Localisation Cardinals on the Generalised Baire Space

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Introduction

The **Baire space** is the set ${}^{\omega}\omega = \{f \mid f : \omega \to \omega\}$. Its topology is generated by clopen sets $[s] = \{f \in {}^{\omega}\omega \mid s \subseteq f\}$ for $s \in {}^{<\omega}\omega$, then ${}^{\omega}\omega$ is homeomorphic to the irrationals. We can define Lebesgue measure on ${}^{\omega}\omega$, and the set $\mathcal{N}$ of Lebesgue null sets.

$$\mathrm{add}(\mathcal{N}) = \min\left\{|\mathcal{A}| \;\middle|\; \mathcal{A} \subseteq \mathcal{N} \text{ and } \bigcup \mathcal{A} \notin \mathcal{N}\right\},$$
$$\mathrm{cof}(\mathcal{N}) = \min\left\{|\mathcal{C}| \;\middle|\; \mathcal{C} \subseteq \mathcal{N} \text{ and } \forall N \in \mathcal{N} \exists C \in \mathcal{C} (N \subseteq C)\right\}.$$

Cichoń Diagram:

$$\begin{tikzcd}
 & \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathbf{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
 & & \mathfrak{b} \arrow[r] \arrow[u] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathbf{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

Let κ be uncountable, then ${}^\kappa\kappa$ is a **generalised Baire space**. We say $f \in {}^\kappa\kappa$ are κ-reals. If κ is strongly inaccessible, we can generalise the middle part of the Cichoń Diagram:

$$\begin{array}{ccccccc}
 & & \mathrm{non}(\mathcal{M}_\kappa) & \longrightarrow & \mathrm{cof}(\mathcal{M}_\kappa) & \longrightarrow & 2^\kappa \\
 & & \uparrow & & \uparrow & & \\
 & & \mathfrak{b}_\kappa & \longrightarrow & \mathfrak{d}_\kappa & & \\
 & & \uparrow & & \uparrow & & \\
\kappa^+ & \longrightarrow & \mathrm{add}(\mathcal{M}_\kappa) & \longrightarrow & \mathrm{cov}(\mathcal{M}_\kappa) & &
\end{array}$$

There is no Lebesgue measure on ${}^\kappa\kappa$, so there is no generalisation of $\mathcal{N}$ to ${}^\kappa\kappa$. We can generalise $\mathrm{add}(\mathcal{N})$ and $\mathrm{cof}(\mathcal{N})$ using a combinatorial definition instead.

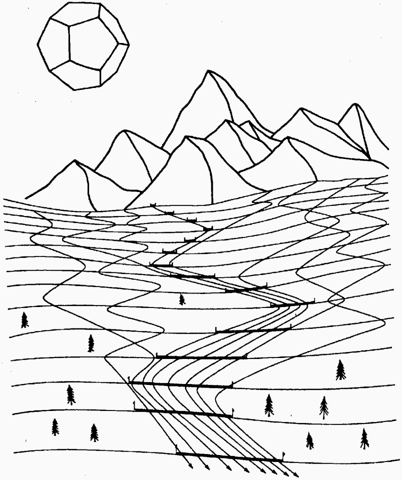
Contents

Let κ be regular strong limit and $h \in {}^{\kappa}\kappa$ be an increasing cofinal cardinal function.

An h-**slalom** is any function $\varphi : \kappa \to [\kappa]^{<\kappa}$ such that $|\varphi(\alpha)| \leq h(\alpha)$ for all $\alpha \in \kappa$.

For $f \in {}^{\kappa}\kappa$, we say $f \in^* \varphi$, or f is **localised** by φ, if there exists some $\xi < \kappa$ such that $f(\alpha) \in \varphi(\alpha)$ for all $\alpha \in [\xi, \kappa)$.

We will let Loc_h be the set of h-slaloms.



[Bartoszyński, 1987]

We define the following cardinal characteristics:

$$\mathfrak{b}^h_\kappa(\in^*) = \min\{|B| \mid B \subseteq {}^\kappa\kappa \text{ and } \forall\varphi \in \mathrm{Loc}_h \exists f \in B(f \notin^* \varphi)\},$$
$$\mathfrak{d}^h_\kappa(\in^*) = \min\{|D| \mid D \subseteq \mathrm{Loc}_h \text{ and } \forall f \in {}^\kappa\kappa \exists\varphi \in D(f \in^* \varphi)\}.$$

These are the **unbounded** and **dominating** h-localisation cardinals.

Proposition *[Brendle et al., 2018] sections 4.3 & 4.4*
$\kappa^+ \leq \mathfrak{b}^h_\kappa(\in^*) \leq \mathfrak{d}^h_\kappa(\in^*) \leq 2^\kappa$, and all relations can consistently be strict inequalities. □

Proposition *[Bartoszyński, 1987] or [Bartoszyński and Judah, 1995]*
$\mathfrak{b}_\omega(\in^*) = \mathrm{add}(\mathcal{N})$ and $\mathfrak{d}_\omega(\in^*) = \mathrm{cof}(\mathcal{N})$ □

Theorem *[Bartoszyński, 1987] or [Blass, 2010] remark 5.15 (for $\kappa = \omega$)*
If $h, g \in {}^{\kappa}\kappa$ are continuous (i.e. $h(\gamma) = \bigcup_{\alpha<\gamma} h(\alpha)$ for limit γ) and unbounded, then $\mathfrak{d}^h_\kappa(\in^*) = \mathfrak{d}^g_\kappa(\in^*)$ and $\mathfrak{b}^h_\kappa(\in^*) = \mathfrak{b}^g_\kappa(\in^*)$.

Proof. Enumerate a club $\langle \xi_\alpha \mid \alpha \in \kappa \rangle$ in κ s.t. $h(\alpha) \le g(\xi_\alpha)$. Let $I_\alpha = [\xi_\alpha, \xi_{\alpha+1})$ and $\pi_\alpha : \kappa \rightarrowtail\!\!\!\twoheadrightarrow {}^{I_\alpha}\kappa$ bijective.

If $f \in {}^{\kappa}\kappa$ let $f' : \alpha \mapsto \pi_\alpha^{-1}(f \restriction I_\alpha)$ and if $\varphi \in \mathrm{Loc}_h$, $\xi \in I_\alpha$, let $\varphi'(\xi) = \{\pi_\alpha(i)(\xi) \mid i \in \varphi(\alpha)\}$. Then $f'(\alpha) \in \varphi(\alpha)$ and $\xi \in I_\alpha$ implies $f(\xi) = f \restriction I_\alpha(\xi) = \pi_\alpha(f'(\alpha))(\xi) \in \varphi'(\xi)$.

So $f' \in^* \varphi$ implies $f \in^* \varphi'$. $\square$

Let pow : $\alpha \mapsto 2^{|\alpha|}$ and id : $\alpha \mapsto |\alpha|$.

Proposition *[Brendle et al., 2018] proposition 65 & 66*

ZFC $+ \exists$ Inaccessible $\vdash$ " $\mathrm{Con}(\mathfrak{d}_\kappa^{\mathsf{pow}}(\in^*) < \mathfrak{d}_\kappa^{\mathsf{id}}(\in^*))$ " □

Proved using the generalised Sacks forcing from [Kanamori, 1980].

Question

Does there exist $h \in {}^\kappa\kappa$ such that $\mathfrak{d}_\kappa^h(\in^*)$ is consistently different from $\mathfrak{d}_\kappa^{\mathsf{pow}}(\in^*)$ and $\mathfrak{d}_\kappa^{\mathsf{id}}(\in^*)$?

Answer. Yes, by using a similar Sacks-like forcing.

Contents

Our goal is to separate $\mathfrak{d}^h_\kappa(\in^*)$ from $\mathfrak{d}^g_\kappa(\in^*)$ for two $h, g \in {}^\kappa\kappa$.

Definition

A forcing notion $\langle \mathbb{P}, \leq \rangle$ has the **(generalised)** h**-Sacks property** if for every $\mathbb{P}$-name $\dot{f}$ and condition $p \in \mathbb{P}$ such that $p \Vdash$ "$\dot{f} \in {}^\kappa\kappa$" there exists a $q \leq p$ and h-slalom $\varphi \in \mathrm{Loc}_h$ such that $q \Vdash$ "$\dot{f}(\alpha) \in \check{\varphi}(\alpha)$" for all $\alpha < \kappa$.

Lemma

If $\mathbb{P}$ has the h-Sacks property, G is $\mathbb{P}$-generic over $\mathbf{V}$, then $\mathbf{V}[G] \vDash$ "$\mathfrak{d}^h_\kappa(\in^*) \leq (2^\kappa)^{\mathbf{V}}$".

Let $T \subseteq {}^{<\kappa}\kappa$ be a tree. For any node $u \in T$ let $\mathrm{suc}(u, T) = \{v \in T \mid \exists \beta < \kappa (v = u^\frown \beta)\}$.

Node u is α-**splitting** in T if $\alpha \leq |\mathrm{suc}(u, T)|$. If u is α-splitting but not $|\alpha|^+$-splitting, then we call u a **sharp** α-splitting node. A **splitting node** is a 2-splitting node, and any other node is **non-splitting**.

We let $u \in \mathrm{Split}_\alpha(T)$ iff u is splitting and $\mathrm{ot}(\{\beta < \mathrm{ot}(u) \mid u \restriction \beta \text{ is splitting}\}) = \alpha$, and we call α the **splitting level** of u.

If $u \in T$, then $T_u = \{v \in T \mid u \subseteq v \text{ or } v \subseteq u\}$.

Forcing notion $\mathbb{S}^h_\kappa$

Let $h \in {}^\kappa\kappa$ be an increasing cofinal cardinal function. The conditions of the forcing $\mathbb{S}^h_\kappa$ are trees $T \subseteq {}^{<\kappa}\kappa$ that satisfy the following properties:

(i) for any $u \in T$ there exists splitting $v \in T$ such that $u \subseteq v$,

(ii) if $\gamma < \kappa$ and $\langle u_\alpha \mid \alpha < \gamma \rangle \in {}^\gamma T$ are splitting nodes with $u_\alpha \subseteq u_\beta$ for $\alpha < \beta$, then $u = \bigcup_{\alpha<\gamma} u_\alpha \in T$ and u is splitting,

(iii) if $u \in \mathrm{Split}_\alpha(T)$, then u is an $h(\alpha)$-splitting node in T.

We say that $T \leq S$ iff $T \subseteq S$ and for every splitting $u \in T$, either $\mathrm{suc}(u,T) = \mathrm{suc}(u,S)$ or $|\mathrm{suc}(u,T)| < |\mathrm{suc}(u,S)|$.

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Proposition *[vdV] lemma 4*

Let $\gamma < \kappa$ and $\langle T_\xi \mid \xi < \gamma \rangle \in {}^\gamma(\mathbb{S}^h_\kappa)$ be decreasing. If $u \in T = \bigcap T_\xi$, then $\exists \eta < \kappa \forall \xi \in [\eta, \gamma)(\mathrm{suc}(u, T) = \mathrm{suc}(u, T_\xi))$.

Proof. Let $\lambda_\xi = |\mathrm{suc}(u, T_\xi)|$, then $\langle \lambda_\xi \mid \xi < \lambda \rangle$ decreases. So there is ξ such that $\lambda_\xi = \lambda_\eta$ for $\eta > \xi$. Then $\mathrm{suc}(u, T_\xi) = \mathrm{suc}(u, T_\eta)$ for all $\eta > \xi$. □

Corollary *[vdV] lemma 4*

$\mathbb{S}^h_\kappa$ is $<\kappa$-closed.

Proof. Check that (i) (ii) and (iii) from the definition hold using the above proposition. □

Let $T \leq_\alpha S$ iff $T \leq S$ and $\mathrm{Split}_\alpha(T) = \mathrm{Split}_\alpha(S)$. A **fusion sequence** is a sequence $\langle T_\alpha \mid \alpha < \kappa \rangle$ s.t. $T_\beta \leq_\alpha T_\alpha$ for all $\beta > \alpha$.

Proposition *[vdV] lemma 6*

$\mathbb{S}^h_\kappa$ is closed under fusion and has the $<(2^\kappa)^+$-cc.

Proof. Let $u \in T$ and $\alpha > \mathrm{ot}(\{\beta \in \kappa \mid u \restriction \beta \text{ splits in } T\}) + 1$, then $\mathrm{Split}_\alpha(T_\alpha) = \mathrm{Split}_\alpha(T)$.

(i), (ii) and (iii) and $T \leq T_\xi$ follow easily. $\square$

If $T \in \mathbb{S}^h_\kappa$ and $u \in T$, then T_u is a condition and $T_u \leq T$.

Every T has a **sharp** $T^* \leq T$ such that $\mathrm{Split}_\alpha(T^*) \subseteq \mathrm{Split}_\alpha(T)$ and each $u \in \mathrm{Split}_\alpha(T^*)$ is a sharp $h(\alpha)$-splitting node.

Theorem *[vdV] theorem 7*

Let $h \in {}^\kappa\kappa$ be increasing cofinal cardinal function, $g : \alpha \mapsto h(\alpha)^{|\alpha|}$, then $\mathbb{S}^h_\kappa$ has the g-Sacks property.

Proof. Let $\dot{f}$ be a name, $T_0 \in \mathbb{S}^h_\kappa$ and $T_0 \Vdash$ "$\dot{f} \in {}^\kappa\kappa$". Find fusion sequence $\langle T_\xi \mid \xi < \kappa \rangle$, $\{B_\xi \subseteq \kappa \mid \xi < \kappa\}$ and $\varphi \in \mathrm{Loc}_g$ s.t.

(a) Each T_ξ is a sharp tree,
(b) $T_{\xi+1} \Vdash$ "$\dot{f}(\xi) \in \check{B}_\xi$" for all $\xi \in \kappa$,
(c) $|B_\xi| \leq g(\xi)$ and $\varphi(\xi) = B_\xi$

Then $T = \bigcap_\xi T_\xi \in \mathbb{S}^h_\kappa$ by fusion, and $T \Vdash$ "$\dot{f} \in^* \check{\varphi}$" $\cdots\!>$

Limit step

Let γ be limit. By $<\kappa$-closure $T'_\gamma = \bigcap_{\xi<\gamma} T_\xi \in \mathbb{S}^h_\kappa$. By fusion, $\mathrm{Split}_\alpha(T'_\gamma) = \mathrm{Split}_\alpha(T_\alpha)$. Let $T_\gamma = (T'_\gamma)^*$, then $\mathrm{Split}_\alpha(T_\gamma) = \mathrm{Split}_\alpha(T'_\gamma)$ for all $\alpha < \gamma$ by sharpness and ordering of $\mathbb{S}^h_\kappa$, so T_γ satisfies the fusion requirements and is sharp.

Successor step

Let $u \in \mathrm{Split}_\xi(T_\xi)$ and $v \in \mathrm{suc}(u, T_\xi)$. Find $T^v \leq (T_\xi)_u$ and $\beta^v_\xi < \kappa$ s.t. $T^v \Vdash$ "$\dot{f}(\xi) = \check{\beta}^v_\xi$". Let $u' \in \mathrm{Split}_\xi(T^v)$ and $v' \in \mathrm{suc}(u', T^v)$ be arbitrary and $T^{v\prime} = (T^v)_{v'}$. Note that $\mathrm{Split}_\alpha(T^{v\prime}) \subseteq \mathrm{Split}_{\xi+1+\alpha}(T^v)$. Let $V_\xi = \bigcup \left\{\mathrm{suc}(u, T_\xi) \mid u \in \mathrm{Split}_\xi(T_\xi)\right\}$ and $T_{\xi+1} = (\bigcup_{v \in V_\xi} T^{v\prime})^*$ and $B_\xi = \left\{\beta^v_\xi \mid v \in V_\xi\right\}$. Now $|B_\xi| \leq |V_\xi| \leq h(\xi)^{|\xi|} = g(\xi)$ since T_ξ is sharp. $T_{\xi+1} \Vdash$ "$\dot{f}(\xi) \in B_\xi$" since $\{T^v \mid v \in V_\xi\}$ is predense below $T_{\xi+1}$. $\square$

Corollary

If $\mathbf{V} \models$ "$2^\kappa = \kappa^+$", then $\mathbb{S}^h_\kappa$ preserves all cardinals and cofinalities.

Proof. $|\mathcal{P}({}^{<\kappa}\kappa)| = |\mathcal{P}(\kappa)| = 2^\kappa = \kappa^+$ and each $T \in \mathbb{S}^h_\kappa$ is a subset of ${}^{<\kappa}\kappa$. So $\mathbb{S}^h_\kappa$ has the κ^+-c.c. and is $<\kappa$-closed. Hence $\mathbb{S}^h_\kappa$ preserves λ if $\lambda \leq \kappa$ or $\kappa^+ < \lambda$.

If $\dot{f}$ is a name and $T \Vdash$ "$\dot{f} : \kappa \to \kappa^+$", use the proof of the Sacks property to find $T' \leq T$ and B_ξ with $|B_\xi| < \kappa$ such that $T' \Vdash$ "$\dot{f}(\xi) \in \check{B}_\xi$" for all ξ. Then $T' \Vdash$ "$\mathrm{ran}(\dot{f}) \subseteq \bigcup_\xi \check{B}_\xi \neq \kappa^+$", so $\dot{f}$ does not name a surjection. Hence $\mathbb{S}^h_\kappa$ preserves κ^+. $\square$

Lemma *[vdV] theorem 9*

The set $C = \{\alpha < \kappa \mid T \cap {}^{\alpha}\kappa = \mathrm{Split}_\alpha(T)\}$ is club.

Theorem *[vdV] theorem 9*

Let $g(\alpha) < h(\alpha)$ for all $\alpha \in S$ stationary, then $\mathbb{S}^h_\kappa$ does not have the g-Sacks property.

Proof. Let G be $\mathbb{S}^h_\kappa$ generic and $f = \bigcap G$, then $f \in {}^{\kappa}\kappa$. Working in the ground model, let $\dot{f}$ name f and $\varphi \in \mathrm{Loc}_h$. If $u \in \mathrm{Split}_\alpha(T)$ and $\alpha_0 \in \kappa$, then take $\alpha_0 < \alpha \in S \cap C$, then $g(\alpha) < h(\alpha) \leq |\mathrm{suc}(u, T)|$. Let $\beta \in \mathrm{suc}(u, T)$ such that $\beta \notin \varphi(\alpha)$.

Then $(T)_{u^\frown\beta} \Vdash$ "$\exists\alpha > \alpha_0(\dot{f}(\alpha) \notin \check{\varphi}(\alpha))$". By density and arbitrariness of α_0 then $T \Vdash$ "$\dot{f} \notin^* \varphi$". □

$\mathbb{S}_\kappa^{\mathsf{pow}}$ has the pow-Sacks property, but not the id-Sacks property.

(in fact $\mathbb{S}_\kappa^2$ where $2 : \alpha \mapsto 2$ already has the pow-Sacks property but not the id-Sacks property)

If S is stationary co-stationary and $F \in {}^\kappa\kappa$ s.t. $F(\alpha) = F(\alpha)^{|\alpha|}$,

$$h \restriction S = F \restriction S \qquad\qquad g \restriction S = 2^F \restriction S$$
$$h \restriction S^c = 2^F \restriction S \qquad\qquad g \restriction S^c = F \restriction S$$

Then $\mathbb{S}_\kappa^h$ has the h-Sacks property, but not the g-Sacks property, and $\mathbb{S}_\kappa^g$ has the g-Sacks property, but not the h-Sacks property.

Contents

Definition

Let A be an index set and $\langle\langle\mathbb{P}_\xi, \leq_\xi\rangle \mid \xi \in A\rangle$ a sequence of forcing notions. Let $\mathcal{C}$ be the set of choice functions $p : A \to \bigcup_{\xi\in A} \mathbb{P}_\xi$. If $p \in \mathcal{C}$ let $\operatorname{supp}(p) = \{\xi \in A \mid p(\xi) \neq \mathbb{1}_\xi\}$. Define the $\leq\kappa$-support product of $\langle\mathbb{P}_\xi \mid \xi \in A\rangle$ as:

$$\overline{\mathbb{P}} = \{p \in \mathcal{C} \mid |\operatorname{supp}(p)| \leq \kappa\}$$

ordered by $q \leq_{\overline{\mathbb{P}}} p$ iff $q(\xi) \leq_\xi p(\xi)$ for all $\xi \in A$.

Lemma

If $\mathbb{P}_\xi$ is $<\kappa$-closed for each $\xi \in A$, then $\overline{\mathbb{P}}$ is $<\kappa$-closed.

$\langle p_\alpha \in \mathbb{P}_\xi \mid \alpha < \kappa \rangle$ is a **fusion sequence** if $p_\beta \leq_\alpha p_\alpha$ for all $\beta > \alpha$.

Given $p, q \in \overline{\mathbb{P}}$, $\alpha \in \kappa$ and $Z \subseteq A$, let $q \leq_{Z,\alpha} p$ iff $q \leq p$ and $q(\xi) \leq_\alpha p(\xi)$ for each $\xi \in Z$. A **generalised fusion sequence** is a sequence $\langle (p_\alpha, Z_\alpha) \mid \alpha < \kappa \rangle$ such that:

(i) $p_\alpha \in \overline{\mathbb{P}}$ and $Z_\alpha \in [A]^{<\kappa}$ for each $\alpha < \kappa$,

(ii) $p_\beta \leq_{Z_\alpha,\alpha} p_\alpha$ and $Z_\alpha \subseteq Z_\beta$ for all $\alpha \leq \beta < \kappa$,

(iii) for limit δ we have $Z_\delta = \bigcup_{\alpha<\delta} Z_\alpha$,

(iv) $\bigcup_{\alpha<\kappa} Z_\alpha = \bigcup_{\alpha<\kappa} \operatorname{supp}(p_\alpha)$.

Lemma *Kanamori [1980] for products of κ-Sacks forcing*

If each $\mathbb{P}_\xi$ is closed under fusion, then $\overline{\mathbb{P}}$ is closed under generalised fusion.

Theorem *[vdV] Lemma 13*

Let A be an index set, $B \subseteq A$ and $B^c = A \setminus B$, and for each $\xi \in A$ let $h_\xi \in {}^\kappa\kappa$ be an increasing cofinal cardinal function. Let $\overline{\mathbb{S}}$ be the $\leq\kappa$-support product of $\langle \mathbb{S}_\kappa^{h_\xi} \mid \xi \in A \rangle$ and G be $\overline{\mathbb{S}}$-generic over $\mathbf{V}$.

If $g : \alpha \to (\sup_{\xi \in B^c} h_\xi(\alpha))^{|\alpha|}$ is well-defined, then for each $f \in \mathbf{V}[G]$ there is $\varphi \in (\mathrm{Loc}_g)^{\mathbf{V}[G \restriction B]}$ such that $f \in^* \varphi$.

Theorem

Let A be an index set, $B \subseteq A$ and $B^c = A \setminus B$, and for each $\xi \in A$ let $h_\xi \in {}^\kappa\kappa$ be an increasing cofinal cardinal function. Let $\overline{\mathbb{S}}$ be the $\leq\kappa$-support product of $\langle \mathbb{S}_\kappa^{h_\xi} \mid \xi \in A \rangle$ and G be $\overline{\mathbb{S}}$-generic over $\mathbf{V}$.

Let S_ξ be a stationary set for each $\xi \in B$ and let $g \in {}^\kappa\kappa$ such that $\xi \in B$ implies $g(\alpha) < h_\xi(\alpha)$ for all $\alpha \in S_\xi$, then $|B| \leq \mathfrak{d}_\kappa^g(\in^*)$.

If S is stationary co-stationary and $F \in {}^{\kappa}\kappa$ s.t. $F(\alpha) = F(\alpha)^{|\alpha|}$,

$$h \restriction S = F \restriction S \qquad g \restriction S = 2^F \restriction S$$
$$h \restriction S^c = 2^F \restriction S \qquad g \restriction S^c = F \restriction S$$

Then $\mathbb{S}^h_\kappa$ has the h-Sacks property, but not the g-Sacks property, and $\mathbb{S}^g_\kappa$ has the g-Sacks property, but not the h-Sacks property.

By the last theorems, if we assume $\mathbf{V} \vDash$ "$2^\kappa = \kappa^+$" and we let $h_\xi = h$ for all $\xi < \lambda$ with $\kappa^+ < \lambda$, then the $\leq\kappa$-support product $\overline{\mathbb{S}}$ of $\langle \mathbb{S}^{h_\xi}_\kappa \mid \xi < \lambda \rangle$ forces that $\kappa^+ = \mathfrak{d}^h_\kappa(\in^*) < \mathfrak{d}^g_\kappa(\in^*) = \lambda = 2^\kappa$.

If each $h_\xi = g$ instead, we can force $\mathfrak{d}^g_\kappa(\in^*) < \mathfrak{d}^h_\kappa(\in^*)$.

Theorem *Solovay, see Jech [2003] theorem 8.10*
There exists a disjoint family of sets $\{S_\xi \mid \xi < \kappa\}$ such that each S_ξ is stationary in κ.

Theorem *[vdV] theorem 17 & corollary 18*
Let $F \in {}^\kappa\kappa$ such that $F(\alpha) = F(\alpha)^{|\alpha|}$, and for $\xi < \kappa$ let

$$g_\xi \restriction S_\xi = F \restriction S_\xi \qquad\qquad g_\xi \restriction S_\xi^c = 2^F \restriction S_\xi$$

If $\lambda : \kappa \to \mathrm{Ord}$ be a cardinal function with $\kappa^+ \leq \lambda(\xi)$ for all $\xi \in \kappa$, then there exists a forcing $\overline{\mathbb{S}}$ that forces for all $\xi < \kappa$ that $\mathfrak{d}_\kappa^{g_\xi}(\in^*) = \lambda(\xi)$.

Contents

If $\varphi \in \mathrm{Loc}_h$ and $f \in {}^\kappa\kappa$, let $f \in^\infty \varphi$ if $\{\alpha \in \kappa \mid f(\alpha) \in \varphi(\alpha)\}$ is cofinal in κ. If $f, g \in {}^\kappa\kappa$, let $f =^\infty g$ if $\{\alpha \in \kappa \mid f(\alpha) = g(\alpha)\}$ is cofinal in κ. We define the h-**anti-localisation** cardinals and the **eventually different** cardinals

$$\mathfrak{b}^h_\kappa(\in^\infty) = \min\left\{|B| \mid B \subseteq {}^\kappa\kappa \text{ and } \forall\varphi \in \mathrm{Loc}_h \exists f \in B(f \notin^\infty \varphi)\right\},$$
$$\mathfrak{d}^h_\kappa(\in^\infty) = \min\left\{|D| \mid D \subseteq \mathrm{Loc}_h \text{ and } \forall f \in {}^\kappa\kappa \exists\varphi \in D(f \in^\infty \varphi)\right\},$$
$$\mathfrak{b}_\kappa(\neq^\infty) = \min\left\{|B| \mid B \subseteq {}^\kappa\kappa \text{ and } \forall g \in {}^\kappa\kappa \exists f \in B(f =^\infty g)\right\},$$
$$\mathfrak{d}_\kappa(\neq^\infty) = \min\left\{|D| \mid D \subseteq {}^\kappa\kappa \text{ and } \forall f \in {}^\kappa\kappa \exists g \in D(f \neq^\infty g)\right\}.$$

If we consider $\kappa = \omega$ and any $h \in {}^\omega\omega$, then:

Theorem *Bartoszyński [1987] or Bartoszyński and Judah [1995]*
$\mathfrak{d}^h_\omega(\in^\infty) = \mathfrak{b}_\omega(\neq^\infty) = \mathrm{non}(\mathcal{M})$ and $\mathfrak{b}^h_\omega(\in^\infty) = \mathfrak{d}_\omega(\neq^\infty) = \mathrm{cov}(\mathcal{M})$

Cichoń Diagram

$$
\begin{tikzcd}
 & & & 2^{\aleph_0} \\
\mathrm{cov}(\mathcal{N}) \arrow[r] & \boxed{\begin{array}{c} \mathbf{non}(\mathcal{M}) \\ \mathfrak{d}^{\overline{\omega},h}_{\omega}(\in^{\infty}) \\ \mathfrak{b}^{\overline{\omega}}_{\omega}(\neq^{\infty}) \end{array}} \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \boxed{\begin{array}{c} \mathbf{cof}(\mathcal{N}) \\ \mathfrak{d}^{\overline{\omega},h}_{\omega}(\in^{*}) \end{array}} \arrow[u] \\
 & \boxed{\begin{array}{c} \mathfrak{b} \\ \mathfrak{b}^{\overline{\omega},\overline{\omega}}_{\omega}(\in^{*}) \\ \mathfrak{d}^{\overline{\omega},\overline{\omega}}_{\omega}(\in^{\infty}) \end{array}} \arrow[u] \arrow[r] & \boxed{\begin{array}{c} \mathfrak{b}^{\overline{\omega},\overline{\omega}}_{\omega}(\in^{\infty}) \\ \mathfrak{d}^{\overline{\omega},\overline{\omega}}_{\omega}(\in^{*}) \\ \mathfrak{d} \end{array}} \arrow[u] & \\
\boxed{\begin{array}{c} \mathfrak{b}^{\overline{\omega},h}_{\omega}(\in^{*}) \\ \mathbf{add}(\mathcal{N}) \end{array}} \arrow[uu] \arrow[r] & \mathrm{add}(\mathcal{M}) \arrow[u] \arrow[r] & \boxed{\begin{array}{c} \mathfrak{d}^{\overline{\omega}}_{\omega}(\neq^{\infty}) \\ \mathfrak{b}^{\overline{\omega},h}_{\omega}(\in^{\infty}) \\ \mathbf{cov}(\mathcal{M}) \end{array}} \arrow[u] \arrow[r] & \mathrm{non}(\mathcal{N}) \arrow[uu] \\
\aleph_1 \arrow[u] & & &
\end{tikzcd}
$$

This generalises to strongly inaccessible κ:

Theorem *Landver [1992] and Blass et al. [2005]*

$\mathfrak{d}_\kappa(\neq^\infty) = \mathrm{cov}(\mathcal{M}_\kappa)$ and $\mathfrak{b}_\kappa(\neq^\infty) = \mathrm{non}(\mathcal{M}_\kappa)$.

Theorem

If $h \in {}^\kappa\kappa$, then $\mathfrak{d}^h_\kappa(\in^\infty) = \mathfrak{b}_\kappa(\neq^\infty)$ and $\mathfrak{b}^h_\kappa(\in^\infty) = \mathfrak{d}_\kappa(\neq^\infty)$.

In particular, the choice of h does not have influence on the cardinality of $\mathfrak{d}^h_\kappa(\in^\infty)$ and $\mathfrak{b}^h_\kappa(\in^\infty)$.

Let $b : \kappa \to \mathrm{Ord}$ be an increasing cardinal function and let $\prod b = \prod_{\alpha\in\kappa} b(\alpha) = \{f : \kappa \to \mathrm{Ord} \mid \forall\alpha < \kappa(f(\alpha) < b(\alpha))\}$. Let Loc^b_h be the set of φ s.t. $\mathrm{dom}(\varphi) = \kappa$ and $\varphi(\alpha) \in [b(\alpha)]^{<h(\alpha)}$.

We define the following cardinal characteristics:

$$\mathfrak{b}^{b,h}_\kappa(\in^*) = \min\left\{|B| \;\middle|\; B \subseteq \prod b \text{ and } \forall\varphi \in \mathrm{Loc}^b_h \exists f \in B(f \notin^* \varphi)\right\},$$

$$\mathfrak{d}^{b,h}_\kappa(\in^*) = \min\left\{|D| \;\middle|\; D \subseteq \mathrm{Loc}^b_h \text{ and } \forall f \in \prod b \exists\varphi \in D(f \in^* \varphi)\right\},$$

$$\mathfrak{b}^{b,h}_\kappa(\in^\infty) = \min\left\{|B| \;\middle|\; B \subseteq \prod b \text{ and } \forall\varphi \in \mathrm{Loc}^b_h \exists f \in B(f \notin^\infty \varphi)\right\},$$

$$\mathfrak{d}^{b,h}_\kappa(\in^\infty) = \min\left\{|D| \;\middle|\; D \subseteq \mathrm{Loc}^b_h \text{ and } \forall f \in \prod b \exists\varphi \in D(f \in^\infty \varphi)\right\},$$

$$\mathfrak{b}^{b}_\kappa(\neq^\infty) = \min\left\{|B| \mid B \subseteq \prod b \text{ and } \forall g \in \prod b \exists f \in B(f =^\infty g)\right\},$$

$$\mathfrak{d}^{b}_\kappa(\neq^\infty) = \min\left\{|D| \mid D \subseteq \prod b \text{ and } \forall f \in \prod b \exists g \in D(f \neq^\infty g)\right\}.$$

Contents

For some choices of b and h, the bounded (anti-)localisation cardinals may be trivial.

Lemma
$\mathfrak{d}^{b,h}_\kappa(\in^*) = 1$ iff $b <^* h$, which implies $\mathfrak{b}^{b,h}_\kappa(\in^*)$ is undefined.
$\mathfrak{d}^{b,h}_\kappa(\in^\infty) = 1$ iff $b <^\infty h$, which implies $\mathfrak{b}^{b,h}_\kappa(\in^\infty)$ is undefined.

Lemma *Cardona and Mejía [2019] & Goldstern and Shelah [1993] ($\kappa = \omega$)*
If $\lambda < \kappa$ exists and is minimal s.t. $D_\lambda = \{\alpha \in \kappa \mid h(\alpha) = \lambda\}$ is cofinal in κ, then $\mathfrak{b}^{b,h}_\kappa(\in^*) = \lambda$ and $2^\kappa \leq \mathfrak{d}^{b,h}_\kappa(\in^*)$. If no such λ exists, $\kappa^+ \leq \mathfrak{b}^{b,h}_\kappa(\in^*)$, and if also $b \leq 2^\kappa$, then $\mathfrak{d}^{b,h}_\kappa(\in^*) \leq 2^\kappa$.

Let increasing $f : \kappa \to \mathrm{Ord}$ be **continuous** at $\gamma \in \kappa$ if $f(\gamma) = \bigcup_{\alpha<\gamma} f(\alpha)$. We call f stationarily continuous there exists S stationary in κ s.t. f is continuous at all limit $\gamma \in S$.

Lemma

For $\lambda < \kappa$ let
$D_\lambda = \{\alpha \in \kappa \mid b(\alpha) \leq \lambda\} \cup \{\alpha \in \kappa \mid h(\alpha) = b(\alpha) \wedge \mathrm{cf}(b(\alpha)) \leq \lambda\}$.

(i) If $\lambda < \kappa$ exists and is minimal s.t. D_λ is cofinal in κ, then $\mathfrak{d}^{b,h}_\kappa(\in^\infty) = \lambda$.

(ii) If all D_λ are bounded, b is stat.cont., then $\mathfrak{d}^{b,h}_\kappa(\in^\infty) = \kappa$.

(iii) If all D_λ are bounded, b is not stat.cont., then $\kappa^+ \leq \mathfrak{d}^{b,h}_\kappa(\in^\infty)$.

A dual result for the relation between $\mathfrak{b}^{b,h}_\kappa(\in^\infty)$ and 2^κ is not known yet.

Say that b **overshadows** h if there exists an interval partition $\langle I_\alpha \mid \alpha < \kappa\rangle$ of κ with $|I_\alpha| = h(\alpha)$ for each $\alpha \in \kappa$ such that $b(\alpha) = b(\xi) = b(\alpha)^{h(\alpha)}$ for all $\xi \in I_\alpha$ and $\alpha \in \kappa$.

Theorem

If b overshadows h, then $\mathfrak{d}_\kappa^{b,h}(\in^\infty) = \mathfrak{b}_\kappa^{b}(\neq^\infty)$ and $\mathfrak{b}_\kappa^{b,h}(\in^\infty) = \mathfrak{d}_\kappa^{b}(\neq^\infty)$.

If $b : \alpha \mapsto \kappa$ for all $\alpha \in \kappa$ and $h \in {}^\kappa\kappa$, then the conditions of the theorem are satisfied, and $\mathfrak{d}_\kappa^{b,h^+}(\in^\infty) = \mathfrak{d}_\kappa^{h}(\in^\infty)$ and $\mathfrak{b}_\kappa^{b,h^+}(\in^\infty) = \mathfrak{b}_\kappa^{h}(\in^\infty)$, where $h^+ : \alpha \mapsto h(\alpha)^+$. In particular, the cardinality of $\mathfrak{d}_\kappa^{h}(\in^\infty)$ and $\mathfrak{b}_\kappa^{h}(\in^\infty)$ does not depend on the choice of $h \in {}^\kappa\kappa$.

Let $b = \kappa$ be the constant κ function and $h \leq h' \leq b$ and $h < b$.

$$
\begin{tikzcd}
\mathfrak{d}^{\overline{\kappa}}_{\kappa}(\leq^*) \arrow[rr] \arrow[rr, dotted, no head] \arrow[dr, dotted, no head] & & \mathfrak{d}^{\overline{\kappa},h'}_{\kappa}(\in^*) \arrow[r] & \mathfrak{d}^{\overline{\kappa},h}_{\kappa}(\in^*) \\
& \mathfrak{b}^{\overline{\kappa},h'}_{\kappa}(\in^\infty) \arrow[ur, dotted, no head] & & \\
\boxed{\begin{array}{c} \mathrm{cov}(\mathcal{M}_\kappa) \\ \mathfrak{b}^{\overline{\kappa},h}_{\kappa}(\in^\infty) \\ \mathfrak{d}^{\overline{\kappa}}_{\kappa}(\neq^\infty) \end{array}} \arrow[uu] \arrow[ur] & & & \boxed{\begin{array}{c} \mathfrak{b}^{\overline{\kappa}}_{\kappa}(\neq^\infty) \\ \mathfrak{d}^{\overline{\kappa},h}_{\kappa}(\in^\infty) \\ \mathrm{non}(\mathcal{M}_\kappa) \end{array}} \arrow[uu] \\
& & \mathfrak{d}^{\overline{\kappa},h'}_{\kappa}(\in^\infty) \arrow[uuu] \arrow[ur] \arrow[dr, dotted, no head] & \\
\mathfrak{b}^{\overline{\kappa},h}_{\kappa}(\in^*) \arrow[uu] \arrow[r] & \mathfrak{b}^{\overline{\kappa},h'}_{\kappa}(\in^*) \arrow[uuu] \arrow[rr] \arrow[rr, dotted, no head] \arrow[ur, dotted, no head] & & \mathfrak{b}^{\overline{\kappa}}_{\kappa}(\leq^*) \arrow[uu] \arrow[uuuulll, bend right]
\end{tikzcd}
$$

The dotted line implies equality if $h' =^* b$.

Nontrivial cases

Assume that $h \le b' \le b \in {}^\kappa\kappa$ and b overshadows h.

$$
\begin{tikzcd}
\mathfrak{b}_\kappa^{b}(\le^*) \arrow[r] & \mathfrak{b}_\kappa^{b}(\neq^\infty) = \mathfrak{d}_\kappa^{b,h}(\in^\infty) \arrow[r] & \mathfrak{d}_\kappa^{b,h}(\in^*) \\
\mathfrak{b}_\kappa^{b'}(\le^*) \arrow[r] & \mathfrak{b}_\kappa^{b'}(\neq^\infty) \arrow[u] & \\
 & \mathfrak{d}_\kappa^{b',h}(\in^\infty) \arrow[r] \arrow[u] & \mathfrak{d}_\kappa^{b',h}(\in^*) \arrow[uu]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathfrak{b}_\kappa^{b',h}(\in^*) \arrow[r] & \mathfrak{b}_\kappa^{b',h}(\in^\infty) & \\
 & \mathfrak{d}_\kappa^{b'}(\neq^\infty) \arrow[r] \arrow[u] & \mathfrak{d}_\kappa^{b'}(\le^*) \\
\mathfrak{b}_\kappa^{b,h}(\in^*) \arrow[r] \arrow[uu] & \mathfrak{b}_\kappa^{b,h}(\in^\infty) = \mathfrak{d}_\kappa^{b}(\neq^\infty) \arrow[r] \arrow[u] & \mathfrak{d}_\kappa^{b}(\le^*)
\end{tikzcd}
$$

Contents

The same forcings $\mathbb{S}_\kappa^h$ that were used to separate cardinals of the form $\mathfrak{d}_\kappa^h(\in^*)$ can be used on the space $\prod b$. That is, if $2^F \in \prod b$ and $h(\alpha), h'(\alpha)$ take the values $F(\alpha)$ or $2^{F(\alpha)}$ dependent on whether $\alpha \in S$ for some stationary costationary set S, then $\mathfrak{d}_\kappa^{b,h}(\in^*) < \mathfrak{d}_\kappa^{b,h'}(\in^*)$ and $\mathfrak{d}_\kappa^{b,h'}(\in^*) < \mathfrak{d}_\kappa^{b,h}(\in^*)$ are both consistent.

Theorem

If $b \in {}^\kappa\kappa$ then $\mathrm{cov}(\mathcal{M}_\kappa) = \mathfrak{b}^h_\kappa(\in^\infty) < \mathfrak{b}^{b,h}_\kappa(\in^\infty)$ is consistent.

The forcing used is $\mathbb{P}^{b,h}_\kappa$ with trees T on Loc^b_h as conditions, i.e. $u \in T$ implies $u : \alpha \to [\kappa]^{<\kappa}$ s.t. $u(\xi) \in [b(\xi)]^{<h(\xi)}$ for each $\xi < \alpha$. If $u \in T$ with $\alpha = \mathrm{ot}(u)$, let $\|u\|_T$ be the least $\nu < \kappa$ such that there exists $A \in [b(\alpha)]^\nu$ such that $A \not\subseteq A'$ for all $A' \in \mathrm{suc}(u, T)$.

Let $T \in \mathbb{P}^{b,h}_\kappa$ iff

(i) for all $u \in T$, $\nu < \kappa$ there is $v \in T$ with $u \subseteq v$ and $\nu \leq \|v\|_T$,

(ii) If $\langle u_\xi \mid \xi < \gamma\rangle$ is a sequence of splitting nodes and $u_\xi \subseteq u'_\xi$ for $\xi < \xi'$, then $\bigcup_{\xi<\gamma} u_\xi$ splits in T,

(iii) if $u \in \mathrm{Split}_\alpha(T)$, then $\max\{|\alpha|, 2\} \leq \|u\|_T$.

Let $S \leq_{\mathbb{P}^{b,h}_\kappa} T$ if $S \subseteq T$ and for each $s \in S$ either $\mathrm{suc}(s, S) = \mathrm{suc}(s, T)$ or $\|s\|_S < \|s\|_T$.

$\mathbb{P}^{b,h}_{\kappa}$ is $<\kappa$-closed, has fusion and is ${}^{\kappa}\kappa$-bounding. Moreover, the $\leq\kappa$-support iteration of $\mathbb{P}^{b,h}_{\kappa}$ is ${}^{\kappa}\kappa$-bounding as well.

Hence, forcing with $\mathbb{P}^{b,h}_{\kappa}$ increases the size of $\mathfrak{b}^{b,h}_{\kappa}(\in^{\infty})$ but keeps $\mathrm{cov}(\mathcal{M}_{\kappa})$ small.

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