On the structure of the blurry HOD hierarchy

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The starting point

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We want to weaken this notion.

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Definition (ZF)

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The set a is hereditarily $<\kappa$-blurrily ordinal definable, denoted $<\kappa$-HOD, iff $\mathrm{TC}(\{a\}) \subseteq {<\kappa}\text{-OD}$. Again, I write $<\kappa$-HOD for the class of all $<\kappa$-HOD sets.

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So an object is $<\kappa$-blurrily ordinal definable if it is one of fewer than κ objects having a property using ordinal parameters.

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The case $\kappa = 2$ is (hereditary) ordinal definability (Gödel, Myhill-Scott).

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The case $\kappa = \omega_1$ was recently proposed and coined (hereditary) "nontypicality" (Tzouvaras).

Three aspects

1. Structural ZF/ZFC results on blurry HOD

2. Interaction with forcing

3. Consistency strength

Basics

Proposition (ZF)

Let $2 \leq \kappa < \lambda$ be cardinals.

1. OD $\subseteq$ $<\kappa$-OD $\subseteq$ $<\lambda$-OD.
2. $<\kappa$-HOD *is transitive, and* HOD $\subseteq$ $<\kappa$-HOD $\subseteq$ $<\lambda$-HOD.
3. *Under* AC, $H_\kappa \subseteq$ $<(2^{<\kappa})^+$-HOD.
4. *So under* AC, V *is the increasing union* $\bigcup_{\kappa \in \mathrm{Card}} <\kappa$-HOD.

$<\kappa$-HOD is an inner model

Proposition (ZF)

Let $\kappa \geq \omega$ be a cardinal. Then $<\kappa$-HOD is an inner model.

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Proof.

Let $\kappa \geq \omega$. It suffices to show that $<\kappa$-HOD satisfies the following condition: for every $u \subseteq {<\kappa}$-HOD, there is a transitive $v \in {<\kappa}$-HOD such that $u \subseteq v$ and $\mathrm{Def}(\langle v, \in \rangle) \subseteq {<\kappa}$-HOD.

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So let $u \subseteq <\kappa$-HOD be given. Let $u \subseteq V_\alpha$, and set $v = V_\alpha \cap <\kappa$-HOD. Clearly, v is transitive, OD and contained in $<\kappa$-HOD, so $v \in <\kappa$-HOD, and $u \subseteq v$.

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So let $u \subseteq <\kappa$-HOD be given. Let $u \subseteq V_\alpha$, and set $v = V_\alpha \cap <\kappa$-HOD. Clearly, v is transitive, OD and contained in $<\kappa$-HOD, so $v \in <\kappa$-HOD, and $u \subseteq v$. To show that $\mathrm{Def}(\langle v, \in\rangle) \subseteq <\kappa$-HOD, let $\varphi(x, \vec{y})$ be a formula, and let $\vec{a} = a_0, \ldots, a_{n-1} \in v$. We have to show that $z = \{x \in v \mid \langle v, \in\rangle \models \varphi(x, \vec{a})\} \in <\kappa$-HOD. Since $z \subseteq v \subseteq <\kappa$-HOD, it suffices to show that z is $<\kappa$-OD. For each $i < n$, let A_i be an OD set containing a_i such that $\mathrm{card}(A_i) < \kappa$.

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Then z is in the set

$$B = \{\{x \in v \mid \langle v, \in\rangle \models \varphi(x, \vec{b})\} \mid \langle b_0, \ldots, b_{n-1}\rangle \in A_0 \times \ldots \times A_{n-1}\}.$$

B is OD, as $A_0, \ldots, A_{n-1}$ and v are, and $\mathrm{card}(B) \leq \mathrm{card}(A)_0 \cdot \ldots \cdot \mathrm{card}(A)_{n-1} < \kappa$. $\square$

Blurry choice

Theorem (ZF)

Let $\kappa > 1$ be a cardinal. Then, whenever $C \in {<}\kappa\text{-HOD}$ is a set consisting of nonempty sets, there is a function $f : C \longrightarrow ([\bigcup C]^{<\kappa})^{\mathrm{V}}$ such that $f \in {<}\kappa\text{-HOD}$, and such that for every $c \in C$, $\emptyset \neq f(c) \subseteq c$.

Blurry Codes

Fix a recursive enumeration $\langle \varphi_n \mid n < \omega \rangle$ of all internal first order formulas in the language of set theory with two free variables. Let $\mathrm{Sat}(x, y, z)$ be the satisfaction relation, so that for a set u, a natural number n and a pair $\langle a, b \rangle \in u^2$, $\mathrm{Sat}(u, n, \langle a, b \rangle)$ holds iff $\langle u, \in \cap\, u^2 \rangle \models \varphi_n(a, b)$.

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For a set a, let $\Sigma(a)$ be least such that there is an OD set A with $a \in A$ and $\mathrm{card}(A) = \Sigma(a)$ (exists assuming AC).

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Let $D(a)$ be the least ordinal of the form $\prec \alpha, n, \beta \succ$ such that the set

$$\{x \in \mathrm{V}_\alpha \mid \mathrm{Sat}(\mathrm{V}_\alpha, \varphi_n, \langle x, \beta \rangle)\}$$

has cardinality $\Sigma(a)$ and contains a.

Observation

We have the following facts about Σ and D.

1. *The functions Σ and D are ordinal definable. Assuming ZFC, both are total.*
2. *Let a be a set for which $\Sigma(a)$ is defined. Let $D(a) = \rho$. Then $D^{-1}(\rho) = \{x \mid D(x) = \rho\}$ is a blurry ordinal definition of a, and $\operatorname{card}(D^{-1}(\rho)) = \Sigma(a)$.*

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So every set $a \in {<}\lambda$-HOD has a "canonical" blurry definition $D^{-1}(\rho)$, which has cardinality $\Sigma(a) < \lambda$.

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Definition

Let $M \subseteq N$ be transitive class models of ZF, and let κ be a cardinal.

M satisfies the (external) κ-cover property in N if for every set $a \in N$ with $a \subseteq M$ and $\mathrm{card}(a) < \kappa$, there is a set $c \in M$ such that $a \subseteq c$ and $\mathrm{card}(c) < \kappa$.

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Let $a \in N$ be a set with $a \subseteq M$. A set of the form $a \cap c$, where $c \in M$ and $\mathrm{card}(c) < \kappa$, is called a κ-approximation to a in M. The set a is said to be κ-approximated in M if every κ-approximation to a in M belongs to M.

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Approximation and cover

Theorem

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Hence, $<\kappa$-HOD is an inner model also for $2 < \kappa < \omega$, and the fact about the approximation property holds for all cardinals $2 \leq \kappa < \lambda$.

A useful fact

Lemma (ZFC)

Let λ be an infinite cardinal, and let $M \subsetneq N$ be inner models such that M satisfies the λ-approximation-property in N.

1. *There is a set $a \in N \setminus M$ with $a \subseteq M$ and $\mathrm{card}(a) < \lambda$.*
2. *If M in addition satisfies the λ-cover property in N, then there are sets a, b such that $a \in N \setminus M$, $a \subseteq b \in M$ and $\mathrm{card}(b) < \lambda$.*

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Note: This situation arises in particular in the case where $M = {<}\kappa\text{-HOD} \subsetneq {<}\lambda\text{-HOD} = N$.

Consequences: Definability

Theorem

Let $\lambda \geq 2$ be a cardinal and let $\kappa \geq \lambda$ be a regular cardinal. Then HOD *is definable in* $<\lambda$-HOD *using* $\mathcal{P}(\kappa) \cap \mathrm{HOD}$ *as a parameter.*

Consequences: No New Large Cardinals

Proposition

Let κ be an infinite cardinal, and let $\lambda \geq \kappa$ be inaccessible in HOD.

1. *If λ weakly compact in $<\kappa$-*HOD*, then it is weakly compact in* HOD.
2. *If λ is measurable in $<\kappa$-*HOD*, then it is measurable in* HOD.

Definition (Bukovský)

Let $M_1 \subseteq M_2$ be transitive models, and let κ be a cardinal in M_2. Then $\mathrm{Apr}_{M_1,M_2}(\kappa)$ says that whenever $f \in M_2$ is a function from an ordinal α to an ordinal β, then there is a function $g : \alpha \longrightarrow \mathcal{P}(\beta)$ in M_1 such that for every $\xi < \alpha$, $f(\xi) \in g(\xi)$ and $\mathrm{card}(g(\xi))^{M_1} < \kappa$.

How close is HOD to blurry HOD?, part 2

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Let $M_1 \subseteq M_2$ be transitive models, and let κ be a cardinal in M_2. Then $\mathrm{Apr}_{M_1,M_2}(\kappa)$ says that whenever $f \in M_2$ is a function from an ordinal α to an ordinal β, then there is a function $g : \alpha \longrightarrow \mathcal{P}(\beta)$ in M_1 such that for every $\xi < \alpha$, $f(\xi) \in g(\xi)$ and $\mathrm{card}(g(\xi))^{M_1} < \kappa$.

Exercise

Let κ be a cardinal. Then $\mathrm{Apr}_{\mathrm{HOD},<\kappa\text{-}\mathrm{HOD}}(\kappa)$ holds.

Consequences: Cardinalities and Cofinalities

Proposition

Let κ be an infinite cardinal. Then HOD *and* $<\kappa$-HOD *have the same cardinals and cofinalities above* κ*, in the following sense:*

1. *If λ is a limit ordinal such that $\mathrm{cf}^{\mathsf{HOD}}(\lambda) \geq \kappa$, then* $\mathrm{cf}^{\mathsf{HOD}}(\lambda) = \mathrm{cf}^{<\kappa\text{-}\mathsf{HOD}}(\lambda)$.

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2. *For $\lambda \geq \kappa$, λ is regular in* HOD *iff λ is regular in* $<\kappa$-HOD.
3. $\mathrm{Card}^{\mathsf{HOD}} \setminus \kappa = \mathrm{Card}^{<\kappa\text{-}\mathsf{HOD}} \setminus \kappa$.

Consequences: Generic Passage from HOD to blurry HOD

Theorem (ZFC, Bukovský 1973)

Suppose M is a transitive inner model of ZFC, and κ is an infinite cardinal. Then the following conditions are equivalent:

1. V *is a forcing extension of M by a κ-c.c. forcing notion.*
2. $\mathsf{Apr}_{M,\mathrm{V}}(\kappa)$ *holds.*

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Interactions with forcing: preserving upwards

Proposition

Suppose that $\mathbb{P}$ *is a notion of forcing,* G *is generic for* $\mathbb{P}$ *over* V*,* κ *is a cardinal in* $\mathrm{V}[G]$*, and* V *is definable in* $\mathrm{V}[G]$ *from a parameter in* $<\kappa\text{-OD}^{\mathrm{V}[G]}$*. Then*

$$<\kappa\text{-OD}^{\mathrm{V}} \subseteq <\kappa\text{-OD}^{\mathrm{V}[G]}$$

and so, $<\kappa\text{-HOD}^{\mathrm{V}} \subseteq <\kappa\text{-HOD}^{\mathrm{V}[G]}$ *as well.*

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Corollary

Let κ *be a cardinal, and let* $\mathbb{P}$ *be a notion of forcing of cardinality* γ, *where* $2^{2^{\gamma}} < \kappa$. *If* G *is* $\mathbb{P}$*-generic over* V, *then*

$$<\kappa\text{-HOD}^{\mathrm{V}} \subseteq <\kappa\text{-HOD}^{\mathrm{V}[G]}.$$

Interactions with forcing: homogeneity

Lemma

Let κ be a regular cardinal, $\mathbb{P}$ a cone homogeneous, $<\kappa$-closed forcing notion, and let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V. Then

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Note how nicely this lemma generalizes the folklore fact that if $\mathbb{P}$ is cone homogeneous and $G \subseteq \mathbb{P}$ is generic, then $\mathrm{HOD}^{\mathrm{V}[G]} \subseteq \mathrm{V}$ - this is the special case $\kappa = \omega$.

Interactions with forcing: preserving downwards

Lemma

Let κ be a regular cardinal, $\mathbb{P}$ a cone homogeneous, $<\kappa$-closed forcing notion, $\bar{\kappa} \leq \kappa$ a cardinal such that $\mathbb{P}$ is $<\bar{\kappa}$-OD, and let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V. Then

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$$<\bar{\kappa}\text{-HOD}^{\mathrm{V}[G]} \subseteq <\bar{\kappa}\text{-HOD}^{\mathrm{V}}.$$

Again, note how nicely this generalizes the fact that if $\mathbb{P}$ is and OD forcing notion and $G \subseteq \mathbb{P}$ is generic, then $\mathrm{HOD}^{\mathrm{V}[G]} \subseteq \mathrm{HOD}^{\mathrm{V}}$.

I have used these general forcing tools in order to study the possible constellations of leaps.

Leaps

Definition

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for every cardinal $\delta < \lambda$.

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Lemma

1. *The class of leaps is closed in the ordinals.*
2. *The least leap, if there is one, is an uncountable successor cardinal.*
3. *Successor leaps are successor cardinals.*

Big leaps

Definition

Say that a leap λ is a big leap if

$$\left(\bigcup_{\delta<\lambda,\delta\in\mathrm{Card}} {<}\delta\text{-HOD}\right) \subsetneq {<}\lambda\text{-HOD}.$$

Big leaps

Definition

Say that a leap λ is a big leap if

$$\left(\bigcup_{\delta<\lambda, \delta\in\mathrm{Card}} {<}\delta\text{-HOD}\right) \subsetneq {<}\lambda\text{-HOD}.$$

Theorem

Every leap is big, and if λ is a limit leap, then ${<}\lambda$-HOD does not satisfy the axiom of choice.

λ is a big leap

Let

$$T = \{\kappa < \lambda \mid \kappa \text{ is a successor leap}\}.$$

For $\kappa \in T$, let κ_- be its predecessor leap, and let τ_κ be least such that

$$B_\kappa = \mathrm{V}_{\tau_\kappa} \cap ({<}\kappa\text{-HOD} \setminus {<}\kappa_-\text{-HOD}) \neq \emptyset.$$

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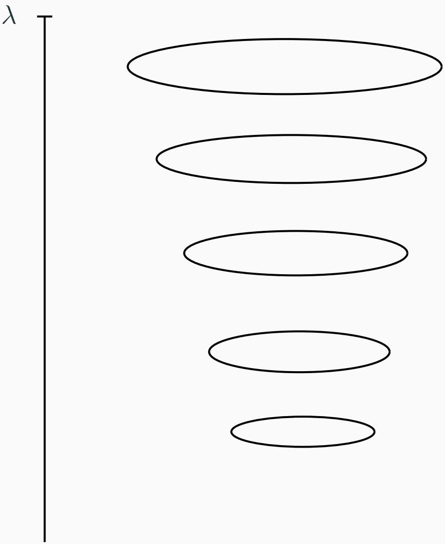
So for every $b \in B_\kappa$, $b \in {<}\kappa$-HOD, $b \notin {<}\kappa_-$-HOD, but $b \subseteq {<}\kappa_-$-HOD. Then $\vec{B} = \langle B_\kappa \mid \kappa \in T\rangle$ is OD, and $\vec{B}$ belongs to ${<}\lambda$-HOD, but it is not in ${<}\kappa$-HOD for any cardinal $\kappa < \lambda$.

$\vec{B}$ has no choice function in $<\lambda$-HOD



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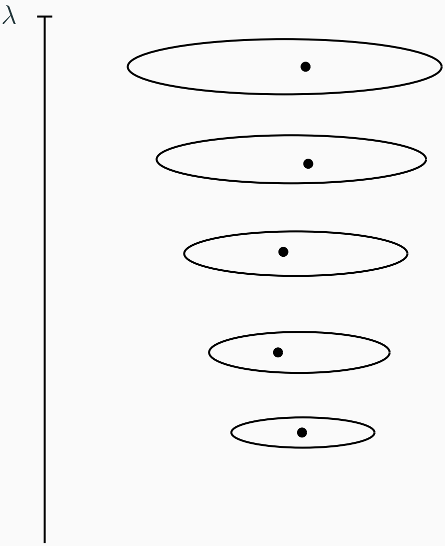

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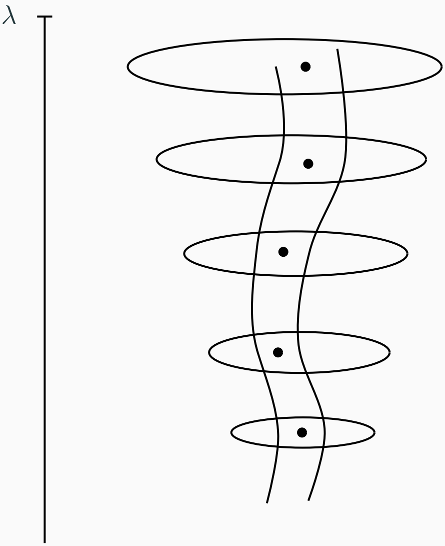

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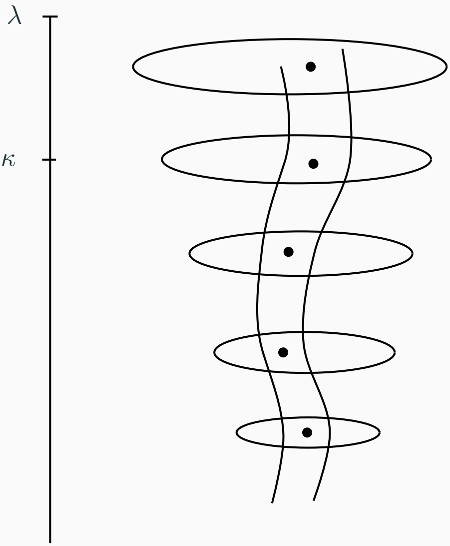

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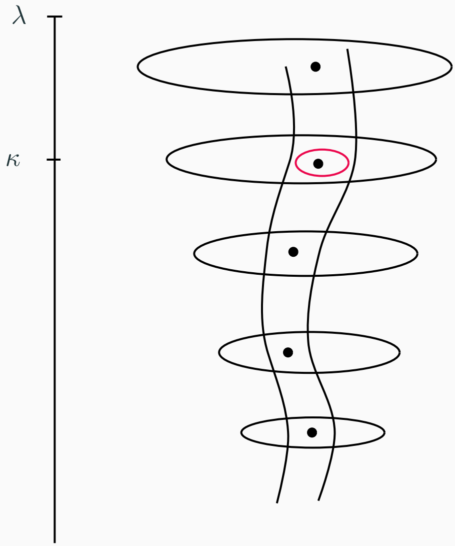

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Strength

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Theorem (F.)

Suppose that λ is a singular strong limit cardinal such that

1. *λ^+ is a leap,*
2. *the leaps are bounded below λ.*

Then there is an inner model with a measurable cardinal.

▸ Skip the proof sketch?

Assume towards a contradiction that there is no inner model with a measurable cardinal.

Let $\bar{\lambda} < \lambda$ be a cardinal which is greater than all the leaps below λ. So $<\lambda$-HOD $= <\bar{\lambda}$-HOD.

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We have to work with blurry codes instead.

So let $P = D[a_0]$, $\pi = \mathrm{otp}(P)$, and let $e : \pi \longrightarrow P$ be the monotone enumeration of P. Since $\mathrm{card}(a_0) \leq \lambda$, it follows that $\pi < \lambda^+$. For $\xi \leq \pi$, let

$$a_0{\restriction}\xi = \{x \in a_0 \mid e^{-1}(D(x)) < \xi\} = a_0 \cap D^{-1}[e[\xi]].$$

So $a_0{\restriction}\pi = a_0$. Let $\Omega \leq \pi$ be least such that $a_0{\restriction}\Omega \notin {<}\lambda$-HOD.

Claim: Ω is a limit ordinal.

Otherwise, say $\Omega = \bar{\Omega} + 1$. Let $a_0' = a_0 \cap D^{-1}(e(\bar{\Omega}))$. Then $a_0 \restriction \Omega = a \restriction \bar{\Omega} \cup a_0'$. Let $\bar{\kappa} = \mathrm{card}(D^{-1}(e(\bar{\Omega})))$. Then $\bar{\kappa} < \lambda$. So $\mathcal{P}(D^{-1}(e(\bar{\Omega})))$ is OD (using $e(\bar{\Omega})$ as a parameter), has cardinality less than λ, and contains a_0', making it a $<\lambda$-blurry ordinal definition of a_0'. Since $a_0' \subseteq {<\lambda}\text{-HOD}$, it follows that $a_0' \in {<\lambda}\text{-HOD}$. But also, by minimality of Ω, $a_0 \restriction \bar{\Omega} \in {<\lambda}\text{-HOD}$. Hence,

$$a_0 \restriction \Omega = (a_0 \restriction \bar{\Omega}) \cup a_0' \in {<\lambda}\text{-HOD},$$

a contradiction.

Claim: $\kappa < \lambda$.

Since Ω is a limit ordinal, it makes sense to consider its cofinality.

As $\Omega \leq \pi < \lambda^+$ and λ is singular, it follows that $\mathrm{cf}(\Omega) < \lambda$. And since $K \subseteq {<}\lambda^+\text{-HOD}$ and K has the covering property, $\bar{\Omega} = \mathrm{cf}^{<\lambda^+\text{-HOD}}(\Omega) < \lambda$.

So let $c : \bar{\Omega} \longrightarrow \Omega$ be monotone and cofinal, $c \in {<}\lambda^+\text{-HOD}$. Let $a_0' = \{a_0 {\restriction} c(\xi) \mid \xi < \bar{\Omega}\}$. Then $a_0' \in {<}\lambda^+\text{-HOD}$, $a_0' \subseteq {<}\lambda\text{-HOD}$ and $a_0' \notin {<}\lambda\text{-HOD}$. So by minimality of κ, $\kappa \leq \mathrm{card}(a_0') \leq \bar{\Omega} < \lambda$, as wished.

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It then follows that $\mathrm{card}(D^{-1}[c']) \leq \bar{\lambda} \cdot \mathrm{card}(c') < \lambda$, and that $a_0 \subseteq D^{-1}[c']$.

But then $\mathcal{P}(D^{-1}[c'])$ is an OD set of cardinality $2^{\mathrm{card}(D^{-1}[c'])} < \lambda$, and it contains a_0, so a_0 has a $<\lambda$-blurry ordinal definition and $a_0 \subseteq {<}\lambda$-HOD. This means that $a_0 \in {<}\lambda$-HOD. This is a contradiction.

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If in L, λ is a regular cardinal that's not weakly compact, then one can force with a very homogeneous Souslin tree to produce a model where λ^+ is the least leap. So if λ was inaccessible, it remains so. Hence, the strength comes from the singularity of λ.

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But one can also produce models where the least leap is λ^+, and λ is singular (of any desired cofinality), without assuming any large cardinals. So the strength also comes from the strength (i.e., from the assumption that λ is a strong limit cardinal).

The reversal

Theorem

By Prikry forcing with respect to U over L[U], where U is on κ, one arrives at a model where κ^+ is a singular strong limit cardinal and κ^+ is the least leap.

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Revised idea: modify Magidor's Prikry forcing with interleaved collapses, so that the even indexed Prikry points remain cardinals but the odd indexed ones get collapsed.

The forcing conditions

$\mathbb{M}$ is the forcing notion consisting of conditions of the form

$$\pi = \langle \langle \kappa_i \mid i < l \rangle, \langle f_i \mid i < l,\ i \text{ is even} \rangle, A, F \rangle$$

with the following properties:

1. $2 \leq l \in \omega$ is an even number, the *length* of π, denoted $\mathrm{lh}(\pi)$.
2. The sequence $\langle \kappa_i \mid i < l \rangle$ is strictly increasing, $\kappa_0 = \omega$, and for $0 < i < l$, κ_i is inaccessible. Note that $l - 1$, the largest index for which κ_i is defined, is *odd*.
3. For even i with $i + 2 < l$,
$$f_i \in \mathrm{Col}(\kappa_i^+, {<}\kappa_{i+2}).$$
4. $f_{l-2} \in \mathrm{Col}(\kappa_{l-2}, < \min(A))$.
5. $A \in U$, and for all $\alpha \in A$, α is inaccessible.
6. For $i < l$, $\kappa_i < \min(A)$.
7. F is a function with domain $[A]^2$, and if $\alpha < \beta < \gamma$ and $\alpha, \beta, \gamma \in A$, then $F(\{\alpha, \beta\}) \in \mathrm{Col}(\alpha^+, {<}\gamma)$.

The ordering

The ordering of $\mathbb{M}$ is defined as follows. Let π be as above, and let

$$\pi' = \langle \langle \kappa'_i \mid i < l' \rangle, \langle f'_i \mid i < l' \text{ is even} \rangle, A', F' \rangle \in \mathbb{M}.$$

Then $\pi' \leq \pi$ if:

1. $l \leq l'$.
2. For $i < l$, $\kappa_i = \kappa'_i$.
3. For $l \leq i < l'$, $\kappa'_i \in A$.
4. For all even $i < l$, $f_i \subseteq f'_i$.
5. For all even $i \in [l, l')$, $F(\kappa'_i, \kappa'_{i+1}) \subseteq f'_i$.
6. $A' \subseteq A$.
7. For all $\{\alpha, \beta\} \in [A']^2$, $F(\alpha, \beta) \subseteq F'(\alpha, \beta)$.

The cardinals in the extension

$$\begin{array}{cccccccccc}
\aleph_0 & \aleph_1 & \aleph_2 & \aleph_3 & \aleph_4 & \aleph_5 & \cdots & \aleph_{2n} & \aleph_{2n+1} & \cdots \\
\kappa_0 & \kappa_0^+ & \kappa_2 & \kappa_2^+ & \kappa_4 & \kappa_4^+ & \cdots & \kappa_{2n} & \kappa_{2n}^+ & \cdots
\end{array}$$

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  $$L[U][G] = L[U][\vec{\kappa}, \vec{f}]$$

  where $\vec{\kappa}$ is the Příkrý sequence and $\vec{f}$ is the sequence of collapsing functions (where for even n and all $\xi \in [\kappa_n^+, \kappa_{n+2})$, $\gamma \mapsto f_n(\xi, \gamma)$ is a surjection from κ_n^+ onto ξ).

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  We want there to be no leap up to $\kappa = \aleph_\omega = \sup_{n<\omega} \kappa_n$, so we have to ensure these surjections are ordinal definable. So:
- Force to code $\vec{f}$ into the continuum function up high (say, above $\aleph_{\omega+2}$). Call the forcing to do this $\mathbb{C}$, and the generic for it H.

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  Note: the forcing $\mathbb{C}$ is not ordinal definable, but it is weakly homogeneous.

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$$\pi \Vdash_{\mathbb{M}} \text{“}1_{\dot{\mathbb{C}}} \Vdash \varphi(\dot{\vec{\kappa}}, \check{\rho})\text{”}.$$

One can now find κ many finite variations of G, all of the form G', where G' results from G by changing its $\vec{\kappa}$ sequence in one odd coordinate, and G' also contains a condition like π.

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One can now find κ many finite variations of G, all of the form G', where G' results from G by changing its $\vec{\kappa}$ sequence in one odd coordinate, and G' also contains a condition like π. In particular, $\vec{f}^G = \vec{f}^{G'}$, and hence, $\dot{\mathbb{C}}^G = \dot{\mathbb{C}}^{G'}$. So $\varphi(\dot{\vec{\kappa}}^{G'}, \rho)$ holds in $\mathrm{V}[G'][H] = \mathrm{V}[G][H]$. So $\vec{\kappa}' \in A$, giving $\mathrm{card}(A) \geq \kappa$.

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By the Dodd-Jensen Covering Lemma for $L[U]$, since $0^\dagger$ does not exist in $\mathrm{V}[G][H]$, there is a maximal Příkrý sequence C over $L[U]$. Each Příkrý sequence is determined by C, a subset of ω and a finite subset of κ. So in total there are κ Příkrý sequences over $L[U]$, and $\vec{\kappa}$ is one of them.

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Let $\bar{\kappa}$ be the least leap in $\mathrm{V}[G][H]$, and assume $\omega < \bar{\kappa} < \kappa$.

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So we get the equiconsistency of the following theories over ZFC:

- There is a measurable cardinal.
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There are lots of open questions in this area. One obvious set of questions has to do with the cofinality of the singular strong limit cardinal below which the leaps are bounded and whose successor is a leap.

A theorem of Shelah

I am grateful to Moti Gitik for pointing out to me the relevance of the following theorem:

Theorem (Shelah)

If κ is a singular strong limit cardinal of uncountable cofinality $\bar{\kappa}$, M is an inner model which contains H_κ along with a well-ordering of it, and $\mathrm{cf}^M(\kappa) = \bar{\kappa}$, then $\mathcal{P}(\kappa) \subseteq M$.

Observation

Suppose that κ is a strong limit cardinal. Then $H_\kappa \subseteq {<}\kappa\text{-HOD}$. Moreover, if κ is singular, then $(\kappa \text{ is singular})^{<\kappa\text{-HOD}}$ iff $\mathrm{cf}^{<\kappa\text{-HOD}}(\kappa) = \mathrm{cf}(\kappa)$.

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Observation

Suppose κ is a strong limit cardinal which is singular in ${<}\kappa$-HOD, and suppose that the leaps below κ are bounded in κ. Then, in ${<}\kappa$-HOD, there is a well-ordering of H_κ in order type κ.

Shelah's theorem gives:

Corollary

Suppose κ is a singular strong limit cardinal of uncountable cofinality, below which the leaps are bounded, and such that $(\kappa \text{ is singular})^{<\kappa\text{-HOD}}$. Then $\mathcal{P}(\kappa) \subseteq {<}\kappa\text{-HOD}$.

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Theorem

Suppose κ is a singular strong limit cardinal of uncountable cofinality below which the leaps are bounded, yet κ^+ is a leap. Then κ is regular in ${<}\kappa$-HOD.

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Suppose κ is a singular strong limit cardinal of uncountable cofinality below which the leaps are bounded and such that κ^+ is a leap. Then $\kappa = \aleph_\kappa$.

In particular, it cannot be that $\aleph_{\omega_1}$ is a strong limit cardinal and $\aleph_{\omega_1+1}$ is the least leap. This is one more sense in which the result at $\aleph_\omega$ is optimal.

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Regarding the consistency strength, all we have at present is a lower bound.

Corollary

Suppose κ is a singular strong limit cardinal of uncountable cofinality $\bar{\kappa}$ below which the leaps are bounded and such that κ^+ is a leap. Then there is an inner model in which κ is measurable with $o(\kappa) = \bar{\kappa}$.

Thank you!