Chang’s Conjecture, generic elementary embeddings and inner models for huge cardinals

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• Much of the early history of logic in general and model theory in particular was tied up with understanding the expressive power of first and second order logic (and their variants).

• One distinguishing feature of first order logic is the Downwards Lowenheim-Skolem Theorem.

• Tremendous effort was put into generalizing the downwards Lowenheim-Skolem theorem so that the elementary substructure had some second order properties

• The coarsest second order properties had to do with cardinality; in this discussion we consider various more subtle second order properties. Among them are being correct for the non-stationary ideal.

Let \( \mathcal{L} \) be a countable language with a distinguished unary predicate \( R \). Then

\[
\mathfrak{A} = \langle A, R^\mathfrak{A}, f_i, R_j, c_k \ldots \rangle_{i,j,k \in \omega}
\]

is said to have type \((\kappa, \lambda)\) if and only iff \(|A| = \kappa\) and \(|R^\mathfrak{A}| = \lambda\).

We often write \( \mathfrak{A} = \langle \kappa; \lambda, f_i \ldots \rangle \) to mean a structure of type \((\kappa, \lambda)\).
Two cardinal Transfer Theorems

We write $$(\kappa, \lambda) \rightarrow (\kappa', \lambda')$$
to mean that if $\mathfrak{A}$ is an $\mathcal{L}$-structure of type $(\kappa, \lambda)$ then there is a $\mathfrak{B} \equiv \mathfrak{A}$ of type $(\kappa', \lambda')$.

Classical Results:
I Vaught's Two Cardinal Theorem: $$(\kappa, \rho) \rightarrow (\omega_1, \omega)$$ for all $\kappa > \rho \geq \omega$.

II The infinite gap two cardinal theorem: $$(\kappa^+ + \gamma, \kappa) \rightarrow (\rho^+ + \delta, \rho)$$ for all infinite cardinals $\kappa, \rho$ and ordinals $\gamma, \delta$.

Jensen developed Morasses to prove:
III Jensen Gap n two cardinal theorem
$$L \models (\forall n \in \omega)(\forall \text{infinite } \kappa, \lambda)((\kappa + n, \kappa) \rightarrow (\lambda + n, \lambda))$$.

More in the spirit of the Lowenheim-Skolem theorem

For $\kappa \geq \kappa'$ and $\lambda \geq \lambda'$, we say $(\kappa, \lambda) \rightarrow (\kappa', \lambda')$ iff

for all $\mathfrak{A}$ of type $(\kappa, \lambda)$ in a countable language there is an elementary substructure $\mathfrak{B} \prec \mathfrak{A}$ of type $(\kappa', \lambda')$.

I Classical Chang's Conjecture $$(\omega_2, \omega_1) \rightarrow (\omega_1, \omega)$$.

This conjecture looks obviously “set theoretical” to modern eyes: If you apply it to $\langle L(\omega_2)^V, (\omega_1)^V, \in \rangle$ one gets an elementary substructure $N \prec L_{\omega_2}$. If $\bar{N}$ is the transitive collapse of $N$ and $j$ is the inverse of the transitive collapse map, then the embedding $j$ yields an $L$-ultrafilter on $\text{crit}(j)$ and hence $O^\#$. As a consequence, it is clear that Chang’s Conjecture cannot be a theorem of ZFC.

II (Silver) Con (ZFC + there is an $\omega_1$-Erdös cardinal) implies Con (ZFC + GCH + $$(\omega_2, \omega_1) \rightarrow (\omega_1, \omega)$$).

In fact the exact consistency strength of Chang’s Conjecture has been shown to be an $\omega_1$-Erdös cardinal.

III (Kunen/Laver) Con (ZFC + there is a huge cardinal) implies that for all $n \geq 1$, Con (ZFC + $$(\omega_{n+1}, \omega_n) \rightarrow (\omega_n, \omega_{n-1})$$)

IV (Foreman) Con (ZFC + there is a 2-huge cardinal) implies Con (ZFC + GCH + $$(\forall m < n)(\omega_{n+1}, \omega_n) \rightarrow (\omega_{m+1}, \omega_m)$$).
Chang’s Conjecture shows up in many situations

For example:

$$\kappa \rightarrow [\mu]^\omega_{\rho^+}$$

is equivalent to

$$(\kappa, \rho^+) \rightarrow (\mu, \rho).$$

An example of a family of deeper results is:

(Foreman) Assume the CH. Then:

$$(\omega_3, \omega_2) \rightarrow (\omega_2, \omega_1)$$

is equivalent to

$$\omega_3 \rightarrow [\omega_2]_{\omega_2}^\omega.$$

Note the infinite exponent.

Ideas from Chang’s Conjecture properties turned out to be crucial for semi-proper forcing type arguments—e.g. “antichain catching”.

What second order properties can you ask for in an elementary substructure?

To investigate this let us give a more modern looking reformulation of Chang’s Conjecture:

**Proposition 1**

$$(\omega_{n+2}, \omega_{n+1}) \rightarrow (\omega_{n+1}, \omega_n) \text{ iff (for all) (there exists) } \theta \gg \omega_{n+2} \text{ and an } N \prec \langle H(\theta), \in, \Delta \rangle \text{ such that}$$

- if $$\pi : N \rightarrow \bar{N}$$ is the transitive collapse then
  - $$\pi \upharpoonright \omega_n = \text{id}$$ and
  - $$\pi(\omega_{n+2}) = \omega_{n+1}.$$

$$\vdash (\Leftarrow)$$ Suppose that the CC fails. Let $$\mathfrak{A} = \langle \omega_{n+2}, \omega_{n+1}, f_i, R_j, c_{i,j,k} \rangle_{i,j,k \in \omega}$$ be the $$\Delta$$-least counterexample. Suppose that $$N \prec \mathfrak{A}$$ is as in the hypothesis. Then: $$\mathfrak{A} \in N$$ so $$N \cap \omega_{n+2}$$ is closed under Skolem functions for $$\mathfrak{A}$$. In particular $$N \cap \omega_{n+2} \prec \mathfrak{A}$$. But $$|N \cap \omega_{n+2}| = \omega_{n+1}$$ and $$N \cap \omega_{n+1} = \omega_n$$. This is a contradiction.

$$(\Rightarrow)$$ Let $$\mathfrak{A} = \langle \omega_{n+2}, \in, f_i \rangle$$ be a fully Skolemized structure such that for all $$z \prec \mathfrak{A}$$ we know that $$sk_{H(\theta)}(z) \cap \omega_{n+2} = z$$. By the CC we know there is a $$z \prec \mathfrak{A}$$ such that the type of $$z$$ is $$(\omega_{n+1}, \omega_n)$$. Let $$N_0 = sk_{H(\theta)}(z)$$ Then $$N_0 \prec H(\theta)$$.

Let $$N = sk_{H(\theta)}(N_0 \cup \omega_n)$$. We claim that:
\begin{itemize}
  \item \(\sup(N \cap \omega_{n+1}) = \sup(N_0 \cap \omega_{n+1})\)
  \item \(\sup(N \cap \omega_{n+2}) = \sup(N_0 \cap \omega_{n+2}).\)
\end{itemize}

To see this, let \(\tau : N_0 \times \omega_n \rightarrow \omega_{n+1}\) be a Skolem function. Since \(\Delta\) is in the language, for \(\vec{x} \in N\), the function \(\tau(\vec{x}, \cdot) : \omega_n \rightarrow \omega_{n+1}\) is definable in \(N_0\). In particular, \(\sup(\tau(\vec{x}, \cdot) \omega_n) \in N_0\). Since

\[
N \cap \omega_{n+1} = \bigcup \{\tau(\vec{x}, \cdot) \omega_n : \tau\text{ is a Skolem function and } \vec{x} \in N_0\},
\]

we see that \(\sup(N_0 \cap \omega_{n+1}) = \sup(N \cap \omega_{n+1})\). The result for \(\omega_{n+2}\) is seen similarly.

Let \(\pi : N \rightarrow \bar{N}\) be the transitive collapse. Since \(\omega_n \subseteq N\), \(\pi \upharpoonright \omega_n\) is the identity map. To see that \(\pi(\omega_{n+2}) = \omega_{n+1}\) we need to see that \(N \cap \omega_{n+2}\) has order type \(\omega_{n+1}\). Note that the order type is at least \(\omega_{n+1}\) by the choice of \(z\). Let \(\alpha \in N \cap \omega_{n+2}\). Then there is a bijection \(f : \omega_{n+1} \rightarrow \alpha\) that lies in \(N\). Hence \(f : N \cap \omega_{n+1} \rightarrow N \cap \alpha\) is a bijection. In particular, \(|N \cap \alpha| = |N \cap \omega_{n+1}|\); thus \(|N \cap \alpha| < \omega_{n+1}\).

In the \textit{Smoke and Mirrors} paper, the following more involved extension is proved:

\begin{proposition}
Let \(\lambda \leq \kappa \ll \theta\) be cardinals with \(\lambda\) and \(\theta\) regular and with \(\text{cf}(\kappa) \geq \lambda\). Let \(\mathfrak{A}\) be a structure expanding \(\langle H(\theta), \in, \Delta, \{\kappa, \lambda\}\rangle\) and \(N_0 \prec \mathfrak{A}\). Let \(N_1 = sk^\mathfrak{A}(N_0 \cup sup(N_0 \cap \lambda))\) and let \(\rho = |sup(N_0 \cap \lambda)|\). Suppose that either:

1. The GCH holds or
2. \(\xi \subseteq N_0\) and \(\kappa \leq \lambda^+\).

Then \(N_1 \cap \lambda = sup(N_0 \cap \lambda)\) and \(|N_1 \cap \kappa| = |N_0 \cap \kappa| \cdot \rho\).
\end{proposition}

\textbf{Note:} I don’t know if the GCH is relevant or necessary.

Can we ask for more properties of \(\bar{N}\)?

For example: can we ask that \((\omega_3, \omega_2) \rightarrow (\omega_2, \omega_1)\) is witnessed by an \(N \prec H(\theta)\) and \(N \cap \omega_3 \in \bar{N}\)?

\textbf{Digression on NS ideals}

Let \(S \subseteq P(X)\). Then \(S\) is \textit{stationary} iff for all \(\mathfrak{A} = \langle X, f_i \rangle_{i \in \omega}\), there is a \(z \prec \mathfrak{A}\) such that \(z \in S\). (In some places this is called “weakly stationary”.)

This is a good generalization of “stationary” in other contexts:
• it coincides with the usual definition of a “stationary set \( S \subseteq \kappa \)” for subsets of \( \kappa \).

• it coincides with the usual definition of a “stationary set \( S \subseteq P_\kappa(\lambda) \)” for subsets of \( P_\kappa(\lambda) \).

With this language we see that all “Chang Conjectures” are simply statements that certain sets are stationary. For example

\[(\omega_{n+2}, \omega_{n+1}) \rightarrow (\omega_{n+1}, \omega_n)\]

is equivalent to

\[\{ z \in [\omega_{n+2}]^{\omega_{n+1}} : |z \cap \omega_{n+1}| = \omega_n \} \text{ is stationary.}\]

Let us see what happens to the non-stationary ideal under elementary substruc-tures:

Suppose that \( N \prec H(\theta) \) and \( |N \cap \omega_i| = \omega_{i-1} \) for \( i = 2, 3, 4 \), and assume that \( \omega_1 \subseteq N \). Let \( A' \subseteq [\omega_4]^{\omega_3} \). Let \( \pi : N \rightarrow \bar{N} \) be the transitive collapse.

Then \( \pi(A') = A \) for some \( A \subseteq [\omega_3]^{\omega_2} \). Moreover,

\[\pi(NS \upharpoonright A') \subseteq (NS \upharpoonright A) \cap \bar{N} .\]

**Definition 3** \( N \) is correct for \( NS \upharpoonright A \) iff \( A \in N \) and there is an \( A', I' \in N \) such that

\[\pi(I' \upharpoonright A') = (NS \upharpoonright A) \cap \bar{N} .\]

(We write \( \pi_N \) for the transitive collapse map of \( N \). The set \( A' \) is mentioned only to make sure that there is a set \( A' \in N \) such that \( \pi_N(A') = A \).)

**Note:** Being correct about \( NS \upharpoonright A \) is a thickness property (or closure property) of \( \bar{N} \).

**Canonically well-ordered stationary sets**

**Proposition 4** (Baumgartner) Let \( M, N \prec H(\theta) \). Suppose that

- \( \sup(M \cap \omega_{n+2}) = \sup(N \cap \omega_{n+2}) \in cof(> \omega) \),
- \( N \cap \omega_{n+1} = M \cap \omega_{n+1} \).
Then \( M \cap \omega_{n+2} = N \cap \omega_{n+2} \).

To use the Baumgartner proposition, we need the following result:

**Proposition 5** (Foreman-Magidor) For \( n \in \omega \) and \( N \prec H(\theta) \):

if \( N \in [\omega_{n+2}]^{\omega_{n+1}}, N \cap \omega_{n+1} \in \omega_{n+1} \), then \( \text{cof}(N \cap \omega_{n+1}) = \omega_n \).

Putting these two propositions together we get the following:

**Corollary 6** Let

\[
p : [\omega_{n+2}]^{\omega_{n+1}} \to \omega_{n+1} \times \omega_{n+2}
\]

be defined by

\[
z \mapsto (z \cap \omega_{n+1}, \sup(z \cap \omega_{n+2})).
\]

Then \( p \) is 1-1 on the collection of \( z \) such that:

1. \( \text{sk}^{H(\theta)}(z) \cap \omega_{n+2} = z \),
2. \( z \cap \omega_{n+1} \in \omega_{n+1} \).

**Upshot:** Relative to a closed unbounded set restricted to \( \{ z \in [\omega_{n+2}]^{\omega_{n+1}} : z \cap \omega_{n+1} \in \omega_{n+1} \} \), \( p \) is 1-1.

**Definition 7** A set \( A \subseteq \{ z \in [\omega_{n+2}]^{\omega_{n+1}} : z \cap \omega_{n+1} \in \omega_{n+1} \} \) is ordinary if \( p \) is 1-1 on \( A \).

We note that ordinary sets have canonical and absolute well-orderings.

**Strong Chang Reflection**

We will say that **Strong Chang reflection** holds for \( (\omega_{n+3}, \omega_n) \) iff for all large enough \( \theta \) there is an ordinary \( A \in [\omega_{n+2}]^{\omega_{n+1}} \) and an \( O' \) such that for some

\[
N \prec \langle H(\theta), \in, \Delta, A, O' \rangle
\]

we have:

1. \( N \cap \omega_{n+2} \in A \) and \( |N \cap \omega_{n+3}| = \omega_{n+2} \),
2. \( \pi_N(O') = N \cap \omega_{n+2} \),
3. \( N \) is correct for \( \text{NS} \upharpoonright A \).

**Informally** SCR says that the collection of \( N \) that are correct about \( \text{NS} \upharpoonright A \) and whose transitive collapse contains \( N \cap \omega_{n+2} \) is stationary and canonically well-ordered.
The Theorems

Theorem 8 Suppose Strong Chang Reflection holds for \((\omega_{n+3}, \omega_n)\). Then there is a transitive inner model for “ZFC + there is a huge cardinal”.

This theorem would not be interesting except for the accompanying theorem:

Theorem 9 Suppose there is a 2-huge cardinal. Then for each \(n\) there is a forcing extension in which SCR holds for \((\omega_{n+3}, \omega_n)\).

Theorem 9 is much harder than Theorem 8, but is less novel. For this reason I will concentrate on a discussion of Theorem 9.

Constructing models with very large cardinals in them

Obstacles:

- If \(U\) is a supercompact measure on \(P_\kappa(\lambda)\), the \(L[U] = L\).
- Even if \(S \subseteq P_\kappa(\lambda)\) we might have \(L[S] = L\).
- If we take \(L(S)\) then we probably don’t get a model of choice.
- Known \(L[E]\) models don’t have very large cardinals.

A “workaround”

If \(A \subseteq P(\lambda)\) and \(W = \langle a_\alpha : \alpha < \gamma \rangle\) is an enumeration of \(A\), let

\[ A^* = \{(\beta, \alpha) : \beta \in a_\alpha \}. \]

Then \(A, A^*, W\) are all elements of \(L[A^*]\) and \(L[A^*] \models ZFC\).

If \(A\) is ordinary then there is a canonical candidate for \(W\)–the lexicographical ordering on the pairs of critical ordinals.

- Our models will be of the form \(L[A^*, NS]\).

Decisive Ideals

Suppose that \(X' \subseteq X\). We get a map \(\Pi : P(X) \to P(X')\) by setting \(\Pi(z) = z \cap X'\). If \(J\) is an ideal on \(P(X)\) we get an ideal \(I = \pi_{X'}(J)\) on \(P(X')\) by setting \(A \in I\) iff \(\Pi^{-1}(A) \in J\).
Definition 10  Let $Z \subseteq P(X)$ and $J$ be an ideal on $Z$. Let $X' \subseteq X$ and $I = \pi_{X'}(J)$. Then $J$ decides $I$ if and only if there is a set $A \in \tilde{I}$ and a well-ordering $W$ of $A$, and $A', W', O', I'$ such that for all generic $G \subseteq P(X)/J$:

1. an initial segment of the ordinals of $V^Z/G$ is well-founded and isomorphic to $(|A'|^+)^V$
   and

2. if $j : V \rightarrow M \cong V^Z/G$ is the canonical embedding, where $M$ is transitive up to $(|A'|^+)^V$, then
   (a) $j(A) = A'$
   (b) $j(W) = W'$
   (c) $j''|A| = O'$
   (d) $I' = j(I) \cap V$.

We will say that $J$ is decisive iff $J$ decides $J$.

- Usually $A$ will be ordinary, in which case the canonical well-ordering is absolute and so clause (b) is vacuous.

- Surprisingly, precipitousness is not part of the definition. In fact, it will follow from the next lemma, that well-foundedness is not a major problem for the applications.

- There seem to be two kinds of ideals yielding good generic elementary embeddings—those ideals that are the remnants of collapsed large cardinals and those ideals that have “antichain catching” properties. This definition is an attempt to characterize the former.

Lemma 11  Suppose that $J$ is a normal and fine ideal on $Z \subseteq P(X)$. Let $G \subseteq P(Z)/J$ be generic and $j : V \rightarrow V^Z/G = M$ be the canonical elementary embedding,

1. If $id : Z \rightarrow Z$ is defined by $id(z) = z$, then $[id]^M_G = j''X$.

2. $V^Z/G$ is well-founded up to $(|X|^+)^V$. 

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The first item is standard. To see the second, note that if there is a well-ordering \(<\) that belongs to \(V^Z/G\) then \(V^Z/G\) is well-founded to the length of \(<\). To see the second item, we note that we can replace \(X\) by \(\lambda = |X|\). Then \([id]^M_G = j^{\lambda}\), and \((j^{\lambda}, E)\) isomorphic to \(\lambda\), where \(E\) is the ultrapower of \(\epsilon\).

Lemma 11 implies that if we have a “local” property of the \(\aleph_n\)’s then by taking generic ultrapowers using the index set \(H(\theta)\) for large enough \(\theta\) we can get all the well-foundedness we want.

The next result is our main tool for building inner models with very large cardinals:

**Theorem 12** Let \(\mu \leq \lambda\) be cardinals. Suppose that \(J\) is a normal, fine ideal on a set \(Z \subseteq P(\lambda)\) that decides a countably complete ideal \(I \subseteq P(Z')\) for some \(Z' \subseteq P(\mu)\). Suppose that \(A, A', W, W', I', O'\) witness that \(J\) decides \(I\). Then either:

a. \(L[A^*, I] \models \bar{I}\) is an ultrafilter on \(A\)
   or for some generic \(G \subseteq P(Z)/I\), if \(j : V \to V^Z/G\) is the ultrapower embedding, then

b. \(L[j(A^*), I'] \models \bar{I}'\) is an ultrafilter on \(j(A)\).

**Note:** The model \(L[j(A^*), I']\) is not the same as \((L[j(A^*)], j(I))]^{V^Z/G}\) if the ultrapower is ill-founded.

**Note:** If \(J\) is \(\kappa\) complete and \(Z'\) is \(P_\kappa(\lambda)\) then this says that there is an inner model with a supercompact. Similarly if \(Z' = [\lambda]^{\kappa}\), then this gives an inner model with a huge cardinal.

**Claim:** For \(S \subseteq P(Z')\):

\[S \in I \iff \text{for all generic } G \subseteq P(Z)/J, \ j^{\mu} \notin j(S)\]

\(\Downarrow \) \(S \in I\) iff \(\pi^{A^*}_\mu(S) \in J\), so it suffices to show that for all \(T \subseteq P(Z)\), for all generic \(G\), we have that \(T \in J\) iff \(j^{\lambda} \notin j(T)\). We compute:

\[(Z \setminus T) \in \bar{J} \iff \{z : z \in Z \setminus T\} \in G \text{ for all generic } G\]
\[\iff \{z : id(z) \in Z \setminus T\} \in G \text{ for all generic } G\]
\[\iff [id]^M \in j(Z \setminus T) \text{ for all generic } G\]
\[\iff j^{\lambda} \in j(Z) \setminus j(T) \text{ for all generic } G\]
\[\iff j^{\lambda} \notin j(T) \text{ for all generic } G.\]
(Theorem 12) Suppose that $a.$ fails. Let $\delta$ be the least ordinal such that $L_{\delta+1}[A^*, I] \models \text{"}\bar{I} \text{ is not an ultrafilter."}$ Then $\delta$ is definable in $L[A^*, I]$ and $\delta < |A'|^+$. Moreover, there is a formula $\phi(w, u, v)$ such that

$$B = \{ z' \in A : L_{\delta+\omega}[A^*, I] \models \phi(z', A^*, I) \} \notin I \cup \bar{I}.$$ 

Let $G \subseteq P(Z)/J$ be generic and $j : V \to M$ be the canonical elementary embedding where $M \cong V^Z/G$ and $M$ is well-founded up to $(|A'|)^V$. Then

$$j \upharpoonright L[A^*, I] : L[A^*, I] \to L^M[j(A^*), j(I)]^M.$$ 

As a caveat to the reader we note that since $M$ may not be well-founded it is likely that $L^M[j(A^*), j(I)]^M \neq L[j(A^*), j(I)]^M$.

Since $j(A) = A', I' = j(I) \cap V, j(W) = W'$ we know that $j(A^*) \in V$. An induction shows that if $\xi \in OR^M$ is well-founded then:

$$L^M_\xi[j(A^*), j(I)] = L_\xi[j(A)^*, j(I)] = L_\xi[(A')^*, I'].$$

**Case 1:** For some $G \subseteq P(Z)/I, j(\delta)$ is not in the well-founded part of $M$.

In this case

$$L_{|A'|^+}[j(A)^*, I'] = \bar{I}'$$

is an ultrafilter.

Since $|A'|^+ = |j(A^*)|^+$,

$$L[j(A)^*, I'] \models \bar{I}'$$

is an ultrafilter.

**Case 2:** For all generic $G \subseteq P(Z)/I, j(\delta)$ belongs to the well-founded part of $M$.

In this case we let $\delta'$ be the least ordinal in $M$ such that

$$L_{\delta'+1}[j(A^*), j(I)] \models j(\bar{I}) \text{ is not an ultrafilter.}$$

Then $j(\delta) = \delta'$ and $\delta'$ is in the well-founded part of $M$. It follows that

$$L_{\delta'+1}[j(A^*), j(I)]^M = L_{\delta'+1}[j(A^*), j(I)]^V.$$ 

Since $\delta'$ is definable in $V$, it follows that for all generic $G, j(\delta) = \delta'$ and

$$j(B) = \{ z' \in j(A) : L_{\delta'+\omega}[j(A)^*, I'] \models \phi(z', j(A)^*, I') \}.$$ 

By hypothesis $j"\mu$ is independent of $G$ (as it is determined by $O'$) and thus we see that either:
The proof of Theorem 8

We will prove a more general theorem:

**Theorem 13** Suppose that $\kappa_2 > \kappa_1 > \kappa_0$ are cardinals and that there is a regular $\theta \gg \kappa_2$ and a stationary set $S \subseteq P(H(\theta))$ and an ordinary $A \subseteq [\kappa_1]^{\kappa_0}$ and an $O'$ such that for all $N \in S$:

1. $N \cap \kappa_1 \in A$, $|N \cap \kappa_2| = \kappa_1$.
2. $\pi_N(O') = N \cap \kappa_1$
3. $N$ is correct for $NS \upharpoonright A$.

Then there is an inner model with a huge cardinal.

**Remarks**

- The usual “proper forcing tricks” show that having a single large $\theta$ where the hypothesis holds is equivalent to the hypothesis holding for all large $\theta$. Moreover the existence of a single $N$ satisfying the hypothesis 1.-3. implies the existence of a stationary set of such $N$.
- If we take $\kappa_0 = \omega_{n+1}$, $\kappa_1 = \omega_{n+2}$ and $\kappa_2 = \omega_{n+3}$, then this is a restatement of Strong Chang Reflection.
- If $\kappa_0, \kappa_1, \kappa_2$ is the cardinal sequence of a 2-huge cardinal then the hypothesis of this theorem hold.

| Since $A$ contains the projection of $S$, $A$ is stationary. A regressive function argument shows that we can find a stationary set $S' \subset S$, an ideal $I'$ and a set $A'$ such that for all $N \in S'$, both $A'$ and $I' \in N$ and $\pi_N(I' \upharpoonright A') = (NS \upharpoonright A) \cap N$. So without loss of generality we assume $S$ has this property. We also assume that $\theta$ is bigger than $(2^{2^{\lambda'}})^{+}$.

We will be done if we can show the following:

**Claim:** $NS \upharpoonright S$ decides $NS \upharpoonright A$.

Let $G \subseteq P(S)/NS$ be generic and $j : V \to M$ be the canonical embedding. Then:
• Since \( \theta \gg \kappa_2 \), \( V^S/G \) is well-founded up to \( \theta \) and hence beyond \( |A'|^+ \). By taking \( \theta \) even larger we can assume that \( M \) is well-founded up to \( (2^{2^{\aleph_1}})^+ \).

• \( \text{crit}(j) = \kappa_0, j(\kappa_0) = \kappa_1 \) and \( j(\kappa_1) = \kappa_2 \).

We need to verify a-d of clause 2 of the definition of \textit{decisiveness}. We use the easy fact that if \( N = j^*H^V((2^{2^{\aleph_1}})^+) \) then \( \pi_N = j^{-1} \).

\[ \text{(a) In } M: \]
\[ \pi_{j^*H^{(2^{2^{\aleph_1}})^+}}(j(A')) = A'. \]

because \( \pi_{j^*H^{(2^{2^{\aleph_1}})^+}} = j^{-1} \). Moreover, for all \( N \in S, \pi_N(A') = A. \) Thus
\[ \pi_{j^*H^{(2^{2^{\aleph_1}})^+}}(j(A')) = j(A), \]

in particular, \( j(A) = A' \).

\[ \text{(b) Since } A \text{ is ordinary, there is a canonical absolute well-ordering } W \text{ that gets sent by } j \text{ to a canonical absolute well-ordering } W'. \]

\[ \text{(c) We do a calculation similar to that of (a). For all } N \in S \]
\[ \pi_N(O') = N \cap \kappa_1. \]

Hence:
\[ O' = \pi_{j^*H^{(2^{2^{\aleph_1}})^+}}(j(O')) \]
\[ = j^*H((2^{2^{\aleph_1}})^+) \cap j(\kappa_1) \]
\[ = j^*\kappa_1. \]

\[ \text{(d) Similarly, since the transitive collapse of } j^*H((2^{2^{\aleph_1}})^+) \text{ is } H((2^{2^{\aleph_1}})^+): \]
\[ I' = \pi_{j^*H^{(2^{2^{\aleph_1}})^+}}(j(I')) \]
\[ = (NS \upharpoonright j(A))^M \cap H^V((2^{2^{\aleph_1}})^+) \]
\[ = j(NS \upharpoonright A) \cap V \]

This finishes the proof of Theorem 13. \( \dashv \)

We have shown that a certain natural second order reflection property of \( \omega_4 \) lies between a huge cardinal and a 2-huge cardinal in consistency strength. While \textit{ad hoc} the technique does provide a way of distinguishing between ideals yielding generic embeddings that are the remnants of large cardinal embeddings and those that are not. One could perhaps hope that similar \textit{ad hoc} techniques might show that the consistency strength of PFA or MM is that of a supercompact cardinal.
References


