Real determinacy and Real Blackwell determinacy

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Gratitude (感謝！)

Thank you very much for the everything, Hiroshi Sakai!

どうもありがとうございました，酒井さん！ m(_ _)m

From now on...

Work in ZF+$\mathrm{DC}_{\mathbb{R}}$.

For a set X,

DC_X: For any relation $R \subseteq X \times X$ such that

$(\forall x \in X)\ (\exists y \in X)\ (x, y) \in R$, there is a function $f : \omega \to X$

such that $(f(n), f(n+1)) \in R$ for any natural number n.

Perfect & imperfect information for games

Perfect information: Players know about the previous moves by both players.

E.g., Gale-Stewart games.

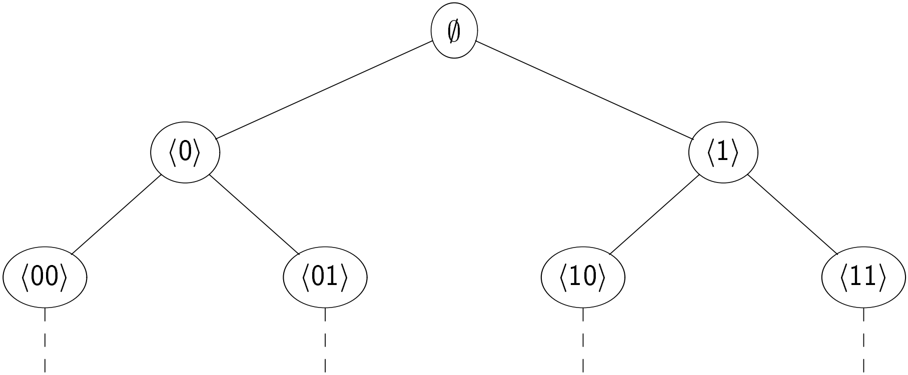
Imperfect information: Players do not know about what the other player did previously.

E.g., Blackwell games.

Gale-Stewart games

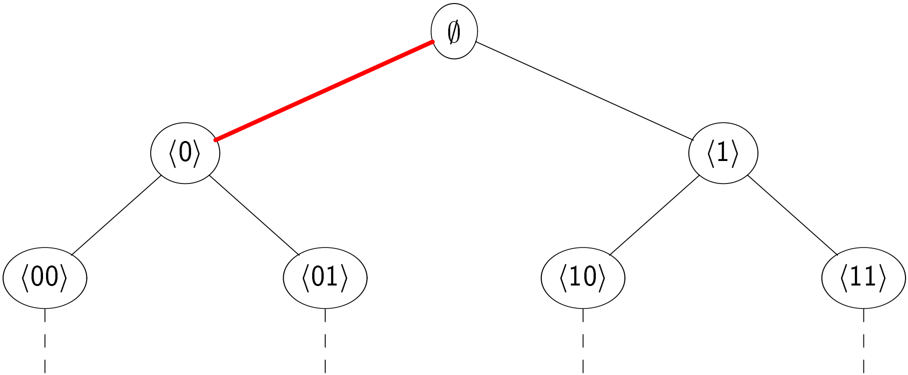
Fix a payoff set $A \subseteq {}^{\omega}2$.

I's turn.



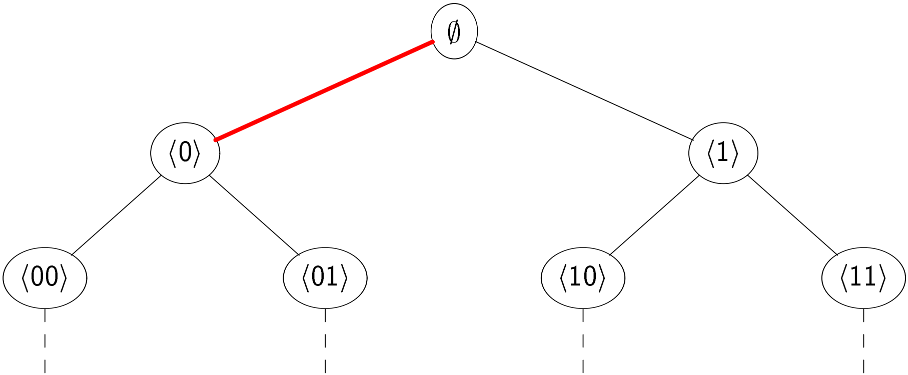
Gale-Stewart games ctd.

I has played.



Gale-Stewart games ctd..

II's turn.



Gale-Stewart games ctd...

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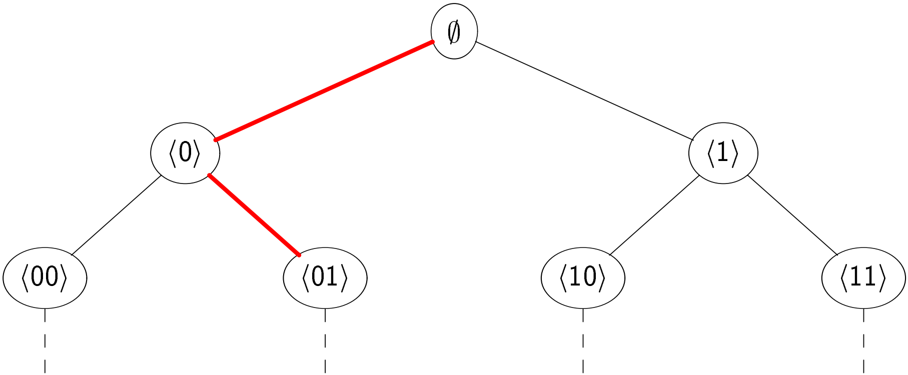
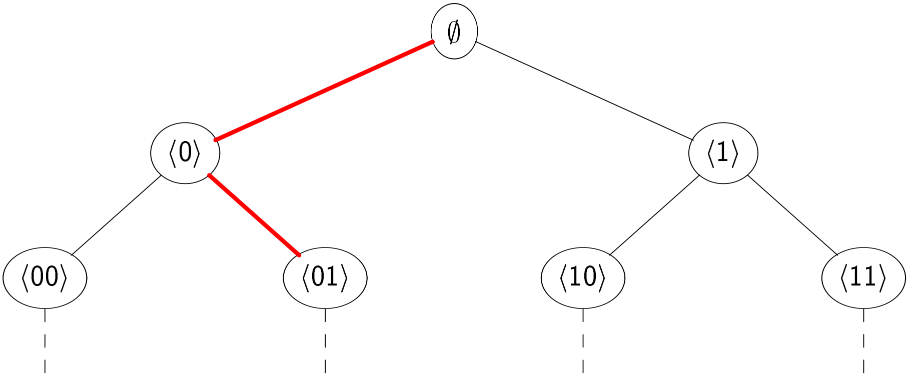


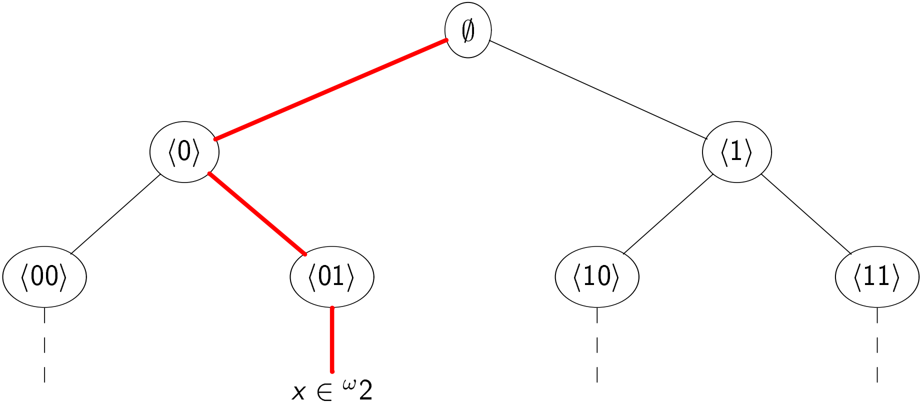
Gale-Stewart games ctd....

I's turn again.



Gale-Stewart games ctd.....

After infinitely many times...



Player I wins if x is in the payoff set A and otherwise Player II wins.

The Axiom of Determinacy

A subset A of ${}^{\omega}2$ is *determined* if one of the players has a winning strategy in the Gale-Stewart game with the payoff set A.

Definition (Mycielski-Steinhaus)

The Axiom of Determinacy (AD) asserts the following:
Every subset A of ${}^{\omega}2$ is determined.

Remark

1. AD contradicts the Axiom of Choice (AC).
2. AD has many beautiful consequences, e.g., every set of reals is Lebesgue measurable.
3. Models of AD (or AD^+) are one of the central objects in set theory.

Extensions of AD

One can define AD_X for any nonempty set X. (Note: $\mathrm{AD} = \mathrm{AD}_2$).

Definition

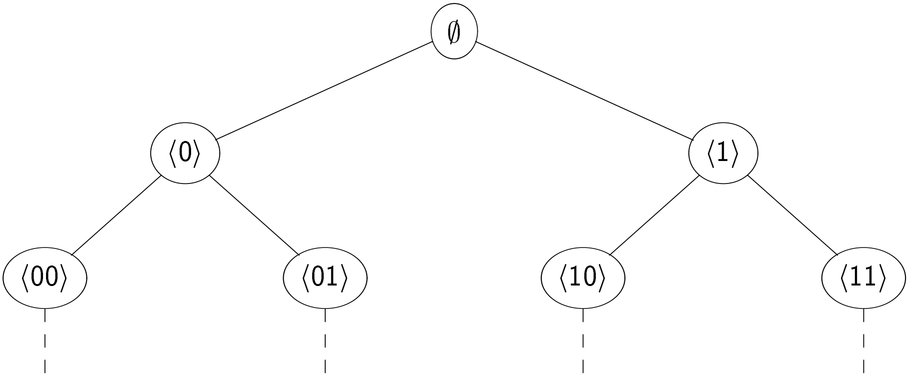
AD_X: Every $A \subseteq {}^{\omega}X$ is determined.

AD_X is inconsistent for most X.

Out interest: AD and $\mathrm{AD}_{\mathbb{R}}$.

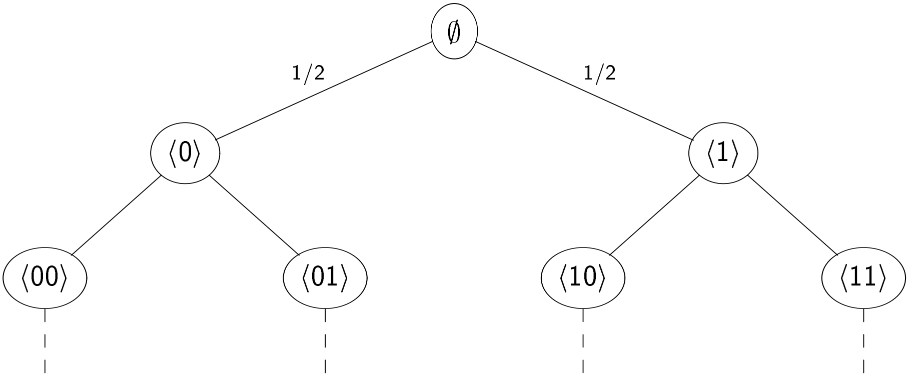
Blackwell games

I's turn.



Blackwell games ctd.

I has played.



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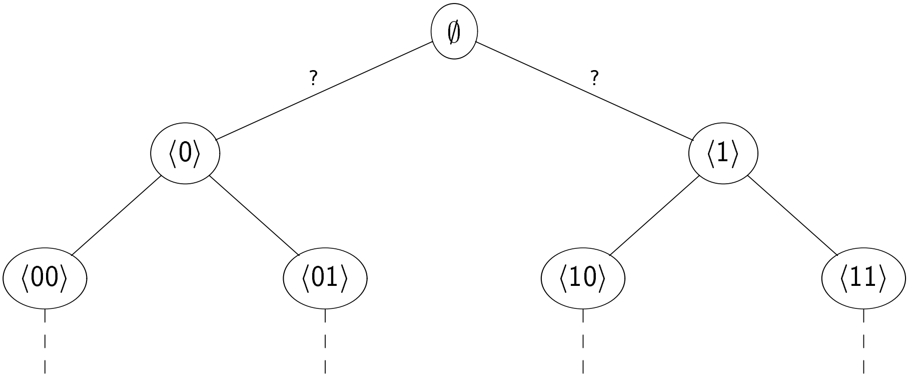

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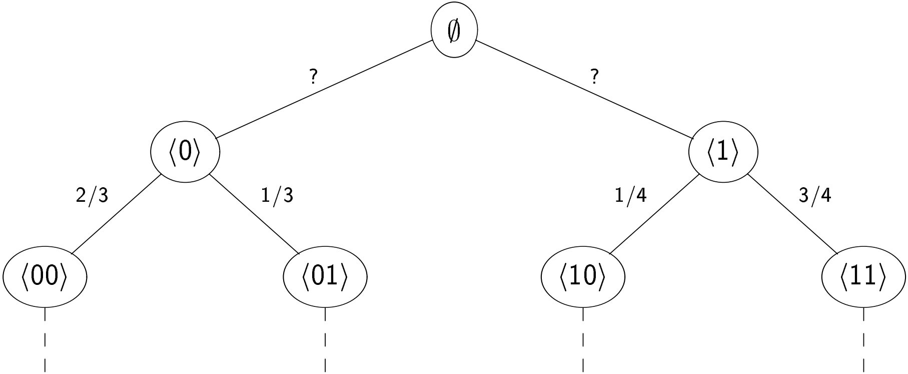

Blackwell games ctd....

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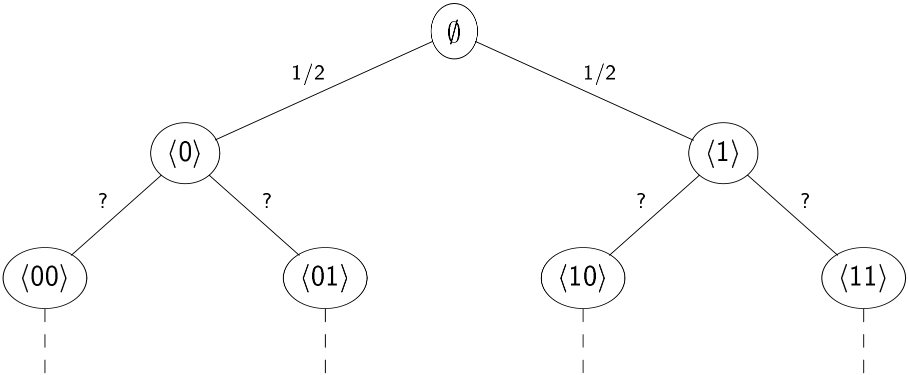
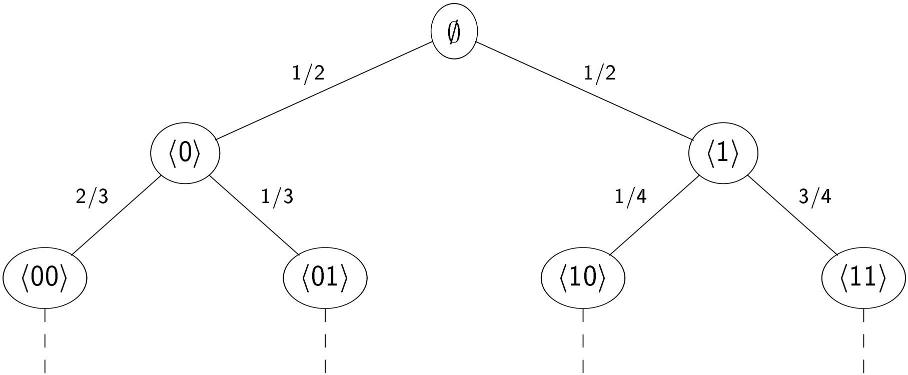


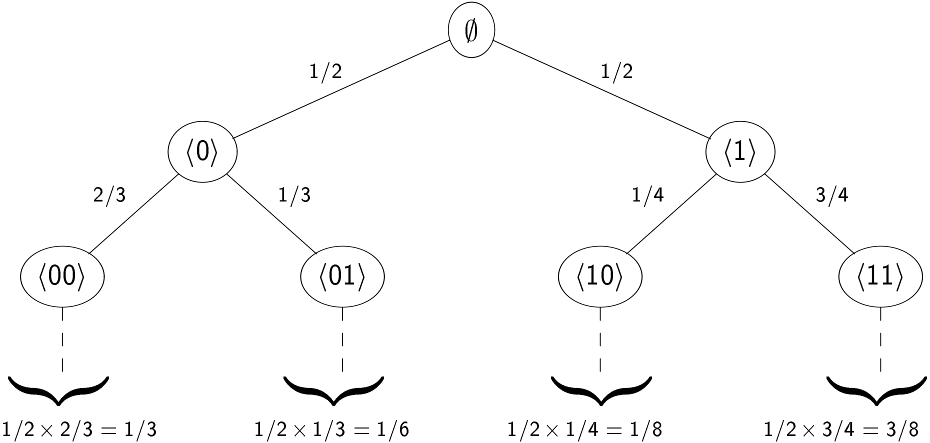
Blackwell games ctd.....

After infinitely many times...



Blackwell games ctd......

Calculate the probability as below.



Blackwell games ctd.......

Player I wins if the probability of the payoff set is 1.
Player II wins if the probability of the payoff set is 0.

Formal definitions; Blackwell games

- σ is a *mixed strategy for I* if $\sigma\colon 2^{\text{Even}} \to \text{Prob}(2)$.
- τ is a *mixed strategy for II* if $\tau\colon 2^{\text{Odd}} \to \text{Prob}(2)$.
- For a mixed strategy σ for I and a mixed strategy τ for II, define $\sigma * \tau\colon {}^{<\omega}2 \to \text{Prob}(2)$ as follows:

$$\sigma * \tau(s) = \begin{cases} \sigma(s) & \text{if } \text{lh}(s) \text{ is even,} \\ \tau(s) & \text{if } \text{lh}(s) \text{ is odd.} \end{cases}$$

Then define $\mu_{\sigma,\tau}\colon {}^{<\omega}2 \to [0,1]$ as follows:

$$\mu_{\sigma,\tau}(s) = \prod_{i<\text{lh}(s)} \sigma * \tau(s \restriction i)(s(i)).$$

With the help of $\mathsf{DC}_{\mathbb{R}}$, one can uniquely extend $\mu_{\sigma,\tau}$ to a Borel probability measure on the Cantor space.

Formal definitions; Blackwell games ctd.

Let $A \subseteq {}^\omega 2$.

- A mixed strategy σ for I is *optimal in A* if for any mixed strategy τ for II, $\mu_{\sigma,\tau}(A) = 1$.
- A mixed strategy τ for II is *optimal in A* if for any mixed strategy σ for I, $\mu_{\sigma,\tau}(A) = 0$.
- A is *Blackwell determined* if either I or II has an optimal strategy in A.
- Bl-AD: Every $A \subseteq {}^\omega 2$ is Blackwell determined.

Note: There is another formulation of Blackwell games coming from game theory.

Formal definitions; Blackwell games ctd..

Let X be a non-empty set.

- σ is a *mixed strategy for I* if $\sigma\colon X^{\mathrm{Even}} \to \mathrm{Prob}_\omega(X)$.
- τ is a *mixed strategy for II* if $\tau\colon X^{\mathrm{Odd}} \to \mathrm{Prob}_\omega(X)$.
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$$\mu_{\sigma,\tau}(s) = \prod_{i<\mathrm{lh}(s)} \sigma * \tau(s \restriction i)(s(i)).$$

If we have $\mathrm{DC}_{(\mathbb{R}\times{}^\omega X)}$, we can uniquely extend $\mu_{\sigma,\tau}$ to a Borel probability measure on ${}^\omega X$.

Note: $\mathrm{DC}_{(\mathbb{R}\times{}^\omega\mathbb{R})}$ follows from $\mathrm{DC}_{\mathbb{R}}$

Formal definitions; Blackwell games ctd...

Let X be a non-empty set and $A \subseteq {}^{\omega}X$.
Assume we have $\mathrm{DC}_{(\mathbb{R}\times{}^{\omega}X)}$.

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- A mixed strategy τ for II is *optimal in A* if for any mixed strategy σ for I, $\mu_{\sigma,\tau}(A) = 0$.
- A is *Blackwell determined* if either I or II has an optimal strategy in A.
- $\mathrm{Bl\text{-}AD}_X$: every $A \subseteq {}^{\omega}X$ is Blackwell determined.

We are mainly interested in Bl-AD and $\mathrm{Bl\text{-}AD}_{\mathbb{R}}$.

Observation 1

Observation

Let X be a nonempty set and A be a subset of ${}^{\omega}X$. If A is determined, then A is Blackwell determined.

Point: Given a strategy σ, one can naturally transform it to a mixed strategy $\hat{\sigma}$. If σ is winning, then $\hat{\sigma}$ is optimal.

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AD implies Bl-AD. $\mathrm{AD}_{\mathbb{R}}$ implies Bl-$\mathrm{AD}_{\mathbb{R}}$.

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Conjecture (Martin)

Bl-AD implies AD.

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If a finite game is Blackwell determined, then it is determined when X is totally ordered.

Finite games $=$ games ending at some fixed round $n < \omega$.

Sketch of proof.

In blackboards. $\square$

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Corollary (Löwe)

Assume $\mathsf{Bl\text{-}AD}_{\mathbb{R}}$. Then Uniformization holds, i.e., every relation on the reals can be uniformized by a function.

Observation 2 ctd.

By the same argument...

Proposition

If a clopen set is Blackwell determined, then it is determined.

Observation 2 ctd.

By the same argument...

Proposition

If a clopen set is Blackwell determined, then it is determined.

Theorem (Neeman)

Assume Bl-AD. Then every Suslin & co-Suslin subset of the Cantor space is determined.

Definition

1. A subset A of the Cantor space is *Suslin* if there is an ordinal γ and a tree T on $2 \times \gamma$ such that $A = \mathrm{p}[T]$.
2. A subset A of the Cantor space is *co-Suslin* if the complement of A is Suslin.

Observation 2 ctd..

Theorem (Kechris & Woodin)

If every Suslin & co-Suslin set is determined, then $\mathrm{AD}^{\mathrm{L}(\mathbb{R})}$ holds.

Observation 2 ctd..

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Corollary (Martin, Neeman & Vervoort)

$\mathrm{L}(\mathbb{R}) \vDash$ "AD $\iff$ Bl-AD". In particular, AD and Bl-AD are equiconsistent.

Observation 3

Observation

Assume BI-AD$_{\mathbb{R}}$. Let $A \subseteq {}^{\omega}\mathbb{R}$. If A is range-invariant, then A is determined.

Definition

A set $A \subseteq {}^{\omega}\mathbb{R}$ is *range-invariant* if for any $\vec{x}, \vec{y} \in {}^{\omega}\mathbb{R}$ with the same range, $\vec{x} \in A \iff \vec{y} \in A$.

Observation 3 ctd.

Theorem (de Kloet, Löwe, I.)

Assume BI-$AD_{\mathbb{R}}$. Then there is a fine, normal, σ-complete ultrafilter on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

Definition

Let U be a filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

1. U is *fine* if for any $x \in \mathbb{R}$, $\{a \in \mathcal{P}_{\omega_1}(\mathbb{R}) \mid x \in a\} \in U$.
2. U is *normal* if for any family $\{A_x \in U \mid x \in \mathbb{R}\}$, $\triangle_{x \in \mathbb{R}} A_x = \{a \in \mathcal{P}_{\omega_1}(\mathbb{R}) \mid (\forall x \in a)\ a \in A_x\} \in U$.

Observation 3 ctd..

Theorem (Solovay)

If there is a fine, normal σ-complete ultrafilter on $\mathcal{P}_{\omega_1}(\mathbb{R})$, then $\mathbb{R}^\#$ exists.

Points:

1. By assumption, ω_1 is measurable and hence $a^\#$ exists for all $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$.
2. Letting U be a fine normal measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$,

$$\phi \in \mathbb{R}^\# \iff \{a \in \mathcal{P}_{\omega_1}(\mathbb{R}) \mid \phi \in a^\#\} \in U.$$

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Corollary (de Kloet, Löwe & I.)

Assume $\mathsf{Bl\text{-}AD}_{\mathbb{R}}$. Then $\mathbb{R}^\#$ exists. Hence $\mathsf{Bl\text{-}AD}_{\mathbb{R}} \vdash \mathrm{Con}(\mathsf{AD})$.

$\mathrm{AD}_{\mathbb{R}}$ vs. $\mathrm{BI\text{-}AD}_{\mathbb{R}}$

Question

Does $\mathrm{BI\text{-}AD}_{\mathbb{R}}$ imply $\mathrm{AD}_{\mathbb{R}}$?

$\mathsf{AD}_{\mathbb{R}}$ vs. $\mathsf{Bl}\text{-}\mathsf{AD}_{\mathbb{R}}$

Question

Does Bl-$\mathsf{AD}_{\mathbb{R}}$ imply $\mathsf{AD}_{\mathbb{R}}$?

95% Theorem (Woodin & I.)

Under ZF+DC, $\mathsf{AD}_{\mathbb{R}}$ and Bl-$\mathsf{AD}_{\mathbb{R}}$ are equivalent.

Remark

Martin's conjecture (Bl-AD$\Rightarrow$AD) implies the above "Theorem" under ZF+DC.

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Martin's conjecture (BI-AD$\Rightarrow$AD) implies the above "Theorem" under ZF+DC.

But!

Remark (Solovay)

$\mathsf{AD}_{\mathbb{R}}$+DC implies the consistency of $\mathsf{AD}_{\mathbb{R}}$.

So assuming DC is not optimal for the above "Theorem".

$\mathsf{AD}_{\mathbb{R}}$ vs. $\mathsf{BI}\text{-}\mathsf{AD}_{\mathbb{R}}$ ctd.

Question

Are $\mathsf{AD}_{\mathbb{R}}$ are $\mathsf{BI}\text{-}\mathsf{AD}_{\mathbb{R}}$ equiconsistent?

$\mathrm{AD}_{\mathbb{R}}$ vs. $\mathrm{BI}\text{-}\mathrm{AD}_{\mathbb{R}}$ ctd.

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Conjecture (Woodin)

Assume the following:

1. Every Suslin & co-Suslin set is determined, and
2. there is a fine, normal σ-complete ultrafilter on $\mathcal{P}_{\omega_1}(\mathbb{R})$.
   Then either there is a model of $\mathrm{AD}_{\mathbb{R}}$ or there is a model M of AD^+ such that $\Theta^M = \Theta$.

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Theorem (Woodin & I.)

If Conjecture is true, then $\mathsf{AD}_{\mathbb{R}}$ and $\mathsf{BI}\text{-}\mathsf{AD}_{\mathbb{R}}$ are equiconsistent.

BI-$AD_{\mathbb{R}}$ and generic embeddings

Theorem (Sargsyan)

Assume CH and that there is a generic embedding $j\colon V \to M$ such that

1. M is transitive and ${}^{\omega}M \cap V[G] \subseteq M$,
2. G is a generic filter of a homogeneous forcing, and
3. $j \restriction \mathrm{Ord}$ is definable in V.

Then there is a model of ZF+$AD_{\mathbb{R}}$+"Θ is regular".

The method: Core Model Induction

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The method: Core Model Induction

Proposition

Assume BI-$\mathrm{AD}_{\mathbb{R}}$. Then for any $\alpha < \Theta$ and $A \subseteq \mathbb{R}$, there is a generic embedding $j \colon \mathrm{L}(A, \mathbb{R}) \to M$ such that

1. M is transitive, $\mathbb{R}^{V[G]} \subseteq M$, and α is countable in M,
2. G is a generic filter of a homogeneous forcing, and
3. $j \restriction \mathrm{Ord}$ is definable in V.

Thank you very much for your attention!

ご清聴ありがとうございました！ お疲れ様です !!