Building generalized indiscernibles in nonelementary classes with set theory

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Kobe University

Outline

- The new* adventures of an old theorem of Morley
- Generalizing the Erdős-Rado Theorem
- The curious case of ordered graphs
- To (large) infinity and beyond!

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Fact (Morley's Omitting Types Theorem)

The class of linear orders is minimal amongst large, finitely accessible categories.

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Given a sentence $\psi \in \mathbb{L}_{\omega_1,\omega}$, if ψ has models of arbitrarily large sizes ($\beth_{\omega_1}$ is enough), then, for any linear order I, we can build a model of ψ that contains I as order indiscernibles.

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- Chang connects this to type omission

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Linear orders is minimal among large, finitely accessible categories. This means that if $\mathbb{K}$ is a large, finitely accessible category, then there is a faithful functor from linear orders to $\mathbb{K}$ that preserves directed colimits.

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(This has been my weak attempt at a joke, so some pity laughter would be appropriate)

Goal

Goal (Talk)

In this talk, I want to talk about how to find other minimal categories, and also a little what we can do with them

- The category theorist in me is really interested in nice diagrammatic ways to express this
- The set theorist in me is really interested in how we find minimal categories
- The model theorist in me is really interested in what we can do in this

Unpacking Makkai-Paré's Phrasing

Notation	What it means

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$\mathcal{K}^{or}$	the category of linear orders
$\mathcal{K}$	Index categories, like linear orders, ordered graphs, trees, etc.
$\mathbb{K}$	Target categories where indiscernibles exist, like $\mathbb{L}_{\infty,\omega}$-elementary classes, AECs, etc. (admit a faithful functor from a finitely accessible category)

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Category theory	Model theory
Finitely accessible categories	Classes axiomatized in $\mathbb{L}_{\infty,\omega}$ (modulo equivalence and Skolemization)
Large	Class many models up to isomorphism; equivalently, arbitarily large models
Faithful functor preserving directed colimits	Blueprints/order indiscernibles

Blueprints as functors

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- Typically, a blueprint Φ (for order indiscernibles) is a set of instructions that tells you how to generate a $\tau(\Phi)$-structure from a given linear order I that contains I as indiscernibles

$$I \hookrightarrow \mathrm{EM}(I, \Phi)$$

Blueprints as functors

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$$I \hookrightarrow \mathrm{EM}(I, \Phi)$$

- These instructions are faithfully functorial, so a map $I \to J$ lifts to $\mathrm{EM}(I, \Phi) \to EM(J, \Phi)$
- These instructions are finitely generated, so commutes with increasing unions/directed colimits

Blueprints as functors

- So Makkai-Paré's observation is that any blueprint generates a faithful functor that preserves directed colimits

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- So Makkai-Paré's observation is that any blueprint generates a faithful functor that preserves directed colimits

$$\Phi : \mathcal{K}^{or} \to \mathbb{K}$$

- With a little work, this can be reversed:

Proposition (Baldwin-B., as would be phrased by Makkai-Paré)

Any faithful functor $\Phi : \mathcal{K}^{or}_{<\omega} \to \mathbb{K}_\kappa$ lifts to a blueprint for order indiscernibles in $\mathbb{K}$.

Prelim wrap-up

Observation

Blueprints for order indiscernibles in $\mathbb{K}$ are (up to natural isomorphism) directed colimit-preserving, faithful functors

$$\Phi : \mathcal{K}^{or} \to \mathbb{K}$$

(Thanks to Tibor Beke for pointing out the necessity of natural isomorphisms.)

Prelim wrap-up

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Some natural questions:

- Can we do this with classes other than linear orders?
- What can we do with these?
- What does this have to do with set theory?

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Generalized Indiscernibles

- Want indiscernibles generalized by structures other than linear order
- Notationally dificult to write out, but functorial definition simplifies it a lot
  - Misha Gavrillovich indexes generalized blueprints by the simplicial category

Generalized Indiscernibles

Definition

- *A blueprint for order indiscernibles in* $\mathbb{K}$ *is a colimit-preserving, faithful functor*

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- *A blueprint for order indiscernibles in* $\mathbb{K}$ *is a colimit-preserving, faithful functor*

$$\Phi : \mathcal{K}^{or} \to \mathbb{K}$$

- *Fix a category* $\mathcal{K}$, *probably a simple finitely accessible category.*
  A blueprint for $\mathcal{K}$*-indiscernibles in* $\mathbb{K}$ *is a colimit-preserving, faithful functor*

$$\Phi : \mathcal{K} \to \mathbb{K}$$

How do we build blueprints?

- "Definitions can't be wrong," but need to actually have blueprints for this to be useful
- For $\mathcal{K}^{or}$, this is what Morley's Omitting Types Theorem tells us!

Fact (Morley-Chang)

Given a theory $T \subset \mathbb{L}_{\kappa^+,\omega}$, if it has models of arbitrarily large sizes ($\beth_{(2^\kappa)^+}$ is enough), then, for any linear order I, we can build a model of T that contains I as order indiscernibles.

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- The proof makes crucial use of the Erdős-Rado Theorem: for every $n < \omega$ and cardinal κ

$$\beth_{n-1}(\kappa)^+ \to (\kappa^+)^n_\kappa$$

The dream construction

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- Iterate ω-many steps to get the indiscernible blueprint

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How big is 'big'?

The ill-founded dream

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- To use the Erdős-Rado Theorem to shrink X_n into homogeneous X_{n+1}, we need

$$|X_n| \geq \beth_n(|X_{n+1}|)^+$$

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- ..but this means

$$|X_0| > |X_1| > |X_2| > |X_3| > \dots$$

- So our dream has turned into an ill-founded nightmare!

Waking from our ill-founded nightmare

- All is not lost! We can go through the construction with some technical bookkeeping that translates as poorly to a talk format as it does to paper
- Essentially, rather than a single linear chain X_n of length ω, you build a well-founded tree of height ω

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  - The indiscernibility is shared across a level, so you can read Φ out of the tree without any ill-foundedness
  - Jiři Rosický has a nice argument that makes this tree idea explicit that removes a lot of the technical details

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  - The indiscernibility is shared across a level, so you can read Φ out of the tree without any ill-foundedness
  - Jiři Rosický has a nice argument that makes this tree idea explicit that removes a lot of the technical details
- In the end, you need to start with a set of size at least

$$\beth_{(2^\kappa)^+}$$

Erdős-Rado Classes

- How do we define generalized blueprints?

Definition (B., Categorical version of Erdős-Rado Class)

$\mathcal{K}$ is an almost Erdős-Rado Class iff for all large, finitely accessible categories $\mathbb{K}$, there is a blueprint

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- To actually build these, we need something like the Erdős-Rado Theorem
  - Structural Partition Relations

Structural partition relations

- Start with a cautionary tale:

Example

Let $\mathcal{K}^{2-or}$ be the class of two disjoint linear orders and let $(I_0, I_1) \in \mathcal{K}^{2-or}$.

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Let $\mathcal{K}^{2-or}$ be the class of two disjoint linear orders and let $(I_0, I_1) \in \mathcal{K}^{2-or}$. Take a coloring of pairs

$$c : [(I_0, I_1)]^2 \to 2$$

given by

$$c(i,j) = \begin{cases} 0 & i \in I_0 \iff j \in I_0 \\ 1 & \text{otherwise} \end{cases}$$

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Want a big part of both linear orders $(I_0^*, I_1^*) \subset (I_0, I_1)$ that is homogeneous.

Structural Partition Relations

Example

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given by

$$c(i,j) = \begin{cases} 0 & i \in I_0 \iff j \in I_0 \\ 1 & \text{otherwise} \end{cases}$$

Two changes:

- Want both parts represented in the homogeneous set: replace cardinality with universality
- Used type to color, so can't get large, homogeneous set from both parts: replace homogeneity with type-homogeneity

Structural Partition Relations

"There are cases in mathematical history when a well-chosen notation can enormously enhance the development of a branch of mathematics and a case in point is the ordinary partition symbol."

András Hajnal and Jean Larson

Definition

Fix $\mathcal{K}$.

$$\lambda \xrightarrow{\mathcal{K}} (\kappa)^n_\mu$$

means: for any $< \lambda$-universal M and coloring

$$c : [M]^n \to \mu$$

there is a $< \kappa$-universal $N \subset M$ that is type-homogeneous; that is, $c \upharpoonright N$ only depends on the type of the input.

Erdős-Rado Classes

Theorem (B., Generlized Omitting Types Theorem)

The following combinatorial statement suffices to build blueprints in large, finitely accesible categories: for every $n < \omega$ and κ, μ, there is a λ so

$$\lambda \xrightarrow{\mathcal{K}} (\kappa)^n_\mu$$

Proof:

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Proof:

- Morally the same argument as before, but with a lot more bookkeeping
  - A **lot** more bookkeeping
  - Typically, $\lambda = \beth_{p(n)}(\kappa)^+$ where $p(x)$ is a polynomial
  - This gives the threshold as the same $\beth_{(2^\kappa)^+}$ as before

Examples!

Example (χ-linear orders)

$\mathcal{K}^{\chi-or}$ is the class of χ disjoint linear orders in the language $(<, P_i)_{i<\chi}$. Erdős-Hajnal-Rado show

$$\beth_{n(n+1)}(\kappa)^+ \xrightarrow{\chi-or} (\kappa)^n_\kappa$$

Example (Convexly-ordered equivalence relations)

$\mathcal{K}^{cer}$ is the class of linear orders with an equivalence relation so each equivalence class is convex. Several uses of the $\mathcal{K}^{\chi-or}$ partition theorem give

$$\beth_{n(n+2)}(\kappa)^+ \xrightarrow{ceq} (\kappa^+)^n_\kappa$$

Examples!

Example (Well-founded trees)

$\mathcal{K}^{wf-tr}$ is the class of trees (in the above language) with no infinite branches. Gruenhut and Shelah show

$$\beth_{1,n}(\kappa) \xrightarrow{wf-tr} (\kappa)^n_\kappa$$

- $\beth_{1,n}(\kappa)$ is very big

Examples!

Example (Trees of height $m < \omega$)

$\mathcal{K}^{m-tr}$ is the class of trees of height n in the language $(P_\ell, <_{tr}, \prec, \wedge)_{\ell<m}$. Shelah proved there is $p(n,m) < \omega$ so

$$\beth_{p(n,m)}(\kappa)^+ \xrightarrow{m-tr} (\kappa^+)^n_\kappa$$

Example (Trees of height ω)

$\mathcal{K}^{\omega-tr}$ is the class of trees of height ω in the language $(P_\ell, <_{tr}, \prec, \wedge)_{\ell<\omega}$.
No (known) combinatorics here! But still build blueprints by seeing an ω-height tree as a union of n-height trees.

Applications!

- Briefly mention:
  - Compactness-like proofs mimicing first-order
  - Defining dividing lines via indiscernible collapse

Shelah trees in AECs

Theorem (Shelah)

Let T be a countable first-order theory. One of the following holds:

1. *T is stable on a tail starting at 2^ω.*
2. *T is unstable in every $\lambda < \lambda^\omega$.*

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Given instability in $\lambda < \lambda^\omega$, build Shelah tree on ${}^\omega\lambda$

- Parametets indexed by ${}^{<\omega}\lambda$, types indexed by branches
- Write down theory T_λ to axiomatize the Shelah tree
- For any μ, finite subsets of T_μ and T_λ are the same!
- Use compactness to build a Shelah tree at μ

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Theorem (Baldwin-Shelah, B.)

Let $\mathbb{K}$ be a κ-tame AEC with amalgamation. One of the following holds:

1. *$\mathbb{K}$ is Galois stable on a tail starting at $\chi < \beth_{(2^\kappa)^+}$.*
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Given Galois instability in big $\lambda < \lambda^\omega$, build Shelah tree on ${}^\omega\lambda$

- Parameters indexed by ${}^{<\omega}\lambda$, types indexed by branches
- Build a $\mathcal{K}^{\omega-tr}$-blueprint Φ patterned on this Shelah tree
- For any μ, $\Phi({}^{<\omega}\mu)$ is a Shelah tree at μ

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Theorem (Vasey)

Let $\mathbb{K}$ be a κ-tame AEC with amalgamation. One of the following holds:

1. *$\mathbb{K}$ is Galois stable on a tail.*
2. *$\mathbb{K}$ is Galois unstable in every sufficiently large λ with cf $\lambda = \omega$ and $\mu < \lambda \implies \mu^{\lambda_0} < \lambda$ (λ_0 is the first Galois stability cardinal).*

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- Generalizing the Erdős-Rado Theorem
- The curious case of ordered graphs
- To (large) infinity and beyond!

The curious case of ordered graphs

- There is an elephant in the room: the class of ordered graphs $\mathcal{K}^{og}$
- This class is well-studied among elementary classes

Fact (Scow)

NIP theories can be characterized by indiscernible collapse from ordered graphs to linear orders

$$\begin{tikzcd} \mathcal{K}^{og} \arrow[rr, "\Phi"] \arrow[dr] & & Mod(T) \\ & \mathcal{K}^{or} \arrow[ur, dotted] & \end{tikzcd}$$

NIP AECs

- This suggests a way to generalize the notion of NIP to infinitary model theory (AECs)
    - Wentao Yang has another notion

Definition

An Abstract Elementary Class is NIP iff ordered graph indiscernibles collapse to order indiscernibles; that is, we have the following lifting diagram for every directed colimit preserving, faithful functor Φ *(suppressing a natural isomorphism):*

$$\begin{tikzcd} \mathcal{K}^{og} \arrow[rr, "\Phi"] \arrow[dr] & & \mathbb{K} \\ & \mathcal{K}^{or} \arrow[ur, dotted] & \end{tikzcd}$$

- But this definition only works if there is an Erdős-Rado Theorem for ordered graphs!

Graph Erdős-Rado

Question

Is there a nice partition calculus for the class of ordered graphs (a la the Erdős-Rado theorem)?

Answer

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- Like all good 'maybe's, this is a question of set theory and consistency

Ordered graphs

Theorem (Deuber, Erdős-Hajnal-Pósa, Nešetřil-Rödl)

Ordered graphs is a Ramsey class.

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For countable H and $k < \omega$, there is G so

$$G \to (H)^2_k$$

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Theorem (Hajnal-Komjáth)

It is consistent with ZFC that: there is a graph H of size ω_1 so for all G

$$G \not\to (H)^2_2$$

The Hajnal-Komjáth proof

Theorem (Hajnal-Komjáth)

It is consistent with ZFC that: there is a graph H of size ω_1 so for all G

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Proof:

- Start with a model of CH and an eventually dominating family $\langle f_\alpha : \omega \to \omega : \alpha < \omega_1 \rangle$
- We force to add a Cohen real, so $V^{\mathbb{P}}$ has a generic $G : \omega \to 2$
- Define X to be a bipartite graph on ω, ω_1 with edge relation

$$\{n, \alpha\} \in E(X) \iff G(f_\alpha(n)) = 1$$

- I don't have a quick explanation for it, but this works

The other half of the maybe

Theorem (Hajnal-Komjáth)

It is consistent with ZFC that: there is a graph H of size ω_1 so for all G

$$G \not\rightarrow (H)^2_2$$

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Theorem (Shelah)

It is consistent with ZFC that: for all H, for all κ, for all $n < \omega$, there is G such that

$$G \to (H)^n_\kappa$$

(and also for colored hypergraphs)

Ordered graphs

Question

Is there a nice partition calculus for the class of ordered graphs (a la the Erdős-Rado theorem)?

Answer

Maybe.

- But this isn't the actual question we care about!
- Looking at the positive proof more, we can extract some information

Ordered graphs

Theorem (Shelah)

It is consistent with ZFC that ordered graphs have nice combinatorics (in terms of structural partition relations).

Ingredients:

- Assume we have GCH to get some nice initial combinatorics

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- Easton support iterate to get 'sort of's everywhere
- String together the 'sort of's to get the actual result.

Outline

- The new* adventures of an old theorem of Morley
- Generalizing the Erdős-Rado Theorem
- The curious case of ordered graphs
- To (large) infinity and beyond!

To (large) infinity...

- We want to use large cardinals to directly imply this combinatorics
- End up with Ramsey-style cardinals

Laver indestructible supercompacts

Theorem (B.-Shelah)

Suppose that κ is Laver indestructible supercompact. If G is a $< \kappa$-universal graph of size κ, then every coloring of n-tuples by $< \kappa$-many colors has a type-homogeneous subset of any κ-sized graph.

Proof:

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- Use the tree property to increase this to κ-sized pieces

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Yes from large cardinals.

Ordered graphs

- In fact, get a Ramsey cardinal-style result

$$\kappa \xrightarrow{og} (\kappa)^n_{<\kappa}$$

- This allows us to build blueprints from models at size κ
- Laver indestructible strong compacts are enough, but not sure if that's a thing
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- But we can do better!

Ordered graphs revisited: Forcing and blueprints

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- So even though blueprints are built around rank $\beth_{(2^\mu)^+}$, Φ itself is in $V_{\mu+\omega}$
  - Easier to see with model theory: blueprints are functions from types over the emptyset in $\mathcal{K}$ to types over the emptyset in $\mathbb{K}$

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Theorem (Shelah)

Ordered graphs and colored hypergraphs are almost Erdős-Rado classes: For any large $\mathbb{K}$, there is a blueprint $\Phi : \mathcal{K}^{og} \to \mathbb{K}$.

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- The verification that Φ is proper is Δ_1 in the parameters, so it passes from $V[G]$ to V

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THANKS!