$AD^{L(\mathbb{R})}$ and a Woodin cardinal in $HOD^{L(\mathbb{R})}$

(Joint work with Nam Trang)

Background

Kechris-Kleinberg-Moschovakis-Woodin

(1) Assume AD. Then

Θ is a limit of cardinals with the strong partition property.

$\Theta = \sup. \{\alpha \in \mathrm{Ord} \mid \exists \pi : \mathbb{R} \longrightarrow \alpha \text{ surjective}\}$

A regular uncountable cardinal κ has the strong partition property if $\kappa \longrightarrow (\kappa)^{\kappa}_{\mu}$ $\forall \mu < \kappa$ holds, i.e., for any partition of $[\kappa]^{\kappa}$ into

μ pieces, there is a set $H \in [\kappa]^\kappa$ s.t. $[H]^\kappa$ is contained in one piece

(2) Assume Θ is a limit of cardinals with the strong partition property. Then every Suslin & co-Suslin set of reals is determined.

$A \subseteq 2^\omega$ is <u>Suslin</u> if for some ordinal γ and some tree T on $2 \times \gamma$,

$A = p[T]$

A is <u>co-Suslin</u> if $2^\omega \setminus A$ is Suslin

Kechris–Woodin

In $L(\mathbb{R})$, T.F.A.E.

(1) AD

(2) Θ is a limit of cardinals with the strong partition property

(3) Every Suslin & co-Suslin set of reals is determined.

Henle–Mathias–Woodin

In general, (2) does not imply (1).

Question

Can one characterize AD in terms of cardinal structure of HOD if $V = L(\mathbb{R})$?

Woodin

AD implies Θ is Woodin in HOD.

Test Question

In $L(\mathbb{R})$, does AD follow from

Θ is Woodin in HOD?

Answer No.

Theorem

Assume ZFC + $\underset{\sim}{\Delta}^1_2$-determinacy.

Then there is a model M of ZFC

s.t. $L(\mathbb{R})^M \models$ "Θ is Woodin in HOD

but AD fails".

Q What is M?

A M is $L[x,G]$

for a Turing cone of $x \subseteq \omega$

& $G \subseteq \mathrm{Coll}(\omega, <\kappa_x)$-gen$/L[x]$

where κ_x is the least inaccessible

in $L[x]$.

ω_1^V

$L[x,G] \models CH$

$$(\kappa_x^+)^{L[x]} = \omega_2^{L[x,G]} \approx \Theta^{L[x,G]}$$

$$\approx \Theta^{L(\mathbb{R})}$$

$$\kappa_x = \omega_1^{L[x,G]} \approx \omega_2^{L(\mathbb{R})}$$

$$L[x] \subseteq L(\mathbb{R})^{L[x,G]} \subseteq L[x,G]$$

So $L(\mathbb{R})^{L[x,G]} \models$ "$\Theta = \omega_2$, so AD fails".

To verify: $L(\mathbb{R})^{L(x,G)} \models$ "$\underbrace{\Theta}_{= \omega_2^{L[x,G]}}$ is Woodin in HOD"

Some background

Kechris–Solovay

Assume Δ^1_2-determinacy.

Then $\exists x_0 \subseteq \omega \ \forall x \geq_T x_0$.

$$L[x] \models \text{"OD-determinacy"}.$$

Kechris–Woodin & Martin

Assume $\underset{\sim}{\Delta}^1_2$-determinacy.

Then $\exists x_0 \subseteq \omega \ \forall x, y \geq_T x_0$

$$(L[x], \in) \equiv (L[y], \in)$$

Woodin

Assume Δ^1_2-determinacy.

Then $\exists x_0 \subseteq \omega \ \forall x \geq_T x_0$

$L[x] \models$ "$\omega_2^{L[x]}$ is Woodin in HOD"?

Open Question

What is $\mathrm{HOD}^{L[x]}$

on a Turing cone of $x \subseteq \omega$?

Woodin

Assume $\underset{\sim}{\Delta}^1_2$-determinacy.

Then on a Turing cone of x,

for $G \subseteq \mathrm{Coll}(\omega, <\kappa_x)$-gen$/L[x]$

(here κ_x is the least inacc. in $L[x]$)

$HOD^{L[x,G]}$ is of the form

$L[M_\infty, \Lambda]$

where M_∞ is the direct limit of

ctble iterates of M_1 and

Λ is a partial informal of

the iteration strategy of M_∞.

Letting δ_∞ be the Woodin cardinal

in M_∞,

$$V_{\delta_\infty}^{M_\infty} = V_{\delta_\infty}^{HOD^{L(x,G)}}$$

$$HOD^{L(x,G)} \models \delta_\infty \text{ is Woodin}$$

$$\delta_\infty = \omega_1^{L(x,G)} = \Theta^{L(x,G)}.$$

Back to $M = L[x,G]$.

$(\kappa_x^+)^{L(x)} = \omega_2^{L(x,G)} = \Theta^{L(\mathbb{R})^M}$.

$\kappa_x = \omega_1^{L(x,G)}$

(*) Want : $L(\mathbb{R})^{L(x,G)} \models \underset{\delta_\infty}{\overset{\shortparallel}{\Theta}}$ is Woodin in HOD"

We have : $L(x,G) \models \underset{\delta_\infty}{\overset{\shortparallel}{\Theta}}$ is Woodin in HOD"

Q $\mathrm{HOD}^{L(\mathbb{R})^{L(x,G)}} = \mathrm{HOD}^{L(x,G)}$?

We know : $\mathrm{HOD}^{L(\mathbb{R})^{L(x,G)}} \subseteq \mathrm{HOD}^{L(x,G)}$

because $L(\mathbb{R})^{L(x,G)}$ is OD in $L(x,G)$

All we need :

$$(**) \quad (V_{\delta_\infty}^{HOD})^{L(\mathbb{R})^{L(x,G)}} = (V_{\delta_\infty}^{HOD})^{L(x,G)}.$$

Q How do we recover $HOD^{L(x,G)}$ in $L(\mathbb{R})^{L(x,G)}$ in an OD fassion?

Note $L(\mathbb{R})^{L(x,G)}$ is a Solovay model$/L[x]$

Lemma (Woodin)

With the above notations, let $\mathbb{P}_x = \{ g \in L(\mathbb{R})^{L(x,G)} \mid \exists \alpha < \kappa_x \ g : Coll(\omega, {<\alpha})\text{-gen}/L[x] \}$

Let $H \subseteq \mathbb{P}_x\text{-gen}/L(\mathbb{R})^{L(x,G)}$.

and $G' = UH$.

Then G' : $Coll(\omega, <\kappa_x)$-gen / $L[x]$

and $\mathbb{R}^{L[x,G]} = \mathbb{R}^{L[x,G']}$

So $L(\mathbb{R})^{L[x,G]} = L(\mathbb{R})^{L[x,G']}$

Moreover $\mathbb{P}_x$ is homogeneous in $L(\mathbb{R})^{L[x,G]}$.

<u>Problem</u> $\mathbb{P}_x$ needs x for a parameter

$$\mathrm{HOD}^{L[x,G]} \subseteq \mathrm{HOD}_x^{L(\mathbb{R})^{L[x,G]}}$$

but $x \notin \mathrm{HOD}^{L(\mathbb{R})^{L[x,G]}}$

even $x \notin \mathrm{HOD}^{L[x,G]}$

<u>Point</u> The above lemma holds with $\mathbb{P}_y$ & $L[y]$ for any $y \geq_T x$ in $L(\mathbb{R})^{L[x,G]}$

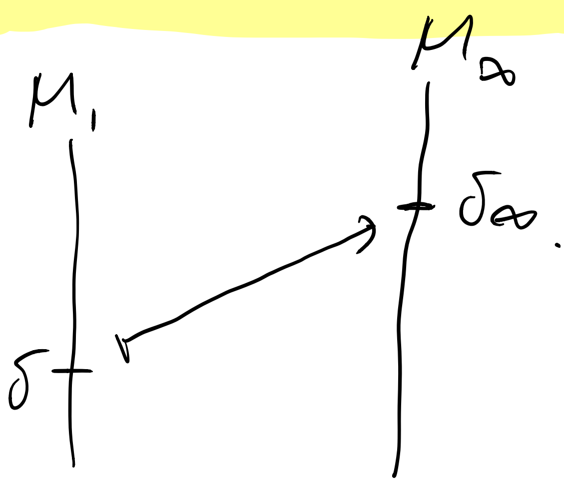
This gives you:

$HOD^{L(x,G)}$ is OD in $L(\mathbb{R})^{L(x,G)}$

Recall

Want: (**) $V_{\delta_\infty}^{HOD^{L(\mathbb{R})^{L(x,G)}}} = V_{\delta_\infty}^{HOD^{L(x,G)}}$.

$$V_{\delta_\infty}^{M_\infty} = V_{\delta_\infty}^{HOD^{L(x,G)}}$$



Steel $V_\delta^{M_1} \models$ "$V = HOD$".

So $V_{\delta_\infty}^{M_\infty} \models$ "$V = HOD$"

Hence $V_{\delta_\infty}^{M_\infty} = V_{\delta_\infty}^{HOD^{L(\mathbb{R},G)}}$

$\subseteq HOD^{L(\mathbb{R})^{L(\mathbb{R},G)}}$

Hence $V_{\delta_\infty}^{HOD^{L(\mathbb{R},G)}} = V_{\delta_\infty}^{HOD^{L(\mathbb{R})^{L(\mathbb{R},G)}}}$

Therefore $L(\mathbb{R})^{L(\mathbb{R},G)} \models$ "$\Theta = \delta_\infty$ is Woodin in HOD"

$\square$ <u>Thm</u>

Questions

(1) Is $HOD^{L(\mathbb{R})^{L(x,G)}} = HOD^{L(x,G)}$?

(2) What is $HOD^{L(\mathbb{R})^{M_1[G]}}$

where $G \subseteq Coll(\omega, <\delta)$-gen$/M_1$,

δ is the Woodin cardinal in M_1?