K}$-subcomplete forcings may not preserve stationary subsets of ω_1: Suppose $\vec{K}$ is not strongly adequate, and let B be a subset of ω_1 such that $S := \{\xi < \omega_1 \mid B \cap \xi \notin K_\xi\}$ is stationary. $\mathbb{C}_{\vec{K},B}$ is $\vec{K}$-subcomplete by Lemma 5.7. But $\mathbb{C}_{\vec{K},B}$ adds a club $C \subseteq \omega_1$ with $C \cap S = \emptyset$ by Lemma 5.6.

By Theorem 5.2 and Proposition 5.3 and 5.4, we have the following corollary.

Corollary 5.8. *Assume there is a supercompact cardinal. Then there is a forcing extension in which* SCFA *and* $\diamondsuit^+_{\omega_1}$ *hold.*

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