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Strongly SSP forcing

Talk based on joint work with B. Velickovic.

Let's recall the following definitions:

Def

▷ given an uncountable cardinal κ,
$\mathbb{P}$ is <u>κ-proper</u> if for every stationary $S \subseteq [\kappa]^\omega$:

$$\mathbb{P} \Vdash S \subseteq [\kappa]^\omega \text{ is stationary.}$$

▷ $\mathbb{P}$ is proper if $\mathbb{P}$ is κ-proper for every unctble card κ,

▷ $\mathbb{P}$ is SSP if $\mathbb{P}$ is ω_1-proper, i.e.
for every $S \subseteq \omega_1$ stationary,
$\mathbb{P} \Vdash S \subseteq \omega_1$ station.

Def Let $M \ni H_\theta$ ctble, $\mathbb{P} \in M$.

▷ $p^* \in \mathbb{P}$ is $(M, \mathbb{P})$-generic if

$$p^* \Vdash M[G] \cap V = M.$$

Eq.: $(\forall q \leq p^*)$

$(\forall D \subseteq \mathbb{P}$ predense with $D \in M)(\exists r \in M \cap D)\ r \| q.$

▷ $p^* \in \mathbb{P}$ is $(M, \mathbb{P})$-semigeneric if

$$p^* \Vdash M[G] \cap \omega_1 = M \cap \omega_1.$$

Eq.: $(\forall q \leq p^*)$

$(\forall D \subseteq \mathbb{P}$ predense with $D \in M)(\exists r \in D)$

$r \| q$ and $r \|_{\omega_1} M.$

$x \|_{\omega_1} M$ means:

$$\operatorname{Hull}(M, x) \cap \omega_1 = M \cap \omega_1$$

$\operatorname{Hull}(M, x) := \{ f(x) : f \in M$ is a function $x \in \operatorname{dom}(f)\}.$

Def

▷ $\mathbb{P}$ is proper for M if

$(\forall p \in \mathbb{P} \cap M)(\exists p^* \leq p)\ p^* : (M, \mathbb{P})$-gen

$(\forall p \in \mathbb{P} \cap M)(\exists p^* \leq p)\ p^*$ is $(M, \mathbb{P})$ gen.

▷ $\mathbb{P}$ is semiproper for M if

$$(\forall p \in \mathbb{P} \cap M)(\exists p^* \leq p)\ p^* \text{ is } (M, \mathbb{P})\text{-semigen.}$$

Fact

$\mathbb{P}$ is proper $\iff$ for club many $M \prec H_\theta$, $\mathbb{P}$ is proper for M.

$\mathbb{P}$ is SP $\iff$ for ??? many $M \prec H_\theta$, $\mathbb{P}$ is semiproper for M.

proj stat.

$\mathbb{P}$ is semiproper $:\iff$ for club many $M \prec H_\theta$, $\mathbb{P}$ semiproper for M.

Def

$\Sigma \subseteq [H_\theta]^\omega$ is projective stat if

for every stat $T \subseteq \omega_1$,

the set $\{M \in \Sigma : M \cap \omega_1 \in T\}$ is stat in $[H_\theta]^\omega$.

An important example

→ shooting an ω_1-chain with finite conditions in X:

let X be a set of ctble elementary submodels of H_θ,

a $\mathbb{P}_X$-condition is a finite set p such that

- every element of p is a pair (M, d_M) with $M \in X$ and $d_M \subseteq H_\theta$ finite,
- $(\forall (M, d_M), (N, d_N) \in p)\ M \cap \omega_1 = N \cap \omega_1 \Rightarrow (M, d_M) = (N, d_N)$,
- $(\forall (M, d_M), (N, d_N) \in p)\ M \cap \omega_1 < N \cap \omega_1 \Rightarrow (M, d_M) \in N$,

and $q \leq p$ if

$$(\forall (M, d_M) \in p)\ (\exists (N, d_N) \in q)\ N = M \text{ and } d_M \subseteq d_N.$$

$$\mathbb{P}_X \ni p = \{(M_0, d_0), (M_1, d_1), (M_2, d_2)\}$$

$q \leq p$

$$q = \{(M_0, d_0'), (M_{\frac{1}{2}}, d_{\frac{1}{2}}), (M_1, d_1), (M_2, d_2')\}$$

→ $\mathbb{P}_X$ is proper for every $N \ni H_\nu$ s.t. $N \cap H_\theta \in X$.
$(\nu >> \theta)$.

(So: if X contains a club, $\mathbb{P}_X$ is proper.
if X is proj stat, $\mathbb{P}_X$ is SSP.)

$\mathbb{P}_X$ is even strongly proper:

Def (Mitchell)

Let $\mathbb{P}$ be a forcing notion with $\mathbb{P} \in M \ni H_\theta$ ctble.

(a) the condition $p^* \in \mathbb{P}$ is strongly $(M, \mathbb{P})$-generic if for every $q \leq p^*$ and every predense subset D of $\mathbb{P} \cap M$ there exists $r \in D$ s.t. $r \| q$.

(b) $\mathbb{P}$ is strongly proper for M if for every $p \in M \cap \mathbb{P}$, there exists $p^* \leq p$ that is strongly $(M, \mathbb{P})$-gen.

(c) $\mathbb{P}$ is strongly proper if for every regular $\theta \gg \mathbb{P}$, there exist club many countable $M \ni H_\theta$ s.t. $\mathbb{P}$ is strongly proper for M.

Lemma For $p^* \in \mathbb{P}$, TFAE:

(a) p^* is strongly $(M, \mathbb{P})$-gen,

(b) for every $q \leq p^*$, there exists $\mathrm{tr}(q|M) \in \mathbb{P} \cap M$ s.t.

$(\forall r \in \mathbb{P} \cap M)\ r \leq \mathrm{tr}(q|M) \Rightarrow r \parallel q.$

Q: Is there a natural notion of "strongly $(M, \mathbb{P})$-semigen" and "strongly ssp"?

Def For $p^* \in \mathbb{P}$,

p^* is strongly $(M,\mathbb{P})$-semigen if

for every $q \leq p^*$, there exists $\underline{\pi(q|M) \in \mathbb{P}}$ s.t.

(i) $\underline{\pi(q|M) \parallel_{\omega_1} M}$,

(ii) $(\forall r \in \mathbb{P} \cap \underbrace{}_{\text{Hull}(M,\pi(q|M))})\ r \leq \pi(q|M) \Rightarrow r \parallel q.$

Def

- We call $\mathbb{P}$ strongly semiproper for M if

$(\forall p \in \mathbb{P} \cap M)(\exists p^* \leq p)$ s.t. p^* is strongly $(M,\mathbb{P})$-semi gen

- We call $\mathbb{P}$ strongly ~~ssp~~ semiproper if there exist projective ~~stat~~ club many countable $M \ni H_\theta$ s.t. $\mathbb{P}$ is strongly semiproper for M.

Note: these notions behave as expected:

p^* strongly $(M,\mathbb{P})$-semigen $\Rightarrow p^*$ is $(M,\mathbb{P})$-semigen

p^* strongly $(M, \mathbb{P})$-gen $\Rightarrow$ p^* is strongly $(M, \mathbb{P})$-semigen

suppose p^* is strongly $(M, \mathbb{P})$-semigen, let $q \le p^*$ and $D \in M$ $D \subseteq \mathbb{P}$ predense,

$$\left(\exists r \in \underbrace{\mathrm{Hull}(M, \mathrm{tr}(q \restriction M))}\cap D \right) \; r \parallel \mathrm{tr}(q \restriction M),$$

then $r \parallel q$ and $r \parallel_{\omega_1} M$.

Strongly ssp forcings share some of the strong properties of strongly proper forcings:

Proposition

(a) If $\mathbb{P}$ is strongly ssp, then $\mathbb{P}$ does not add any fresh functions $\omega_1^{\beta} \longrightarrow V$.

a function $f : \beta \to V$ in $V[G] \setminus V$ is fresh if $f \restriction \alpha$ belongs to V

(b) If $\mathbb{P}$ is strongly ssp:

(b) If $\mathbb{P}$ is strongly (...) $\forall \alpha < \beta$.

then $\mathbb{P}$ preserves ω_1-Souslin trees.

Pf of (a)

Let $\dot{f}$ be a $\mathbb{P}$-name and suppose $r \in \mathbb{P}$ s.t.

$$r \Vdash \dot{f} : \omega_1 \to V \text{ and } (\forall \alpha < \omega_1)\ \dot{f} \restriction \alpha \in V.$$

Pick $r^* \leq r$ which is strongly $(M, \mathbb{P})$-semigen

and

$q \leq r^*$, $g : M \cap \omega_1 \to V$ s.t.

$$q \Vdash \dot{f} \restriction M \cap \omega_1 = g.$$

Claim $tr(q|M)$ decides $\dot{f}$.

Suppose not, then pick $s_1, s_2 \in \mathbb{P} \cap \text{Hull}(M, tr(q|M))$

$\alpha \in M \cap \omega_1$ and $x \neq y$ s.t.

$s_1, s_2 \leq tr(q|M)$,

$s_1 \Vdash \dot{f}(\alpha) = x$, $s_2 \Vdash \dot{f}(\alpha) = y$.

$\lambda_1 \Vdash$ …

$\perp$

What applications of ssp forcing can be redone using strongly ssp forcing?

Some examples of useful ssp forcings:

ssp

Yes

▷ **Namba:** $\quad Nm_{\kappa} \Vdash cf(\kappa)=\omega \quad$ for $\kappa>\omega_1$ regular.

▷ **Jensen:** suppose κ is inaccessible and assume that (extended Namba problem) GCH holds below κ,

let $\underset{\sim}{K} \subseteq \,]\omega_1, \kappa[$ be a set of regular cardinals,

then there exists an ssp forcing $\mathbb{P}$ s.t.:

$$\mathbb{P} \Vdash \kappa = \omega_2^{V[G]} \text{ and } (\forall \lambda \in \underset{\sim}{K})\ cf(\lambda)=\omega$$

almost

and for every regular $\lambda \in \kappa \setminus \underset{\sim}{K}$: $\quad \mathbb{P} \Vdash cf(\kappa) = \omega_1.$

$\mathbb{P} \Vdash \mathrm{cf}(\lambda) = \omega_1.$

▷ Jensen; Cld-Sch; Ket-Far-Zan: given a reg card λ;

Suppose NS is precipitous, there exists an ssp forcing $\mathbb{P}$ s.t.:

$\mathbb{P}$ adds a generic iteration

$$It = (M_\beta, g_\alpha, j_{\alpha\beta} : \alpha < \beta \leq \omega_1)$$

of the ctble transitive model M_0 of length $\omega_1 + 1$) with $M_{\omega_1} = H_\lambda^V$.

If also $Y^\#$ exists for a sufficiently big set Y, then there exists an ssp forcing $\mathbb{P}$ s.t.

$$\mathbb{P} \Vdash \underset{\sim}{\delta}^1_2 \geq \lambda.$$

$\boxed{\text{Can we do all this using strongly ssp forcing?}}$

General idea for building such strongly ssp posets:

- the forcings are very similar to the forcings $\mathbb{P}_X$, for some projective stat X,

with the exception that models in a condition

- ... are now allowed to "grow" when the condition becomes stronger, as long as they don't gather new ordinals $< \omega_1$.

- to guide this "growth" of the models, we use certain games.

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suppose $(G(M) : M \ni H_\theta)$ is a family of closed games
 (where)
where P1 plays elements of X,

P2 plays elements of Y,

P1	x_0	x_1	x_2	...	x_n	...
P2	y_0	y_1	y_2	...	y_n	...

the rules can vary according to the application,

but if in round n, P2 has played $(y_0, \dots, y_n) = \bar{y}$

and $\mathrm{Hull}(M, \bar{y}) \cap \omega_1 > M \cap \omega_1$,

then P2 loses.

Notation: if $z = (\bar{x}, \bar{y}) \in X^n \times Y^n$ is a finite run of such
$p_2(z) := \bar{y}$.

game, write $P2(\Sigma)$ ...

Let X be the set of $M \ni H_\theta$ s.t. P2 wins $G(M)$.

$$r = \{(M_0, \vec{z}_0, \Sigma_0), (M_1, \vec{z}_1, \Sigma_1), (M_2, \vec{z}_2, \Sigma_2)\}$$

$(M_{\frac{1}{2}}, \vec{z}_{\frac{1}{2}}, \Sigma_{\frac{1}{2}})$

$$M_\alpha^\delta = \bigcup \{ Hull(M, P2(\vec{z})) : (M, \vec{z}, \Sigma) \in r, \ M \cap \omega_1 = \delta_\alpha \}$$

Consider then the following forcing $\mathbb{P}$,

a condition r is a finite set that satisfies the following conditions:

- every element of r is a triple $(M, \vec{z}, \Sigma)$ with

$M \in X$, Σ a winning strategy for P2 in the game $\mathcal{G}(\ldots)$ and z a finite run of the game $\mathcal{G}(M)$ in which P2 follows Σ.

- $(\forall (M_1, z_1, \Sigma_1), (M_2, z_2, \Sigma_2) \in \mathbb{P})$
  $M_1 \cap \omega_1 = M_2 \cap \omega_1 \Rightarrow (M_1, z_1, \Sigma_1) = (M_2, z_2, \Sigma_2)$
- $(\forall (M_1, z_1, \Sigma_1), (M_2, z_2, \Sigma_2) \in \mathbb{P})$
  $M_1 \cap \omega_1 < M_2 \cap \omega_1 \Rightarrow (M_1, z_1, \Sigma_1) \in \mathrm{Hull}(M_2, P2(z_2))$.

If $p, q \in \mathbb{P}$, then

$q \leq p \Leftrightarrow (\forall (M, z, \Sigma) \in p)(\exists (M', z', \Sigma') \in q)$ with $M = M'$, $\Sigma = \Sigma'$ and z' extends z.

If X is projective stationary, then the above $\mathbb{P}$ is strongly sp.

The precise behaviour of $\mathbb{P}$ depends of course on the game $\mathcal{G}$.

Example: the Namba game

X set of reg cardinals $\geq \omega_2$

$\in M$

Definition

$\mathbb{K}$

For every countable $M \ni H_\theta$, define the game $G^{\text{Namba}}(M, K)$:

- set $M_0 = M$,

$K_n \in \mathbb{K} \cap M_n$ and $V_n \in K_n$

- in round n,
  - $\to$ P1 plays $V_n \in K$,

    $\xi_n \in K_n \setminus V_n$
  - $\to$ P2 plays $\xi_n \in K \setminus V_n$,
  - $\to$ define $M_{n+1} = Hull(M_n, \xi_n)$.

The game has ω-many rounds and P2 wins iff

$$\forall n \ \xi_n \parallel_{\omega_1} M_n.$$

Proposition:

$\mathbb{K}$

The set of ctble $M \ni H_\theta$ for which P2 wins $G^{\text{Namba}}(M, K)$ is projective stat.

Corr

There exists a strongly ssp forcing $\mathbb{P}$ s.t.

$$\mathbb{P} \Vdash \mathrm{cf}(\kappa) = \omega$$

$$(\forall \kappa \in \mathcal{K})$$

Thanks!