Good ultrafilters and universality properties of forcing

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Introduction

1. Good ultrafilters and saturation
2. Universality properties of forcing (joint with Matteo Viale)

Boolean-valued structures

Let L be a first-order language and let $\mathbb{B}$ be a Boolean algebra. A $\mathbb{B}$-valued structure $\mathfrak{M}$ for L generalizes the usual Tarski semantics by assigning to each L-formula φ a Boolean value $[\![\varphi]\!]^{\mathfrak{M}} \in \mathbb{B}$.

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Given an ultrafilter U on $\mathbb{B}$, let $\equiv_U$ be the equivalence relation on $\mathfrak{M}$ defined by

$$\tau \equiv_U \sigma \iff [\![\tau = \sigma]\!]^{\mathfrak{M}} \in U.$$

The quotient of $\mathfrak{M}$ by the above equivalence relation gives rise to a 2-valued structure $\mathfrak{M}/U$.

Fullness

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Definition

A $\mathbb{B}$-valued structure $\mathfrak{M}$ is full if for every L-formula $\varphi(x, y_1, \ldots, y_n)$ and parameters $\sigma_1, \ldots, \sigma_n \in M$ there exist $\tau_1, \ldots, \tau_m \in M$ such that

$$[\![\exists x \varphi(x, \sigma_1, \ldots, \sigma_n)]\!]^{\mathfrak{M}} = \bigvee_{i=1}^{m} [\![\varphi(\tau_i, \sigma_1, \ldots, \sigma_n)]\!]^{\mathfrak{M}}.$$

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Definition

Let δ be a cardinal; a $\mathbb{B}$-valued structure $\mathfrak{M}$ satisfies the δ-mixing property if for every antichain $A \subset \mathbb{B}$ with $|A| < \delta$ and every set $\{\tau_a \mid a \in A\} \subseteq M$ there exists $\tau \in M$ such that for all $a \in A$

$$a \leq [\![\tau_a = \tau]\!]^{\mathfrak{M}}.$$

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Proposition

If $\mathfrak{M}$ satisfies the δ-mixing property for all cardinals δ, then $\mathfrak{M}$ is full.

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combinatorial properties of $U \longleftrightarrow$ model-theoretic properties of $\mathfrak{M}/U$

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For the purpose of this talk, we shall be concerned with the model-theoretic property of saturation.

Good ultrafilters

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Let κ be a cardinal, $\mathbb{B}$ a Boolean algebra, and $f\colon [\kappa]^{<\aleph_0} \to \mathbb{B}$.

- f is monotonic if for all $S, T \in [\kappa]^{<\aleph_0}$, $S \subseteq T$ implies $f(T) \leq f(S)$.
- f is multiplicative if for all $S, T \in [\kappa]^{<\aleph_0}$, $f(S \cup T) = f(S) \wedge f(T)$.

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Note that every multiplicative function is monotonic.

Definition (Mansfield [1971])

Let δ be a cardinal. An ultrafilter U on a Boolean algebra $\mathbb{B}$ is δ-good if for every $\kappa < \delta$ and every monotonic function $f\colon [\kappa]^{<\aleph_0} \to U$, there exists a multiplicative function $g\colon [\kappa]^{<\aleph_0} \to U$ such that $g(S) \leq f(S)$ for all $S \in [\kappa]^{<\aleph_0}$.

Good ultrafilters

Theorem (Mansfield [1971])

Let δ be an uncountable cardinal, $\mathbb{B}$ a δ-complete Boolean algebra, and L a language with $|L| < \delta$. Suppose $\mathfrak{M}$ is a full $\mathbb{B}$-valued structure for L which satisfies the δ-mixing property, and U is an countably incomplete δ-good ultrafilter on $\mathbb{B}$; then the quotient $\mathfrak{M}/U$ is δ-saturated.

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Question

What about the converse implication? If an ultrafilter U makes every quotient δ-saturated, must U necessarily be δ-good?

Capturing saturation

Three years later, Benda [1974] observed that, using Mansfield's definition, "it is not straightforward to prove the other implication". To get around this problem, Benda introduced another class of ultrafilters, which we shall call δ-Benda, and proved an ultrafilter is δ-Benda if and only if every quotient by U is δ-saturated.

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Theorem (P. [2021])

A countably incomplete ultrafilter U is δ-good if and only if it is δ-Benda. Thus, the combinatorial property of goodness precisely captures the model-theoretic property of saturation.

Further work

Moreno Pierobon, in his master's thesis [2019] and his joint work with Matteo Viale [2022], further deepens this analysis of saturated quotients of Boolean-valued structures, by connecting the fullness and mixing properties with topological properties of presheaves and étalé spaces.

Motivation

Explain the "unreasonable effectiveness" of forcing in constructing models of set theory.

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This part is joint work with Matteo Viale,

available at `arXiv:2310.11691 [math.LO]`

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Let $\mathrm{trcl}(x)$ denote the transitive closure of a set x. For each cardinal λ, we define

$$H_\lambda = \{x \mid |\mathrm{trcl}(x)| < \lambda\},$$

so that

$$V = \bigcup_\lambda H_\lambda.$$

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Definition

Let $\in_{\Delta_0}$ be the expansion of the language $\{\in\}$ containing an n-ary relation symbol R_φ for each Δ_0 $\{\in\}$-formula $\varphi(x_1, \ldots, x_n)$, as well as constant symbols for $\emptyset$ and ω.

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- $\mathrm{Th}(H_\lambda, \in_{\Delta_0})$ is the set of $\in_{\Delta_0}$-sentences which are true in H_λ
- $\mathrm{Th}_\forall(H_\lambda, \in_{\Delta_0})$ is its universal fragment

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Question

Are models constructed by forcing somehow "canonical" or "universal" among all the models of the above theories?

Boolean-valued models of set theory

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Let δ be a cardinal such that $[\![\check{\delta} \text{ is a cardinal}]\!]^{V^{\mathbb{B}}} = \mathbb{1}$. We define

$$H_{\check{\delta}}^{\mathbb{B}} = \left\{ \tau \in V^{\mathbb{B}} \;\middle|\; [\![|\mathrm{trcl}(\tau)| < \check{\delta}]\!]^{V^{\mathbb{B}}} = \mathbb{1} \right\}$$

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$$H^{\mathbb{B}}_{\check{\delta}} = \left\{ \tau \in V^{\mathbb{B}} \;\middle|\; [\![|\mathrm{trcl}(\tau)| < \check{\delta}]\!]^{V^{\mathbb{B}}} = \mathbb{1} \right\}$$

with caution. Whenever G is $\mathbb{B}$-generic over V, we have

$$H^{V[G]}_{\delta} = \left\{ \tau_G \;\middle|\; \tau \in H^{\mathbb{B}}_{\check{\delta}} \right\}.$$

In other words, $H^{\mathbb{B}}_{\check{\delta}}$ is a canonical set of names for H_δ as computed in the forcing extension by $\mathbb{B}$.

Boolean-valued models of set theory

Proposition

If $\mathbb{B}$ is a complete Boolean algebra then $H_{\check{\delta}}^{\mathbb{B}}$, with the Boolean values inherited from $V^{\mathbb{B}}$, is a $\mathbb{B}$-valued structure for $\in_{\Delta_0}$ which satisfies the λ-mixing property for all cardinals λ.

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As usual, we wish to quotient $H^{\mathbb{B}}_{\check{\delta}}$ by an ultrafilter U on $\mathbb{B}$ to recover a 2-valued structure $H^{\mathbb{B}}_{\check{\delta}}/U$.

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As usual, we wish to quotient $H_{\check{\delta}}^{\mathbb{B}}$ by an ultrafilter U on $\mathbb{B}$ to recover a 2-valued structure $H_{\check{\delta}}^{\mathbb{B}}/U$.

In fact, we shall show that, for carefully chosen $\mathbb{B}$, there exist densely many ultrafilters U on $\mathbb{B}$ such that the model $H_{\check{\delta}}^{\mathbb{B}}/U$ obtained by forcing is universal among the models of $\mathrm{Th}_{\forall}(H_{\kappa^+}, \in_{\Delta_0})$.

Lévy's collapsing algebra

Let δ be an inaccessible cardinal. For every infinite regular cardinal $\kappa < \delta$, let $\mathrm{Coll}(\kappa, {<}\delta)$ be the completion of the poset

$$P = \big\{ p \;\big|\; p \text{ is a function, } |p| < \kappa,\ \mathrm{dom}(p) \subset \delta \times \kappa, \\ \forall \langle \alpha, \xi \rangle \in \mathrm{dom}(p)(p(\alpha, \xi) \in \alpha \cup \{0\}) \big\}$$

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In other words, $\mathrm{Coll}(\kappa, {<}\delta)$ is the complete Boolean algebra which generically adjoins surjections $\kappa \to \alpha$ for any $\alpha < \delta$.

First universality result

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Theorem (P. and Viale)

Let δ be an inaccessible cardinal and $\kappa < \delta$ be an infinite regular cardinal. Then there exist densely many ultrafilters U on $\mathsf{Coll}(\kappa, {<}\delta)$ such that:

- *the quotient $H_{\check{\delta}}^{\mathsf{Coll}(\kappa,<\delta)}/U$ is a model of $\mathsf{Th}_\forall(H_{\kappa^+}, \in_{\Delta_0})$;*
- *for every model $\mathfrak{M} \models \mathsf{Th}_\forall(H_{\kappa^+}, \in_{\Delta_0})$ such that $|M| \leq \delta$, there exists an embedding $e \colon \mathfrak{M} \to H_{\check{\delta}}^{\mathsf{Coll}(\kappa,<\delta)}/U$ of $\in_{\Delta_0}$-structures.*

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Woodin's stationary tower

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Let δ be a Woodin cardinal; for every infinite $\kappa < \delta$ the stationary tower of height δ and critical point κ^+ is defined as

$$\mathbb{Q}^{\kappa}_{<\delta} = \left\{ a \in V_\delta \;\middle|\; a \text{ is stationary, } \forall Z \in a \, (|Z| \leq \kappa \text{ and } Z \cap \kappa^+ \in \kappa^+) \right\}$$

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ordered by

$$b \leq a \iff \bigcup a \subseteq \bigcup b \text{ and } \forall Z \in b \, (Z \cap \bigcup a \in a).$$

In fact, if defined in terms of towers of normal ideals, $\mathbb{Q}^{\kappa}_{<\delta}$ inherits the structure of δ-complete Boolean algebra.

Generic ultrapower

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Definition

If $\mathcal{A}$ is any set of relations on H_{κ^+}, let us say $\mathcal{A} \subseteq \bigcup_{n<\omega} \mathcal{P}({}^n(H_{\kappa^+}))$, then we let $\in_{\mathcal{A}}$ be the expansion of $\in_{\Delta_0}$ obtained naming each element of $\mathcal{A}$ by a new relation symbol.

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Definition
Let δ be a Woodin cardinal; for every infinite $\kappa < \delta$ we define

$$\mathrm{Ult}(H_{\kappa^+}, \delta) = \{f : \mathcal{P}(X) \to H_{\kappa^+} \mid X \in V_\delta\}.$$

Generic ultrapower

Proposition

For every set of relations $\mathcal{A}$ on H_{κ^+}, the set $\mathsf{Ult}(H_{\kappa^+}, \delta)$ is a full $\mathbb{Q}^{\kappa}_{<\delta}$-valued structure for $\in_{\mathcal{A}}$ with the δ-mixing property.

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Theorem (Woodin)

If G is $\mathbb{Q}^{\kappa}_{<\delta}$-generic over V, then $\mathrm{Ult}(H_{\kappa^+}, \delta)/G$ is well founded and thus may be identified with its transitive collapse.
Modulo this identification, we obtain

$$H_{\delta}^{V[G]} \subseteq \mathrm{Ult}(H_{\kappa^+}, \delta)/G.$$

Second universality result

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Theorem (P. and Viale)

Let δ be a Woodin cardinal and $\kappa < \delta$ be an infinite cardinal. Then there exist densely many ultrafilters U on $\mathbb{Q}^{\kappa}_{<\delta}$ such that for every set of relations $\mathcal{A}$ on H_{κ^+}:

- *the quotient* $\mathrm{Ult}(H_{\kappa^+}, \delta)/U$ *is a model of* $\mathrm{Th}(H_{\kappa^+}, \in_{\mathcal{A}})$;
- *for every model* $\mathfrak{M} \models \mathrm{Th}_{\forall}(H_{\kappa^+}, \in_{\mathcal{A}})$ *such that* $|M| \leq \delta$, *there exists an embedding* $e\colon \mathfrak{M} \to \mathrm{Ult}(H_{\kappa^+}, \delta)/U$ *of* $\in_{\mathcal{A}}$*-structures;*

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- *for every model* $\mathfrak{M} \models \mathrm{Th}(H_{\kappa^+}, \in_{\mathcal{A}})$ *such that* $|M| \leq \delta$*, there exists an elementary embedding* $j\colon \mathfrak{M} \to \mathrm{Ult}(H_{\kappa^+}, \delta)/U$ *of* $\in_{\mathcal{A}}$*-structures.*

Further work

In the first universality result, the Lévy collapse $\mathrm{Coll}(\kappa, {<}\delta)$ is a δ-c.c. complete Boolean algebra. On the other hand, in the second result the stationary tower $\mathbb{Q}^{\kappa}_{<\delta}$ is not δ-c.c. and not complete, which makes this and other arguments technically awkward.

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In his PhD thesis, Tsvetlin Marinov constructed an ideal I on $\mathbb{Q}^{\kappa}_{<\delta}$ such that (modulo a game-theoretic conjecture) the quotient $\mathbb{Q}^{\kappa}_{<\delta}/I$ is δ-c.c. and complete, while still enjoying the desirable properties of the stationary tower.

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Problem

Analyze the (universal fragments of) the theories of H_λ for λ limit.

Thank you!