Projective Colourings

a gentle introduction

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Kobe Set Theory Seminar, October 15, 2025

Rheinland-Pfälzische Technische Universität
Kaiserslautern
Landau

Preliminaries

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About this talk

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- joint work with Luke Serafin

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Assumption

- AC

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Every uncountable Polish space has cardinality $\mathfrak{c}$.

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Graph: pair (V, E) with $E \subseteq V \times V$.

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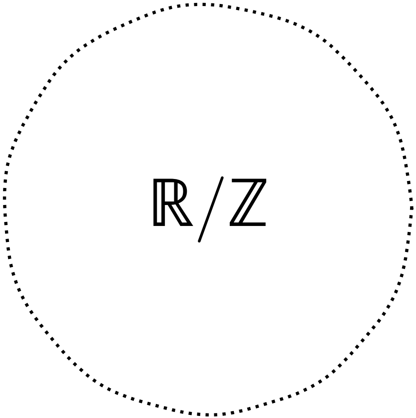

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$$\alpha \in [0,1) \setminus \mathbb{Q}$$

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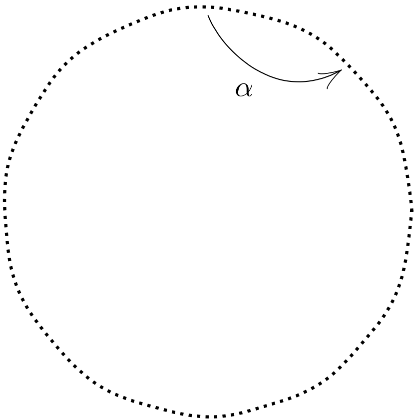

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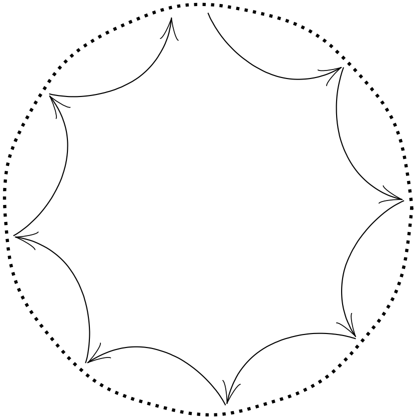

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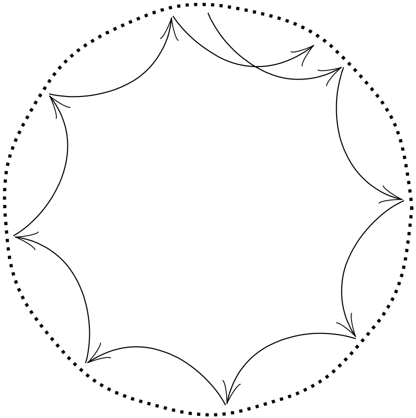

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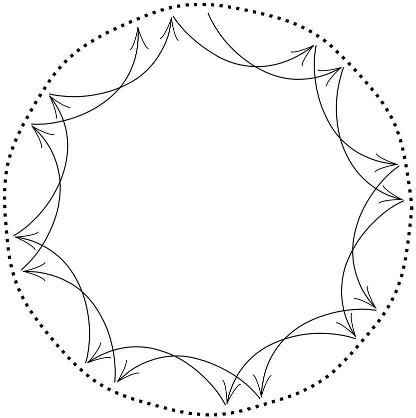

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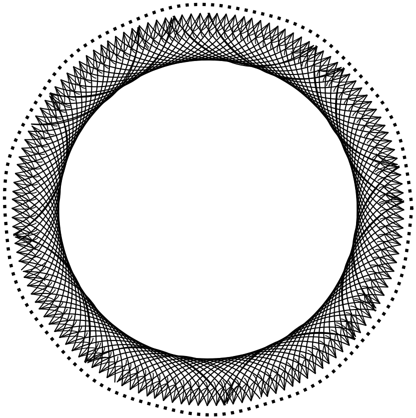

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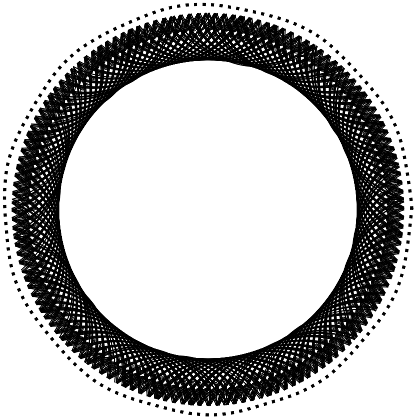

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$A \subseteq X$ is $\mathbf{\Delta}^1_n$ if it is both $\mathbf{\Pi}^1_n$ and $\mathbf{\Sigma}^1_n$.

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Theorem (Suslin)

A set is Borel if and only if it is $\mathbf{\Delta}^1_1$.

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Let $\mathbf{P}, \mathbf{Q}$ be pointclasses. If G is a $\mathbf{P}$ class graph and $\mathbf{P} \subseteq \mathbf{Q}$, then G is a $\mathbf{Q}$ class graph as well.

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Let $\mathbf{P}$ be a pointclass and G any graph. The $\mathbf{P}$ *class chromatic number* of G, written $\chi_{\mathbf{P}} G$, is the smallest cardinal α such that G admits a $\mathbf{P}$ class colouring.

Projective Colourings

Facts (easy)

RPTU

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Question

What is the consistency strength of

$$\exists G\,(\chi G < \chi_{\mathbf{\Delta}^1_2} G < \chi_B G)?$$

What if G must be a Borel graph?

Well-Ordered Spaces

Well-Ordered Spaces

Theorem (R., Serafin)

Let $n \in \mathbb{N}$, $n \geq 2$, and let $\mathbf{P} \in \{\, \mathbf{\Delta}, \mathbf{\Pi}, \mathbf{\Sigma} \,\}$. In a model of ZFC in which there is a $\mathbf{P}^1_n$ well-order of ${}^\omega\omega$ (and hence of any Polish space), for any locally countable $\mathbf{P}^1_n$ class graph G we have $\chi_{\mathbf{P}^1_n} G = \chi G$.

Well-Ordered Spaces

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RPTU

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$\omega^{\text{目}_v}$ is a Polish space well-ordered by $\sqsubseteq$.

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- V well-ordered by $\preceq$
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- $C_v \ni \{\, (x, n) : x \in$ 𐌇$_v$uniq., $n \in \omega \,\}$

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define new well-order $\sqsubseteq$ on C_v by stratification over the second coordinate:
$a \sqsubset b$ if and only if the lowest upper bound on $\pi_2 a$ is strictly smaller than that on $\pi_2 b$ or if they are equal but $a \sqsubset b$.

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Proof.

$$\begin{aligned}\downarrow a := \kappa \in \omega + 1 : \forall x \in \boxminus_v \forall n \in \omega \\ ((x, n) \in a \to n \leq \kappa) \\ \wedge \forall N \in \omega \, (\forall x \in \boxminus_v \forall n \in \omega \\ ((x, n) \in a \to n \leq N) \to N \geq \kappa),\end{aligned}$$

$$a \sqsubset b \leftrightarrow \downarrow a < \downarrow b \vee (\downarrow a = \downarrow b \wedge a \sqsubset b)$$

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- $\forall x \in \text{目}_v \forall n \in \omega$
  $((x,n) \in a \to \forall y \in \mathsf{N}x\, ((y,n) \notin a))$.

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Proof.

$a \in C_v$ is a v-*palette* if:

- $\forall x \in \text{日}_v \, (\exists n \in \omega \, ((x,n) \in a)$.
- $\forall x \in \text{日}_v \forall n \in \omega$
  $((x,n) \in a \to \forall y \in \mathsf{N}x \, ((y,n) \notin a))$.

Know:

RPTU

Well-Ordered Spaces

Proof.

- V well-ordered by $\preceq$
- 目_v: conn. comp. of v
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Know:

- classical $\aleph_0$-colouring of G restricted to 目_v is a v-palette

Well-Ordered Spaces

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Know:

- classical $\aleph_0$-colouring of G restricted to 日_v is a v-palette
- by assumption, at least one such colouring exists $\to$ every vertex has at least one palette

Well-Ordered Spaces

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Know:

- both conditions are $\mathbf{P}^1_n$-definable $\to$ set of all v-palettes is a $\mathbf{P}^1_n$ subset of C_v.

WELL-ORDERED SPACES

Proof.

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Proof.

define C_v to be the $\sqsubseteq$-least v-palette.

Well-Ordered Spaces

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- $\boldsymbol{C}_v$ is the $\sqsubseteq$-least v-palette
- $\boldsymbol{C} := \{ \boldsymbol{C}_v : v \in V \}$

Proof.

define $\boldsymbol{C}_v$ to be the $\sqsubseteq$-least v-palette.

RPTU

Well-Ordered Spaces

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- V well-ordered by $\preceq$
- $\boxminus_v$: conn. comp. of v
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- C_v is the $\sqsubseteq$-least v-palette
- $C := \{ C_v : v \in V \} \approx V$

Proof.

define C_v to be the $\sqsubseteq$-least v-palette. identify $C := \{ C_v : v \in V \}$ with V in the natural way, turning it into a Polish space.

Well-Ordered Spaces

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Proof.

- $d : V \to V \times V, v \mapsto (v, v)$

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- $d : V \to V \times V, v \mapsto (v, v)$
- $\varphi : V \to V, v \mapsto \min_{\preceq} \boxminus_v$

Well-Ordered Spaces

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Well-Ordered Spaces

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- $\bar{\varphi} : V \times V \to V \times V, (v, w) \mapsto (v, \varphi w)$
- $p : V \to C, v \mapsto C_v$

Well-Ordered Spaces

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- $\bar{p} : V \times V \to V \times \boldsymbol{C}, (v, w) \mapsto (v, pw)$

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- $\bar{p} : V \times V \to V \times \boldsymbol{C}, (v, w) \mapsto (v, pw)$
- $m : V \times \boldsymbol{C} \to \omega + 1$ maps $(x, \boldsymbol{C}_v)$ to the unique n s.t. $(x, n) \in \boldsymbol{C}_v$ or to ω if no such n exists

Well-Ordered Spaces

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- $c := m\bar{p}\bar{\varphi}d : V \to \omega + 1$

Summary

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Theorem (R., Serafin)

Let $n \in \mathbb{N}$, $n \geq 2$, and let $\mathbf{P} \in \{\, \mathbf{\Delta}, \mathbf{\Pi}, \mathbf{\Sigma} \,\}$. In a model of ZFC in which there is a $\mathbf{P}^1_n$ well-order of ${}^\omega\omega$ (and hence of any Polish space), for any locally countable $\mathbf{P}^1_n$ class graph G we have $\chi_{\mathbf{P}^1_n} G = \chi G$.

Summary

Theorem (R., Serafin)

Let $n \in \mathbb{N}$, $n \geq 2$, and let $\mathsf{P} \in \{ \boldsymbol{\Delta}, \boldsymbol{\Pi}, \boldsymbol{\Sigma} \}$. *In a model of ZFC in which there is a* P^1_n *well-order of* ${}^\omega\omega$ *(and hence of any Polish space), for any locally countable* P^1_n *class graph G we have* $\chi_{\mathsf{P}^1_n} G = \chi G$.

Theorem (R., Serafin)

Let $\mathsf{P} \in \{ \boldsymbol{\Delta}^1_2, \boldsymbol{\Sigma}^1_2 \}$. *In a model of ZFC in which there is a* P *well-order of* ${}^\omega\omega$ *(and hence of any Polish space), for any* P *class graph G we have* $\chi_{\mathsf{P}} G = \chi G$.

Summary

Theorem (R., Serafin)

Let $n \in \mathbb{N}$, $n \geq 2$, and let $\mathbf{P} \in \{ \mathbf{\Delta}, \mathbf{\Pi}, \mathbf{\Sigma} \}$. In a model of ZFC in which there is a $\mathbf{P}^1_n$ well-order of ${}^\omega\omega$ (and hence of any Polish space), for any locally countable $\mathbf{P}^1_n$ class graph G we have $\chi_{\mathbf{P}^1_n} G = \chi G$.

Theorem (R., Serafin)

Let $\mathbf{P} \in \{ \mathbf{\Delta}^1_2, \mathbf{\Sigma}^1_2 \}$. In a model of ZFC in which there is a $\mathbf{P}$ well-order of ${}^\omega\omega$ (and hence of any Polish space), for any $\mathbf{P}$ class graph G we have $\chi_{\mathbf{P}} G = \chi G$.

Conjecture

Let G be any Borel graph. In any model of ZFC+PD, we have $\chi_{\mathbf{\Delta}^1_2} G \in \{ \chi G, \chi_B G \}$.

Summary

Question

Is there a model where a Borel graph G exists with $\chi G < \chi_{\mathbf{\Delta}^1_2} G < \chi_B G$? If so, what is the consistency strength of the existence of such a graph?