Structural Reflection on the edge and beyond

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Sequential *ESR*

Let $\mathcal{L}$ be a first-order language containing unary predicate symbols $\vec{P} = \langle \dot{P}_m \mid m < \omega \rangle$.

Given a strictly increasing sequence $\vec{\lambda} = \langle \lambda_m \mid m < \omega \rangle$ of cardinals with supremum λ, an $\mathcal{L}$-structure A has *type $\vec{\lambda}$ (with respect to $\vec{P}$)* if the universe of A has rank λ and $\mathrm{rank}(\dot{P}_m^A) = \lambda_m$ for all $m < \omega$.

Definition:

Given a class $\mathcal{C}$ of $\mathcal{L}$-structures and a strictly increasing sequence $\vec{\lambda} = \langle \lambda_m \mid m < \omega \rangle$ of cardinals, let **Exact Structural Reflection for $\mathcal{C}$ and $\vec{\lambda}$** (written $ESR_{\mathcal{C}}(\vec{\lambda})$) be the following assertion:

For every A in $\mathcal{C}$ of type $\langle \lambda_{m+1} \mid m < \omega \rangle$, there is an elementary embedding of some B in $\mathcal{C}$ of type $\langle \lambda_m \mid m < \omega \rangle$ into A.

Recall that $C^{(n)}$ is the closed unbounded Π_n-definable class of ordinals α that are Σ_n-correct in V, i.e., $V_\alpha \preceq_{\Sigma_n} V$.

Definition:

Given a natural number $n > 0$ and a strictly increasing sequence $\vec{\lambda} = \langle \lambda_m \mid m < \omega \rangle$ of cardinals with supremum λ, a cardinal $\kappa < \lambda_0$ is *n*-**exact for** $\vec{\lambda}$ if for every $A \in V_{\lambda+1}$, there exists a cardinal $\lambda < \theta \in C^{(n)}$, a cardinal $\lambda < \theta' \in C^{(n+1)}$, an elementary submodel X of V_θ with $V_\lambda \cup \{\lambda\} \subseteq X$, and an elementary embedding $j : X \longrightarrow V_{\theta'}$ with $A \in \mathrm{ran}(j)$, $j(\kappa) = \lambda_0$ and $j(\lambda_m) = \lambda_{m+1}$ for all $m < \omega$.

If, for $z \in V_\kappa$ we further require that $z \in X$ and $j(z) = z$, then we say that κ is *n*-**exact for** $\vec{\lambda}$ **and** z.

We say that κ is **parametrically** *n*-**exact for** $\vec{\lambda}$ if κ is *n*-exact for $\vec{\lambda}$ and z, for each $z \in V_\kappa$.

n-**exact cardinals**

If κ is *n*-exact for $\vec{\lambda} = \langle \lambda_i : i < \omega \rangle$, witnessed by some j such that $j(\text{crit}(j)) = \kappa$, then $\text{crit}(j)$ is parametrically *n*-exact for the sequence $\langle \kappa \rangle ^\frown \langle \lambda_i : i < \omega \rangle$.

Moreover, if κ is the least parametrically *n*-exact cardinal for some sequence $\vec{\lambda}$, then this is witnessed by some j with $j(\text{crit}(j)) = \kappa$.

Theorem

Let $\vec{\lambda} = \langle \lambda_m \mid m < \omega \rangle$ be a strictly increasing sequence of cardinals.

1. *The cardinal λ_0 is n-exact for $\langle \lambda_{m+1} \mid m < \omega \rangle$ if and only if Σ_{n+1}-ESR$(\vec{\lambda})$ holds.*

2. *If λ_0 is parametrically n-exact for $\langle \lambda_{m+1} \mid m < \omega \rangle$ iff $\Sigma_{n+1}(V_{\lambda_0})$-ESR$(\vec{\lambda})$ holds.*

The strength of n-exact cardinals

We have following lower bounds on the strength of 0-exact and 0-parametrically exact cardinals:

- Let λ be the supremum of $\vec{\lambda}$. If $\kappa < \lambda_0$ is 0-exact for $\vec{\lambda}$, then there exists an I3-embedding $j : V_\lambda \to V_\lambda$.

- If κ is the least cardinal that is parametrically 0-exact for $\vec{\lambda}$, then the set of I3-cardinals is stationary in κ.

We also have the following upper bound:

- If κ is the critical point of an I1 embedding $j : V_{\lambda+1} \to V_{\lambda+1}$, then κ is parametrically 0-exact for the sequence $\vec{\lambda} = \langle j^{i+1}(\kappa) : i < \omega \rangle$. Hence, $\Sigma_1(V_\kappa)$-_ESR_$(\vec{\lambda})$ holds.

The following was an open question:

Question

Does ZFC _prove that_ Σ_2-_ESR_$(\vec{\lambda})$ _fails for every_ $\vec{\lambda}$ _of length_ ω?

The main theorem

Theorem

If $j : L(V_{\lambda+1}) \longrightarrow L(V_{\lambda+1})$ be an I0-embedding with critical sequence $\langle \kappa_m \mid m < \omega \rangle$, then there is a transitive model M of ZFC with $M \cap \mathrm{Ord} = \lambda^+$, $\vec{\kappa} \in M$ and κ_0 is parametrically n-exact for $\vec{\kappa} = \langle \kappa_{m+1} \mid m < \omega \rangle$ in M for every natural number n.

Hence $ESR_{\mathcal{C}}(\vec{\kappa})$ holds in M for every class $\mathcal{C}$ of $\mathcal{L}$-structures that is definable in M with parameters in V_{κ_0}.

Proof:

We start with the following observation:

Claim

There exists a well-ordering $\lhd$ of V_λ of order-type λ, with $j(\lhd) = \lhd$.

Proof of claim:

Pick a wellordering $\lhd_0$ of V_{κ_0}, and let $\lhd_1 = j(\lhd_0) \setminus \lhd_0$. Given $\lhd_n$, $n \geqslant 1$, let $\lhd_{n+1} = j(\lhd_n)$. Finally, let $\lhd = \bigcup_{n < \omega} \lhd_n$. Then $\lhd$ is as required. $\qquad\square$

Proof continued:

Now, set $\Gamma = V_\lambda \cup \{\vec{\kappa}, \lhd\}$ and note that it belongs to $L(V_{\lambda+1})$.

By using $\lhd$, in $L(\Gamma)$ we may easily well-order Γ in order-type λ, so that $L(\Gamma)$ is a model of ZFC.

Moreover, since $j(\lhd) = \lhd$, we have that $j(\Gamma) = V_\lambda \cup \{j(\vec{\kappa}), \lhd\}$, hence $L(\Gamma) = L(j(\Gamma))$, and so j restricted to $L(\Gamma)$ yields an elementary embedding $L(\Gamma) \to L(\Gamma)$. Thus by a classical result of Kunen, $(\Gamma)^\sharp$ exists, and therefore λ^+ is inaccessible in $L(\Gamma)$.

Hence, letting $M = L_{\lambda^+}(\Gamma)$ we have that M is a model of ZFC.

Moreover, since $j(\lambda^+) = \lambda^+$, the restriction map $j \restriction M : M \longrightarrow M$ is an elementary embedding.

Proof continued:

Now fix a natural number n and assume, aiming for a contradiction, that, in M, the cardinal κ_0 is not parametrically n-exact for $\langle \kappa_{m+1} \mid m < \omega \rangle$.

Pick an ordinal $\lambda < \theta < \lambda^+$ such that $j(\theta) = \theta$ and $\theta \in (C^{(n+1)})^M$.

Working in M, let $A \in V_{\lambda+1}$ be such that, for every elementary substructure Y of H_θ with $V_\lambda \cup \{\lambda\} \subseteq Y$, there is no elementary embedding $i : Y \longrightarrow H_\theta$ with $i(\operatorname{crit}(i)) = \kappa_0$, $A \in \operatorname{ran}(i)$ and $i(\kappa_m) = \kappa_{m+1}$ for all $m < \omega$.

Without loss of generality, we may assume that $A \notin \kappa_0 \cup \{\kappa_m \mid m < \omega\}$.

Proof continued:

The elementarity of $j \restriction M$ then implies that, in M, for every elementary substructure Y of H_θ with $V_\lambda \cup \{\lambda\} \subseteq Y$, there is no elementary embedding $i : Y \longrightarrow H_\theta$ with $i(\text{crit}(i)) = \kappa_1$, $j(A) \in \text{ran}(i)$ and $i(\kappa_m) = \kappa_{m+1}$ for all $0 < m < \omega$.

Still in M, let X_0 be an elementary substructure of H_θ of cardinality λ, containing A, and with $V_\lambda \cup \{\lambda\} \subseteq X_0$. Pick a bijection $b_0 : \lambda \longrightarrow X_0$ with $b_0(0) = A$, $b_0(m+1) = \kappa_m$ for all $m < \omega$ and $b_0(\omega + \alpha) = \alpha$ for all $\alpha < \kappa_0$.

Proof continued:

Set $X_1 = j(X_0)$ and $b_1 = j(b_0)$.

The set X_1 is an elementary substructure of H_θ of cardinality λ with $V_\lambda \cup \{\lambda\} \subseteq X_0$ and $b_1 : \lambda \longrightarrow X_1$ is a bijection with $b_1(0) = j(A)$, $b_1(m+1) = \kappa_{m+1}$ for all $m < \omega$ and $b_1(\omega + \alpha) = \alpha$ for all $\alpha < \kappa_1$.

Moreover, note that

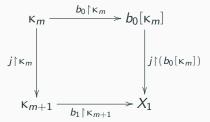$$b_1 \circ (j \restriction \lambda) \;=\; (j \restriction X_0) \circ b_0 \tag{1}$$

holds. I.e., the following diagram commutes:

$$
\begin{array}{ccc}
\lambda & \xrightarrow{\;b_0\;} & X_0 \\
{\scriptstyle j \restriction \lambda} \downarrow & & \downarrow {\scriptstyle j \restriction X_0} \\
\lambda & \xrightarrow[\;b_1\;]{} & X_1
\end{array}
$$

Proof continued:

Now, note that in the restricted diagram

$$
\begin{array}{ccc}
\kappa_m & \xrightarrow{\ b_0 \restriction \kappa_m \ } & b_0[\kappa_m] \\
{\scriptstyle j \restriction \kappa_m} \downarrow & & \downarrow {\scriptstyle j \restriction (b_0[\kappa_m])} \\
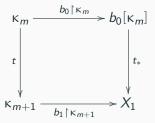\kappa_{m+1} & \xrightarrow[\ b_1 \restriction \kappa_{m+1} \ ]{} & X_1
\end{array}
$$

The map $j \restriction \kappa_m$ is the identity on κ_0, and the map $j \restriction (b_0[\kappa_m])$ yields a partial elementary embedding from X_0 to X_1.

Proof continued:

So let us define T to be the set of all functions $t : \kappa_m \longrightarrow \kappa_{m+1}$, some $m < \omega$, such that $t \restriction \kappa_0 = \mathrm{id}_{\kappa_0}$ and the partial function

$$t_* : b_0[\kappa_m] \longrightarrow X_1; \ x \mapsto (b_1 \circ t \circ b_0^{-1})(x)$$

is a partial elementary embedding from X_0 to X_1. I.e.,

$$
\begin{CD}
\kappa_m @>{b_0 \restriction \kappa_m}>> b_0[\kappa_m] \\
@V{t}VV @VV{t_*}V \\
\kappa_{m+1} @>>{b_1 \restriction \kappa_{m+1}}> X_1
\end{CD}
$$

By ordering T under end-extensions, we can turn T into a tree of height at most ω.

Proof continued:

Since $j \upharpoonright X_0 : X_0 \longrightarrow X_1$ is an elementary embedding, we can conclude that $j \upharpoonright \kappa_m \in T$ for all $m < \omega$.

This shows that T has a cofinal branch in V and hence it has a cofinal branch B in M. Then $\bigcup B$ is a function from λ to λ and, if we define

$$i = b_1 \circ \left(\bigcup B \right) \circ b_0^{-1} : X_0 \longrightarrow X_1$$

then, in M, i is an elementary embedding of X_0 into H_θ with $j(A) \in \mathrm{ran}(i)$, $i \upharpoonright \kappa_0 = \mathrm{id}_{\kappa_0}$ and $i(\kappa_m) = \kappa_{m+1}$ for all $m < \omega$. This contradicts our earlier conclusions. $\square$

Sequential *ESR* beyond Choice

Definition: (ZF)

A cardinal κ is **Reinhardt** if it is the critical point of an elementary embedding $j : V \to V$.

Theorem: (ZF)

If κ is a Reinhardt cardinal, then there exists a strictly increasing sequence $\vec{\lambda} = \langle \lambda_i \mid i < \omega \rangle$ of cardinals such that $ESR_{\mathcal{C}}(\vec{\lambda})$ holds for all classes $\mathcal{C}$ that are definable with parameters in V_κ.

Proof:

Let $j : V \to V$ be an elementary embedding with critical point κ. Let $\vec{\lambda} = \langle \lambda_i : i < \omega \rangle$ be the critical sequence. Thus, $\lambda_0 = \kappa$.

Fix a formula $\varphi(v_0, v_1)$ and $z \in V_\kappa$ such that the class $\mathcal{C} = \{A \mid \varphi(A, z)\}$ consists of $\mathcal{L}$-structures.

Pick a structure A in $\mathcal{C}$ of type $\langle \lambda_{i+1} \mid i < \omega \rangle$.

Proof continued:

Then the elementarity of j implies that $\varphi(j(A), z)$ holds.

Thus, $j(A)$ is an $\mathcal{L}$-structure of type $j(\langle \lambda_{i+1} \mid i < \omega \rangle)$ and the restriction map $j \restriction A : A \to j(A)$ is an elementary embedding of structures in $\mathcal{C}$.

Since we have $j(\vec{\lambda}) = \langle \lambda_{i+1} \mid i < \omega \rangle$, the elementarity of j yields that there is an $\mathcal{L}$-structure B of type $\vec{\lambda}$ with the property that $\varphi(B, z)$ holds and there exists an elementary embedding $j : B \to A$.

This shows that $ESR_{\mathcal{C}}(\vec{\lambda})$ holds. $\quad\square$

Definition: (ZF)

An ordinal δ is a **proto-Berkeley cardinal** if for all transitive sets M with $\delta \in M$, there exists a non-trivial elementary embedding $j : M \to M$ with crit$j < \delta$.

An ordinal δ is a **Berkeley cardinal** if for all transitive sets M with $\delta \in M$, for every $\eta < \delta$ there exists a non-trivial elementary embedding $j : M \to M$ with $\eta < \mathrm{crit}j < \delta$.

Theorem: (ZF)

If δ is the least Berkeley cardinal, then there exists a strictly increasing sequence $\vec{\lambda} = \langle \lambda_i \mid i < \omega \rangle$ of cardinals, with supremum less than δ, such that $ESR_{\mathcal{C}}(\vec{\lambda})$ holds for every class $\mathcal{C}$ of $\mathcal{L}$-structures that is definable with parameters.

Proof:

Let $\mathcal{C}$ be Σ_n-definable with parameter z.

Pick a cardinal $\theta > \delta$ in $C^{(n)}$, with $z \in V_\theta$, θ large enough. There exists a non-trivial elementary embedding $j : V_\theta \to V_\theta$ with $\mathrm{crit}(j) < \delta$, $j(\delta) = \delta$ and $j(z) = z$.

Let $\vec{\lambda} = \langle \lambda_i : i < \omega \rangle$ be the critical sequence. Since we picked θ large enough, we may assume that the supremum of the sequence is less than δ.

Fix a Σ_n-formula $\varphi(v_0, v_1)$ such that $\mathcal{C} = \{A \mid \varphi(A, z)\}$.

Pick a structure A in $\mathcal{C} \cap V_\theta$ of type $\langle \lambda_{i+1} \mid i < \omega \rangle$.

Proof continued:

We have that $\varphi(A, z)$ holds in V, and therefore also in V_θ. Then the elementarity of j implies that $\varphi(j(A), z)$ holds in V_θ too.

Thus, $j(A)$ is an $\mathcal{L}$-structure of type $j(\langle \lambda_{i+1} \mid i < \omega \rangle)$ and the restriction map $j \restriction A : A \to j(A)$ is an elementary embedding of $\mathcal{L}$-structures that is an element of V_θ.

Since we have $j(\vec{\lambda}) = \langle \lambda_{i+1} \mid i < \omega \rangle$, the elementarity of j now allows us to conclude that, in V_θ, there is an $\mathcal{L}$-structure B of type $\vec{\lambda}$ with the property that $\varphi(B, z)$ holds and there exists an elementary embedding $j : B \to A$.

This shows that $ESR_{\mathcal{C}}(\vec{\lambda})$ holds in V_θ. But since $\theta \in C^{(n)}$, it holds also in V. $\quad \square$

Corollary: (ZF)

Let δ be the least Berkeley cardinal. Given $\eta < \delta$, there exists a strictly increasing sequence $\vec{\lambda} = \langle \lambda_i \mid i < \omega \rangle$ of cardinals greater than η and with supremum less than δ such that $ESR_{\mathcal{C}}(\vec{\lambda})$ holds for every class $\mathcal{C}$ of $\mathcal{L}$-structures that is definable with parameters.

Proposition

Suppose that for every class $\mathcal{C}$ of structures in the language $\{\in, \dot{P}\}$ that is definable by a Σ_0-formula, with parameters, there exists an ordinal $\lambda < \delta$ with the property that for every structure A in $\mathcal{C}$ with $\mathrm{rank}(\dot{P}^A) = \lambda$, there exists a structure B in $\mathcal{C}$ with $\mathrm{rank}(\dot{P}^B) < \lambda$ and an elementary embedding of B into A. Then δ is a proto-Berkeley cardinal.

Proof:

Fix a transitive set M with $\delta \in M$. Define $\mathcal{C}$ to be the class of all $\mathcal{L}_{\in,\dot{P}}$-structures A with domain M such that $\dot{P}^A$ is an ordinal in M. Then $\mathcal{C}$ is definable by a Σ_0-formula with parameter M.

By our assumption, there exists an ordinal $\lambda < \delta$ with the property that for every A in $\mathcal{C}$ with $\text{rank}(\dot{P}^A) = \lambda$, there exists a structure B in $\mathcal{C}$ with $\text{rank}(\dot{P}^B) < \lambda$ and an elementary embedding of B into A.

Let A denote the unique structure in $\mathcal{C}$ with $\dot{P}^A = \lambda$. Then there exists a structure $B \in \mathcal{C}$ such that such that $\dot{P}^B = \eta$ with $\eta < \lambda$ and an elementary embedding $j : B \to A$.

But then $\eta \in \dot{P}^A \setminus \dot{P}^B$ and therefore the elementarity of j implies that $\eta < \lambda \leqslant j(\eta) \in M \cap \text{Ord}$. Thus, j is a non-trivial elementary embedding with critical point less than δ. $\quad\square$

Corollary

The following are equivalent:

1. *$ESR_{\mathcal{C}}(\vec{\lambda})$ holds for some increasing sequence $\vec{\lambda} = \langle \lambda_i : i < \omega \rangle$ of cardinals, for all Σ_0-definable, with parameters, classes $\mathcal{C}$ of structures of type $\langle \dot{P}_i : i < \omega \rangle$.*

2. *There exists a Berkeley cardinal.*