Operations vs. $*$-tactics

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Posets preserving PFA

Thm (König - Y. 2003)

PFA is preserved under any ω_2-closed forcing.

Since then, several broader classes of posets preserving PFA have been found. Such classes are defined in terms of Banach-Mazur games.

The Banach-Mazur game

For a (separative) poset $\mathbb{P}$ and an ordinal α, $G_\alpha(\mathbb{P})$ denotes the following two-player game: Players choose smaller and smaller $\mathbb{P}$-conditions in turn. At each limit stage, Player II goes first to choose a condition smaller than the preceding moves.

II wins iff she was able to make α moves without breaking the rule.

$$
\begin{array}{lccccccccccc}
\mathrm{I} & a_0 & & & & a_1 & & \cdots & & & a_{\omega+1} & & \cdots \\
 & & \geq & & \geq & & \geq \quad \geq & & & \geq & & \geq \quad \geq & \\
\mathrm{II} & & & b_0 & & & b_1 & \cdots & \geq b_\omega & & & b_{\omega+1} & \cdots
\end{array}
$$

In this talk, we only consider the case $\alpha = \omega_1 + 1$.

Strategies, tactics and operations

Def A <u>strategy</u> (for Player II) in $G_{\omega+1}(\mathbb{P})$ is a mapping which suggests a move, knowing the full information on the sequence of preceding moves of Player I (called as "the record"), at each turn of Player II.

$$\begin{array}{llllll} \mathrm{I} & a_0 & a_1 & \cdots & a_{\omega+1} & \cdots \\ \mathrm{II} & \tau(\langle a_0 \rangle) & \tau(\langle a_0 a_1 \rangle) \cdots & \tau(\langle a_n \rangle_{n<\omega}) & \tau(\langle a_n \rangle_{n<\omega} {}^\frown \langle a_{\omega+1} \rangle) & \cdots \end{array}$$

Def A <u>tactic</u> is a strategy whose suggestions depend only on "the current position" (= the Boolean infimum of "the record").

I	a_0	a_1	$\cdots$		$a_{\omega+1}$	$\cdots$
II	$\tau(a_0)$	$\tau(a_1)$	$\cdots$	$\tau(\bigwedge_{n<\omega} a_n)$	$\tau(a_{\omega+1})$	$\cdots$

Def An <u>operation</u> is a strategy whose suggestions depend only on "the current position" and "the number of the turn" ($\equiv$ the length of the record).

I	a_0	a_1	$\cdots$		$a_{\omega+1}$	$\cdots$
II	$\tau(0, a_0)$	$\tau(1, a_1)$	$\cdots$	$\tau(\omega, \bigwedge_{n<\omega} a_n)$	$\tau(\omega+1, a_{\omega+1})$	$\cdots$

The modified game $G^*(\mathbb{P})$

In this game, Player I chooses a countable set of $\mathbb{P}$-conditions, instead of a single condition at each turn.

$$
\begin{array}{llllllll}
\text{I} & A_0 \, (\in [\mathbb{P}]^{\leq\omega}) & & A_1 & \cdots & & A_{\omega+1} & \cdots \\
\text{II} & & b_0 & & b_1 \ \cdots & b_\omega & & b_{\omega+1} \cdots
\end{array}
$$

They must obey the following inequalities:

$$\bigwedge A_0 \geq b_0 \geq \bigwedge(A_0 \cup A_1) \geq b_1 \geq \cdots \geq \bigwedge\Big(\bigcup_{n<\omega} A_n\Big) \geq b_\omega \geq \bigwedge\Big(\Big(\bigcup_{n<\omega} A_n\Big) \cup A_{\omega+1}\Big) \geq b_{\omega+1} \geq \cdots$$

That is, the Boolean infimum of the union of preceding moves of Player I at each moment plays the role of I's move in the usual B-M game.

II wins iff she was able to make ω_1 moves without breaking the rule.

$*$-tactics and $*$-operations

The notion of a strategy for $G^*(\mathbb{P})$ is defined in the same way as that of $G_{\omega+1}(\mathbb{P})$.

<u>Def</u> A <u>$*$-tactic</u> (for $G^*(\mathbb{P})$) is a strategy whose suggestions depend only on the union of "the record".

$$
\begin{array}{llllll}
\mathrm{I} & A_0 & A_1 & \cdots & A_{\omega+1} & \cdots \\
\mathrm{II} & \tau(A_0) & \tau(A_0 \cup A_1) & \cdots \ \tau(\bigcup_{n<\omega} A_n) & \tau(\bigcup_{n<\omega} A_n \cup A_{\omega+1}) & \cdots
\end{array}
$$

A <u>$*$-operation</u> is a strategy whose suggestions depend only on the union of the record and "the number of the turn".

Game closure properties

Def A poset $\mathbb{P}$ is (ω_1+1)-$\left\{\begin{array}{l}\text{strategically}\\ \text{tactically}\\ \text{operationally}\\ \text{*-tactically}\\ \text{*-operationally}\end{array}\right\}$ closed if II has a winning $\left\{\begin{array}{l}\text{strategy}\\ \text{tactic}\\ \text{operation}\\ \text{*-tactic}\\ \text{*-operation}\end{array}\right\}$ (for $G_{\omega_1+1}(\mathbb{P})$, or $G^*(\mathbb{P})$ respectively).

Thm (Y., 2011) PFA is preserved under any (ω_1+1)-*-operationally closed forcing.

preserving PFA | not preserving PFA

$\square_{\kappa,\omega_2}$

$$\begin{tikzcd} & & *\text{-tact. closed} \arrow[dr, Rightarrow] & & \\ \omega_2\text{-closed} \arrow[r, Rightarrow] & \text{tact. closed} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & *\text{-oper. closed} \arrow[r, Rightarrow] & \text{str. closed} \\ & & \text{oper. closed} \arrow[ur, Rightarrow] & & \end{tikzcd}$$

$\neg CC$

(ω_1+1)-

Question Compare (ω_1+1)-oper. closed and (ω_1+1)-$*$-tact. closed.

MA_{ω_2}

Def CP $\overset{}{\underset{\text{def}}{\iff}}$ $\exists f : \omega_2 \to \omega_1 \ \forall \beta \in S^2_1 \ (= \{\xi < \omega_2 \mid \mathrm{cf}\,\xi = \omega_1\})$

$\exists C \subseteq_{\text{club}} \beta \ [\mathrm{o.t.}\, C = \omega_1 \wedge \forall \xi \in C\, (f(\xi) = \mathrm{o.t.}(C \cap \xi))]$.

$SCP^- \underset{\text{def}}{\iff} \exists \langle z_\alpha \mid \alpha \in S^2_0 \rangle$ s.t.

- $\forall \alpha \in S^2_0 \ [z_\alpha \subseteq_{\text{cofinal}} \alpha \wedge |z_\alpha| = \omega]$,
- $\forall \beta \in S^2_1 \ \exists C \subseteq_{\text{club}} \beta \cap S^2_0$ s.t.

  $\mathrm{o.t.}\, C = \omega_1 \wedge \langle z_\alpha \mid \alpha \in C \rangle$ is a $\subseteq$-continuous increasing sequence.

Fact $CP \equiv MA_{\omega_1}((\omega_1{+}1)\text{-oper.closed})$.

$SCP^- \equiv MA_{\omega_1}((\omega_1{+}1)\text{-}*\text{-tact.closed})$.

Remark A natural poset forcing $\left\{\begin{matrix} CP \\ SCP^- \end{matrix}\right\}$ is

$(\omega_1{+}1)\text{-}\left\{\begin{matrix} \text{oper.} \\ *\text{-tact.} \end{matrix}\right\}\text{-closed}$.

Main Results

Thm (Y., 2012) Under $MA^+(\omega_1\text{-closed})$

(1) $\Vdash_{\mathbb{P}} \neg CP$ for any (ω_1+1)-$*$-tact. closed $\mathbb{P}$.

(2) $\Vdash_{\mathbb{P}} \neg SCP^-$ for any (ω_1+1)-oper. closed $\mathbb{P}$.

Cor Under $MA^+(\omega_1\text{-closed})$.

(1) There exists an (ω_1+1)-oper. closed poset which is not (ω_1+1)-$*$-tact. closed.

(2) There exists an (ω_1+1)-$*$-tact. closed poset which is not (ω_1+1)-oper. closed.

Remark

As for (1) of the main theorem, in fact we can show that under $MA^+(\omega_1\text{-closed})$,

$\Vdash_{\mathbb{P}} CC$ for any (ω_1+1)-$*$-tact. closed $\mathbb{P}$.

(Note that CC negates CP).

This can be proved by imitating Miyamoto's proof of the consistency of CC + the Strong Non Reflection using $MA^+(\omega_1\text{-closed})$, which was inspired by Sakai's proof of the consistency of $\square_{\omega_1,2}$ and CC.

Some ideas for proof of (2)

$\mathbb{P}$: (ω_1+1)-oper. closed poset, σ: winning operation for $G_{\omega_1+1}(\mathbb{P})$.

$\mathbb{R}$: the poset of all partial plays (in $G_{\omega_1+1}(\mathbb{P})$) ending with II's move, where II follows σ.

$\mathbb{R}$ has a natural projection to $\mathbb{P}$ ($\pi(r)$ is the last move of II in r for $r \in \mathbb{R}$) and thus is forcing equivalent to $\mathbb{P} * \dot{\mathbb{Q}}$ for some $\dot{\mathbb{Q}}$.

It is not hard to see that (2) follows from the following lemma:

Lemma Let $\langle \dot{z}_\alpha : \alpha \in S^2_0 \rangle$ be a sequence of $\mathbb{P}$-names s.t.

$\Vdash_{\mathbb{P}} \forall \alpha \in S^2_0 \,(\dot{z}_\alpha \subseteq_{\text{cofinal}} \check{\alpha} \wedge |\dot{z}_\alpha| = \omega)$.

Then in $V^{\mathbb{P} * \dot{\mathbb{Q}}} = V^{\mathbb{R}}$,

$S = \{ x \in [\omega_2^V]^\omega \mid \dot{z}_{\sup(x)} \not\subseteq \check{x} \}$ is stationary in $[\omega_2^V]^\omega$.

Note $\dot{\mathbb{Q}}$ is not necessarily proper in $V^{\mathbb{P}}$.

<u>Sketch of proof of Lemma</u> Work in V.

θ: large regular. $\mathfrak{A} = \langle H_\theta, \in, \cdots \rangle$ enough structured.

It is enough to show that for each $r \in \mathbb{R}$, there exists a countable elementary substructure N and an $(N, \mathbb{R})$-generic condition $s \leq_{\mathbb{R}} r$ s.t. $s \Vdash_{\mathbb{R}} \dot{Z}_{\sup(\check{N} \cap \omega_2^V)} \nsubseteq \check{N}$ ($\Leftrightarrow \pi(s) \Vdash_{\mathbb{P}} \sim\!\sim\!\sim$).

<u>Sublemma</u> Suppose $N \prec_{\text{ctble}} \mathfrak{A}$, $r \in \mathbb{R}$ and $\langle p_n \mid n < \omega \rangle$ is an $(N, \mathbb{P})$-generic sequence with $\pi(r) \geq p_0$. Then for each $q \in \mathbb{P}$ extending $\sigma(N \cap \omega_1, \bigwedge_{n<\omega} p_n)$ there exists an $(N, \mathbb{R})$-generic $s \leq r$ s.t. $\pi(s) \leq q$.

Using a variation of Velickovic games, we can find $N_0, N_1 \prec_{\text{ctble}} \mathfrak{A}$ and $\langle p_n^0 \mid n<\omega \rangle, \langle p_n^1 \mid n<\omega \rangle$ s.t.

(a) $\sup(N_0 \cap \omega_2) = \sup(N_1 \cap \omega_2)$ $(=: \eta)$.

(b) $N_0 \cap \omega_1 = N_1 \cap \omega_1$.

(c) $N_0 \cap N_1 \cap \omega_2$ is bounded in η.

(d) $\langle p_n^i \mid n<\omega \rangle$ is an $(N_i, \mathbb{P})$-generic sequence (for $i = 0, 1$).

(e) $\pi(r) \geq p_0^0 \geq p_0^1 \geq p_1^0 \geq p_1^1 \geq p_2^0 \geq p_2^1 \geq \cdots$

By (b) and (e) we have $\sigma(N_0 \cap \omega_1, \bigwedge_{n\in\omega} p_n^0) = \sigma(N_1 \cap \omega_1, \bigwedge_{n\in\omega} p_n^1)$ $(=: q)$.

By (a) and (c), $q \Vdash_{\mathbb{P}}$ "$\dot{Z}_{\check{y}} \nsubseteq \check{N}_0$ or $\dot{Z}_{\check{y}} \nsubseteq \check{N}_1$." Thus w.m.a.

there exists $q' \leq_{\mathbb{P}} q$ s.t. $q' \Vdash_{\mathbb{P}}$ "$\dot{Z}_{\check{y}} \nsubseteq \check{N}_0$." By sublemma with (d)

and the fact that $q = \sigma(N_0 \cap \omega_1, \bigwedge_{n\in\omega} p_n^0)$, there exists an $(N_0, \mathbb{R})$-generic

$s \leq_{\mathbb{R}} r$ with $\pi(s) \leq q' \Vdash$ "$\dot{Z}_{\check{y}} \nsubseteq \check{N}_0$". //