The dominating number and the unbounded number for singular cardinals

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July 6, 2022

- This is joint work with H.Sakai.

Theorem 1(H., Sakai).

If λ is a strong limit singular cardinal, then $\mathfrak{d}_\lambda = 2^\lambda$.

Theorem 2(H., Sakai)(Done).

$\mathfrak{b}_\lambda = \mathrm{cf}\,(\lambda)^+$.

Theorem 3(H., Sakai).

$\mathfrak{d}_{\aleph_\omega} \geq \max \mathrm{pcf}\,(\{\aleph_n \mid n < \omega\})$.

Today, we will prove Theorem 1 and Theorem 3.

Each proof consists of two steps.

Step

1. We show that if there is a "good" unbounded family $\mathcal{P} \subseteq {}^{\lambda}(\kappa^+)$, then $\mathfrak{d}_\lambda$ is large.
2. We construct the "good" unbounded family $\mathcal{P} \subseteq {}^{\lambda}(\kappa^+)$.

Last week, we constructed the map $f_\bullet \colon {}^{\lambda}(\kappa^+) \longrightarrow {}^{\lambda}\lambda$ and showed the following lemma.

Definition($\Psi(\mu)$).

Let $\mathcal{P} \subseteq {}^{\lambda}(\kappa^+)$. $\mathcal{P}$ has the property $\Psi(\mu)$ if $\mu = |\mathcal{P}| \in (\kappa, \lambda)$ is a regular cardinal and $\{\alpha < \lambda \mid |\{p(\alpha) \mid p \in \mathcal{Q}\}| \geq \kappa\}$ has the size λ for all $\mathcal{Q} \in [\mathcal{P}]^{\mu}$.

Lemma 1.

If $\mathcal{P}$ has $\Psi(\mu)$, then $\{f_p \mid p \in \mathcal{P}\}$ is an unbounded family.

In this talk, let λ be a singular cardinal and κ be its cofinality. Also, fix an increasing cofinal sequence $\langle \lambda_i \mid i < \kappa \rangle$ in λ.
Today, we will prove the stronger form of Theorem 1.

Theorem 4.

If $\mu^\kappa < \lambda$ for all $\mu < \lambda$, then $\mathfrak{d}_\lambda \geq \lambda^\kappa$.

If λ is a strong limit, then $\lambda^\kappa = 2^\lambda$. Hence it Theorem 1 follows from Theorem 4.
In this section, we assume that $\mu^\kappa < \lambda$ for all $\mu < \lambda$.

Proposition 4.

Let $\mathcal{P} \subseteq {}^{\lambda}(\kappa^+)$. Assume that $\mathcal{P}$ satisfies

- $|\mathcal{P}| = \lambda^\kappa$ and
- $\{f_p \mid p \in \mathcal{Q}\}$ is unbounded for all $\mathcal{Q} \in [\mathcal{P}]^{(2^\kappa)^+}$.

Then $\mathfrak{d}_\lambda \geq \lambda^\kappa$.

Proof.

Let $\mathcal{G} = \{g_\xi \mid \xi < \mathfrak{d}_\lambda\} \subseteq {}^{\lambda}\lambda$ be a dominating family. Let $\mathcal{F}_\xi$ be $\{f_p \mid p \in \mathcal{P}, f_p <^* g_\xi\}$ for all $\xi < \mathfrak{d}_\lambda$.
Then $\bigcup_{\xi<\mathfrak{d}_\lambda} \mathcal{F}_\xi = \{f_p \mid p \in \mathcal{P}\}$.
On the other hand, $\mathcal{F}_\xi$ is a bounded family. Hence, $|\mathcal{F}_\xi| < (2^\kappa)^+$ by the assumption. Thus,

$$\lambda^\kappa = |\mathcal{P}| = |\{f_p \mid p \in \mathcal{P}\}| = \left|\bigcup_{\xi<\mathfrak{d}_\lambda} \mathcal{F}_\xi\right| \leq \mathfrak{d}_\lambda \cdot (2^\kappa)^+ = \mathfrak{d}_\lambda.$$

□

Let us construct $\mathcal{P} \subseteq {}^{\lambda}(\kappa^+)$ as in Proposition 4.
We will define a function $p_\bullet : \prod_{i<\kappa} \lambda_i \longrightarrow {}^{\lambda}(\kappa^+)$ and let $\mathcal{P} = \{p_x \mid x \in \prod_{i<\kappa} \lambda_i\}$.
For each $j < \kappa$, let $T_j = \prod_{i<j} \lambda_i$ and $\mathcal{H}_j = [T_j]^{\kappa}$. Note that $|T_j|, |\mathcal{H}_j| < \lambda$ since $\mu^\kappa < \lambda$ for all $\mu < \lambda$.
Let $T = \bigcup_{i<\kappa} T_i$ and $\mathcal{H} = \bigcup_{i<\kappa} \mathcal{H}_i$.
Enumerate $\mathcal{H} = \{H_\alpha \mid \alpha < \lambda\}$ such that each member of $\mathcal{H}$ appears λ-many times.
Now, we will construct $\varphi : \bigcup \{T_i \times [\lambda_i, \lambda_{i+1}) \mid i < \kappa\} \longrightarrow \kappa^+$ and define $p_\bullet$ by

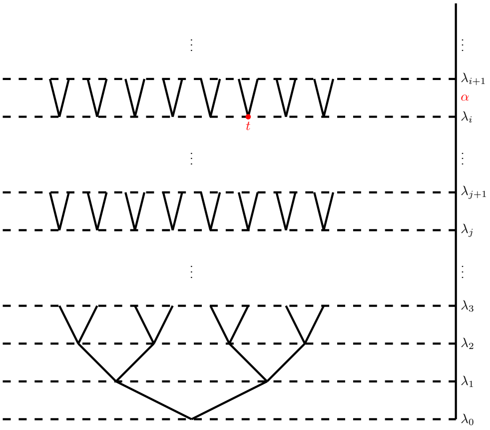
$$p_x(\alpha) = \varphi(x \restriction i, \alpha)$$

where $\alpha \in [\lambda_i, \lambda_{i+1})$.

Let $t \in T_i$ and $\alpha \in [\lambda_i, \lambda_{i+1})$. Let $j < \kappa$ be such that $H_\alpha \in \mathcal{H}_j$. Now, if $i < j$, then we define $\varphi(t, \alpha) = 0$. If $i \geq j$, then we define φ by

$$\varphi(t, \alpha) = \begin{cases} k & (\exists k < \kappa\, [t \restriction j = t_k]) \\ 0 & (otherwise) \end{cases}$$

where $H_\alpha = \{t_k \mid k < \kappa\}$ is an injective enumeration.



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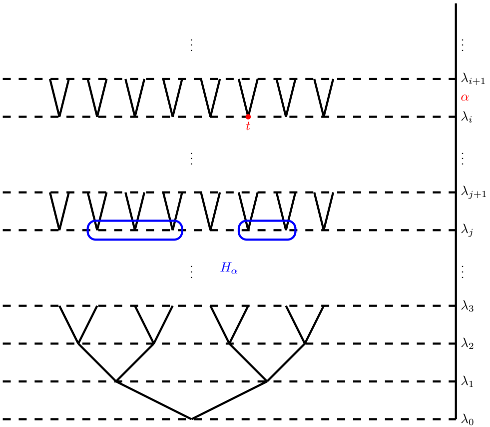

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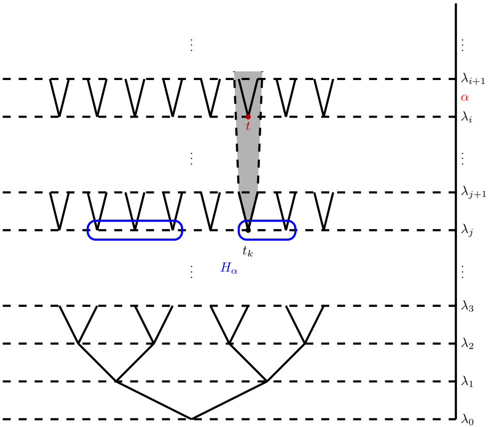


In order to prove that $\mathcal{P} = \left\{ p_x \mid x \in \prod_{i<\kappa} \lambda_i \right\}$ satisfies the assumptions of Theorem 4, we will show some lemma.

Lemma 2.

$p_\bullet$ is injective.

Proof.

Let $x, y \in [T_j]^\kappa$ and $x \neq y$. We will show that $p_x \neq p_y$. There is $j < \kappa$ such that $x \upharpoonright j \neq y \upharpoonright j$. Take $H \in \mathcal{H}_j$ such that $x \upharpoonright j, y \upharpoonright j \in H$. By the choice of $\langle H_\alpha \mid \alpha < \lambda \rangle$, there is some $\alpha < \lambda$ such that $\alpha \geq \lambda_j$ and $H = H_\alpha$. Take $i < \kappa$ with $\alpha \in [\lambda_i, \lambda_{i+1})$. Thus,

$$p_x(\alpha) = \varphi(x \upharpoonright i, \alpha) \neq \varphi(y \upharpoonright i, \alpha) = p_y(\alpha)$$

since $(x \upharpoonright i) \upharpoonright j = x \upharpoonright j \neq y \upharpoonright j = (x \upharpoonright i) \upharpoonright j$. □

Lemma 3.

If $\mathcal{X} \subseteq T$ has the size $(2^\kappa)^+$, then there exists some $i < \kappa$ such that $\{x \upharpoonright i \mid x \in \mathcal{X}\}$ has the size κ^+.

Proof.

Suppose, for a contradiction, that $S_i = \{x \upharpoonright i \mid x \in \mathcal{X}\}$ has the size at most κ for all $i < \kappa$. Then $S = \bigcup \{S_i \mid i < \kappa\}$ also has the size at most κ.
Define a function σ from $\mathcal{X}$ to $[S]^\kappa$ by

$$\sigma(x) = \{x \upharpoonright i \mid i < \kappa\}.$$

Then σ is an injection since $\bigcup \sigma(x) = x$. Therefore $(2^\kappa)^+ = |\mathcal{X}| \leq |[S]^\kappa| \leq 2^\kappa$. This is a contradiction. $\square$

Proposition 5.

$\mathcal{P} = \{p_x \mid x \in T\}$ satisfies the assumptions in Proposition 4.

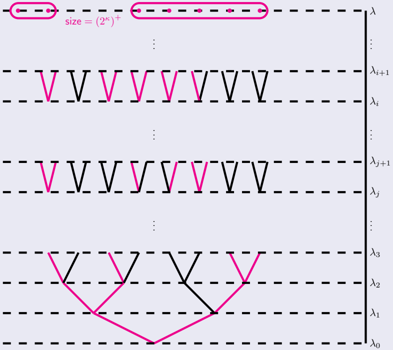
Proof.

By Lemma 2, $\mathcal{P}$ has the size $\left|\prod_{i<\kappa} \lambda_i\right| = \lambda^\kappa$. We will show that $\{f_p \mid p \in \mathcal{Q}\}$ is unbounded for all $\mathcal{Q} \in [\mathcal{P}]^{(2^\kappa)^+}$.

By Lemma 1, it suffices to show that $\mathcal{Q}$ has $\Psi\left((2^\kappa)^+\right)$.

In order to prove this, let $\mathcal{R} \in [\mathcal{Q}]^{(2^\kappa)^+}$.

Let $\mathcal{X} \in [T]^{(2^\kappa)^+}$ such that $\mathcal{R} = \{p_x \mid x \in \mathcal{X}\}$. Then we can find $j < \kappa$ such that $\{x \restriction j \mid x \in \mathcal{X}\}$ has the size κ^+ by Lemma 3. In particular, there is $\mathcal{Y} \subseteq \mathcal{X}$ such that $H = \{x \restriction j \mid x \in \mathcal{Y}\}$ has the size κ.



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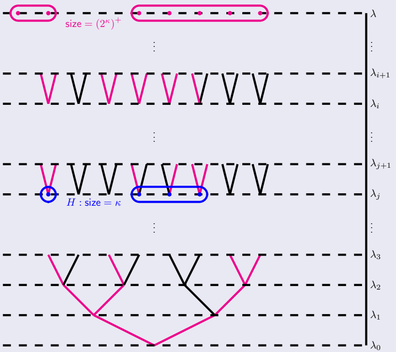
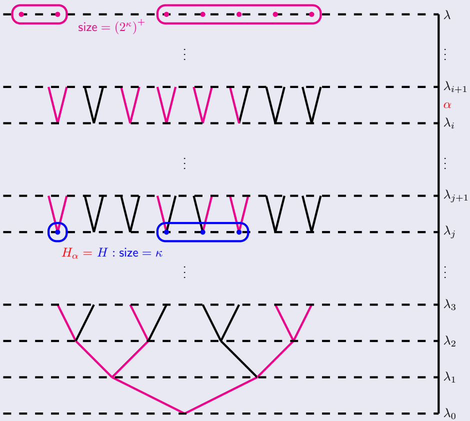


Proof.

Now, by the choice of $\langle H_\alpha \mid \alpha < \lambda \rangle$,
$S = \{\alpha < \lambda \mid H_\alpha = H, \alpha \geq \lambda_j\}$ has the size λ.
Let $\alpha \in S$ and $\alpha \in [\lambda_i, \lambda_{i+1})$. Thus,

$$\begin{aligned} \{p_x(\alpha) \mid x \in \mathcal{X}\} &\supseteq \{p_x(\alpha) \mid x \in \mathcal{Y}\} \\ &= \{\varphi(x \restriction i, \alpha) \mid x \in \mathcal{Y}\} = \kappa. \end{aligned}$$

Hence, $\{p_x \mid x \in \mathcal{X}\}$ satisfies $\Psi\left((2^\kappa)^+\right)$.



$\square$

Pcf theory is a theory introduced by Saharon Shelah. It has many applications about combinatorics for singular cardinals.

Definition

Let A be a set of ordinals. Let $\prod A = \{f : A \longrightarrow \mathrm{On} \mid \forall a \in A(f(a) \in a)\}$. Let F be a filter over A and $f, g \in \prod A$. We define $f \leq_F g$, $\mathrm{cf}\,(\prod A, I)$, and $\mathrm{pcf}\,(A)$ by

$$f <_F g \Longleftrightarrow \{a \in A \mid f(a) < g(a)\} \in F$$
$$\mathrm{cf}\,(\prod A, F) = \min\{|X| | \forall f \in \prod A, \exists g \in X(f \leq_F g)\}$$
$$\mathrm{pcf}\,(A) = \{\mathrm{cf}\,(\prod A, D) \mid D \text{ is an ultrafilter over } A\}$$

"pcf" means the **p**ossible **cof**inality.

Definition.

Let A be a set of ordinals. Suppose that F is a filter over A and $\langle f_\xi \mid \xi \in L \rangle$ is a $<_F$ -increasing sequence, where L is a set of ordinals. Then f is said to be strongly $<_F$-increasing if there are sets $S_\xi \in F$ for all $\xi \in L$ such that

$$\xi, \zeta \in L, \xi < \zeta \longrightarrow \forall a \in S_\xi \cap S_\zeta \left(f_\xi(a) < f_\zeta(a) \right).$$

Definition.

Suppose A is a set of ordinals and I is an ideal over A. Let μ be a regular cardinal and $\langle f_\xi \mid \xi < \mu \rangle$ be a $<_I$ -increasing sequence.
We say that $\langle f_\xi \mid \xi < \mu \rangle$ satisfies $(*)_\nu$ for a regular cardinal $\nu \leq \mu$ if for all unbounded $X \subseteq \mu$, there is $X_0 \subseteq X$ such that

- $\mathrm{otp}(X_0) = \nu$ and
- $\langle f_\xi \mid \xi \in X_0 \rangle$ is strongly increasing.

In this talk, we consider $\mathrm{pcf}\left(\{\aleph_n \mid n < \omega\}\right)$. Next facts are important for the proof of Theorem 3. They are proved in [1].

Fact(Shelah).

There exists $\mathrm{mp}\left(\aleph_\omega\right)$.

Hence, we can find an ultrafilter D over ω such that $\mathrm{cf}\left(\prod_{n\in\omega}\aleph_n, D\right) = \max\mathrm{pcf}\left(\{\aleph_n \mid n < \omega\}\right)$ and fix it. In the rest of this talk, we denote $\max\mathrm{pcf}\left(\{\aleph_n \mid n < \omega\}\right)$ by $\mathrm{mp}\left(\aleph_\omega\right)$. By the definition, $\mathrm{mp}\left(\aleph_\omega\right)$ is a regular cardinal.

Fact(Shelah).

There exists a $<_D$-cofinal $<_D$-increasing sequence $\langle f_i \mid i < \mathrm{mp}\left(\aleph_\omega\right)\rangle$ in $\prod_{n<\omega}\aleph_n$ satisfying $(*)_{\omega_1}$.

First, we will give a sufficient condition for that $\mathfrak{d}_{\aleph_\omega} \geq \mathrm{mp}\left(\aleph_\omega\right)$.

Proposition 6.

Let $\mathcal{P} \subseteq {}^{\aleph_\omega}\omega_1$. Assume that $\mathcal{P}$ satisfies

- $|\mathcal{P}| = \mathrm{mp}\left(\aleph_\omega\right)$ and
- $\{f_p \mid p \in \mathcal{Q}\}$ is unbounded for all $\mathcal{Q} \in [\mathcal{P}]^{\mathrm{mp}(\aleph_\omega)}$.

Then $\mathfrak{d}_{\aleph_\omega} \geq \mathrm{mp}\left(\aleph_\omega\right)$.

Proof.

Let $\mathcal{G} = \{g_\xi \mid \xi < \mathfrak{d}_{\aleph_\omega}\} \subseteq {}^{\aleph_\omega}\aleph_\omega$ be a dominating family. Suppose, for a contradiction, that $\mathfrak{d}_{\aleph_\omega} < \max\mathrm{pcf}\left(\{\aleph_n \mid n < \omega\}\right)$. Let $\mathcal{F}_\xi = \{f_p \mid p \in \mathcal{P}, f_p <^* g_\xi\}$ for all $\xi < \mathfrak{d}_{\aleph_\omega}$. Then $\bigcup_{\xi < \mathfrak{d}_{\aleph_\omega}} \mathcal{F}_\xi = \{f_p \mid p \in \mathcal{P}\}$.

By the assumption, regularity of $\mathrm{mp}\left(\aleph_\omega\right)$, and the pigeonhole principle, there exists $\xi < \mathfrak{d}_{\aleph_\omega}$ such that $|\mathcal{F}_\xi| = \mathrm{mp}\left(\aleph_\omega\right)$. On the other hand, $\mathcal{F}_\xi$ is an unbounded family by the assumption. This is a contradiction. $\square$

We would like to construct $p_\bullet$ as we did in section 2.

We will define a function $p_\bullet : \prod_{n<\omega} \aleph_n \longrightarrow {}^{\aleph_\omega}\omega_1$ and let $\mathcal{P} = \left\{p_x \mid x \in \prod_{n<\omega} \aleph_n\right\}$.
For each $m < \omega$, let $T_m = \prod_{n<m} \aleph_n$ and $\mathcal{H}_m = [T_m]^\omega$. Note that $|T_m|, |\mathcal{H}_m| < \aleph_\omega$ since $\mu^\omega < \aleph_\omega$ for all $\mu < \aleph_\omega$. Let $T = \bigcup_{n<\omega} T_n$ and $\mathcal{H} = \bigcup_{n<\omega} \mathcal{H}_n$.
Enumerate $\mathcal{H} = \{H_\alpha \mid \alpha < \aleph_\omega\}$ such that each member of $\mathcal{H}$ appears $\aleph_\omega$-many times.
Now, we will construct $\varphi : \bigcup \{T_n \times [\aleph_n, \aleph_{n+1}) \mid n < \omega\} \longrightarrow \omega_1$ and define $p_\bullet$ by

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where $\alpha \in [\aleph_n, \aleph_{n+1})$.$\cdots$

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Enumerate $\mathcal{H} = \{H_\alpha \mid \alpha < \aleph_\omega\}$ such that each member of $\mathcal{H}$ appears $\aleph_\omega$-many times.
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However, it should be noted that we do not have the exponential assumptions we had in section 2. In particular, we need to modify this parts.

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For each $m < \omega$, let $T_m = \prod_{n<m} \aleph_n$ and $\mathcal{H}_m$ be a cofinal set in $\left([T_m]^\omega, \subseteq\right)$. Note that $|T_m|, |\mathcal{H}_m| < \aleph_\omega$. Let $T = \bigcup_{n<\omega} T_n$ and $\mathcal{H} = \bigcup_{n<\omega} \mathcal{H}_n$.

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However, it should be noted that we do not have the exponential assumptions we had in section 2. In particular, we need to modify this parts.

Lemma 4.

$\mathrm{cf}\left([\aleph_n]^{\aleph_0}, \subseteq\right) = \aleph_n$ for all $n \geq 1$.

Proof.

It suffices to show that $\mathrm{cf}\left([\aleph_{n+1}]^{\aleph_0}, \subseteq\right) \leq \mathrm{cf}\left([\aleph_n]^{\aleph_0}, \subseteq\right) \cdot \aleph_{n+1}$. This is because $[\aleph_0, \aleph_1)$ is a cofinal set in $\mathrm{cf}\left([\aleph_1]^{\aleph_0}, \subseteq\right)$ and thus

$$\begin{aligned}\mathrm{cf}\left([\aleph_n]^{\aleph_0}, \subseteq\right) &\leq \mathrm{cf}\left([\aleph_{n-1}]^{\aleph_0}, \subseteq\right) \cdot \aleph_n \\ &\leq \cdots \leq \mathrm{cf}\left([\aleph_1]^{\aleph_0}, \subseteq\right) \cdot \aleph_2 \cdots \aleph_n = \aleph_n.\end{aligned}$$

Also, the reverse inequality is clear.

Proof.

Now, let X be a cofinal set in $([\aleph_n]^{\aleph_0}, \subseteq)$ and we construct a cofinal set in $([\aleph_{n+1}]^{\aleph_0}, \subseteq)$. For each $\gamma \in [\aleph_n, \aleph_{n+1})$, let π_γ be a bijective function from γ to $\aleph_n$. We denote $\bigcup_{\gamma \in [\aleph_n, \aleph_{n+1})} \{\pi_\gamma^{-1} B \mid B \in X\}$ by Y. Then $|Y| \leq |X| \cdot \aleph_{n+1}$ We show that Y is cofinal in $([\aleph_{n+1}]^{\aleph_0}, \subseteq)$.

Let $A \in [\aleph_{n+1}]^{\aleph_0}$, then there exists $\gamma < \aleph_{n+1}$ such that $A \subseteq \gamma$. Denote π_γ"A by A'. Since X is a cofinal set, there is some $B \in X$ such that $A' \subseteq B$. Hence $A \subseteq \pi_\gamma^{-1} B$. □

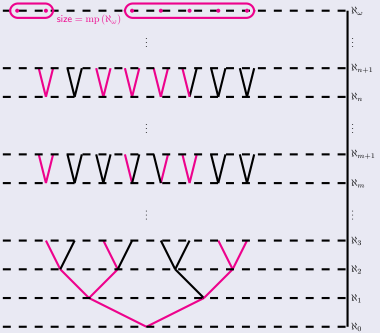
Proposition 7.

Let $\mathcal{X}$ be a $<_D$-increasing $<_D$-cofinal sequence in $\prod_{n<\omega} \aleph_n$ atisfying $(*)_{\omega_1}$. Then $\mathcal{P} = \{p_x \mid x \in \mathcal{X}\}$ satisfies the assumptions in Proposition 6.

Proof.

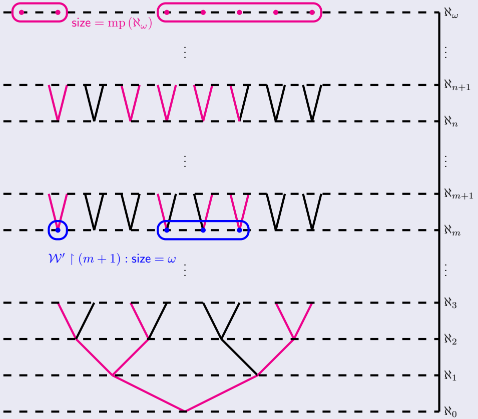
This proof is similar to Proposition 5.
By Lemma 2, $\mathcal{P}$ has the size $\mathrm{mp}(\aleph_\omega)$. We will show that $\{f_p \mid p \in \mathcal{Q}\}$ is unbounded for all $\mathcal{Q} \in [\mathcal{P}]^{\mathrm{mp}(\aleph_\omega)}$.
Let $\mathcal{Y} \in [\mathcal{X}]^{\max \mathrm{pcf}(\{\aleph_n | n<\omega\})}$. Since $\mathcal{X}$ satisfies $(*)_{\omega_1}$, we can find a strongly increasing sequence $\mathcal{Z} = \langle x_\xi \mid \xi < \omega_1 \rangle$ in $\mathcal{Y}$. Thus, We can take $S_\xi \in D$ such that

$$\xi < \zeta < \omega_1 \longrightarrow \forall n \in S_\xi \cap S_\zeta \left(x_\xi(n) < x_\zeta(n) \right).$$



Proof.

We will show that $\{p_x \mid x \in \mathcal{Z}\}$ holds $\Psi(\omega_1)$. In order to prove this, let $\mathcal{Z}' \in [\mathcal{Z}]^{\omega_1}$. Take $n_\xi \in S_\xi$ for each $\xi < \omega_1$. By the pigeon hole principle, there exists $m < \omega$ such that $\mathcal{W} = \{x_\xi \in \mathcal{Z}' \mid n_\xi = m\}$ has the size ω_1. Thus, $\{x_\xi(m) \mid x_\xi \in \mathcal{W}\}$ has the size ω_1 since if $x_\xi, x_\zeta \in \mathcal{W}$ and $\xi < \zeta$, then $x_\xi(m) < x_\zeta(m)$. Therefore, we can find $\mathcal{W}' \subseteq \mathcal{W}$ such that $\{x \restriction (m+1) \mid x \in \mathcal{W}'\}$ has the size ω. Take $H \in \mathcal{H}$ such that $\{x \restriction (m+1) \mid x \in \mathcal{W}'\} \subseteq H$.



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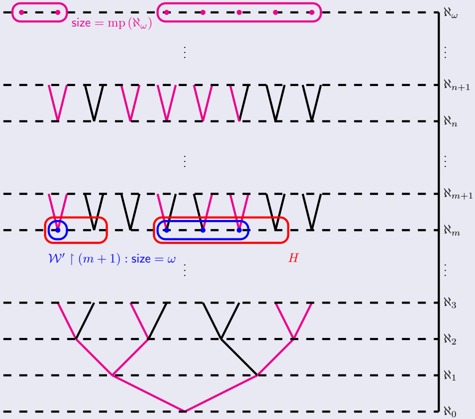
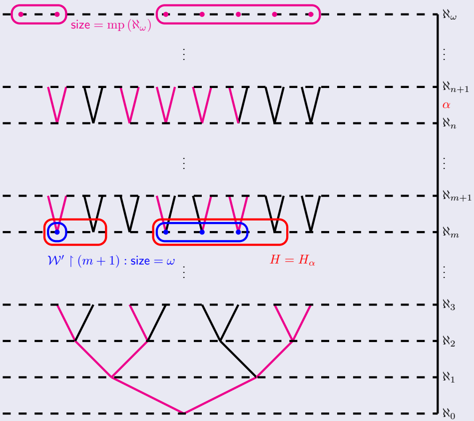


Proof.

Now, by the choice of $\langle H_\alpha \mid \alpha < \aleph_\omega \rangle$,
$S = \{\alpha < \aleph_\omega \mid H_\alpha = H, \alpha \geq \aleph_{m+1}\}$ has the size $\aleph_\omega$.
Let $\alpha \in S$ and $\alpha \in [\aleph_n, \aleph_{n+1})$. Thus,

$$\left|\{p_{x_\xi}(\alpha) \mid x_\xi \in \mathcal{Z}'\}\right| \geq \left|\{p_{x_\xi}(\alpha) \mid p_{x_\xi} \in \mathcal{W}'\}\right| = \left|\{\varphi(p_{x_\xi} \restriction n, \alpha) \mid p_{x_\xi} \in \mathcal{W}'\}\right| = \omega.$$

Then $\{f_{p_x} \mid x \in \mathcal{Z}\}$ is an unbounded family by Lemma 1. Hence, $\{f_{p_x} \mid x \in \mathcal{Y}\}$ is also an unbounded family and this completes the proof.



$\square$

Thank you for your attention!

Reference I

[1] Uri Abraham and Menachem Magidor.
Cardinal arithmetic.
In *Handbook of set theory*, pp. 1149–1227. Springer, 2010.