Proposition 1. Assume ZFC is consistent

then there is a formula $\psi(\underline{x})$ s.t. $\underbrace{\text{is a formula}}_{\text{concretely given}}$

$\psi(\underline{x})$ <u>represents</u> even numbers $\subseteq \mathbb{N}$

(i.e. s.t. for a number n we have — numeral of n

n is even $\Rightarrow$ ZFC $\vdash \psi(\bar{n})$

not even $\Rightarrow$ ZFC $\vdash \neg\psi(\bar{n})$)

But (*) ZFC $\nvdash (\forall x \in \underline{\omega})(\psi(x) \vee \psi(x+1))$

We assume the language of ZFC is $\{\in, \underline{\emptyset}, \underbrace{\{\cdot,\cdot\}}_{(\cdot \cup \cdot)}, \underline{\omega}\}$

$\mathcal{L}_{\{\}}$

$x \cup \{1, x\}$

Proof

$\psi(\underline{x}) \leftrightarrow \forall \underline{y} \in \underline{x}$ ($\underline{y}$ is not a Gödel number of a proof of inconsistency from ZFC

$\wedge$ $\underbrace{\underline{x} \text{ is even}}_{(\exists z \in \underline{x})(\underline{x} = 2z)}$).

(*) holds, otherwise

ZFC $\vdash$ Consis($\ulcorner$ZFC$\urcorner$) so ZFC is inconsistent

↑

2. Incompleteness Thm!

$\boxtimes$

Thm[2] (Tarski's Undefinability of Truth Theorem)

Let T be a consistent theory with a (Gödel) numbering i.e. any concretely given (recursive) mapping $\varphi \mapsto \ulcorner\varphi\urcorner$

$\uparrow$ L formula $\quad$ $\uparrow$ closed term in L

Let $w = w(\underline{x})$ be an L-formula s.t. for any L-formula φ $T \vdash w(\ulcorner\varphi\urcorner)$. (✡✡) [For the real Gödel numbering $w(x)$ can be taken as "$x \in \omega$"]

Then there is no L-formula $\tau = \tau(\underline{x}, \underline{y})$ s.t.

(**) $T \vdash \forall x (w(x) \rightarrow \tau(x, \ulcorner\varphi\urcorner) \leftrightarrow \varphi(x))$

Proof Suppose there would be τ with (**).

Let $\rho(\underline{x}) :\equiv \neg\tau(\underline{x}, \underline{x})$

Then we have (✡) $T \vdash \forall x (w(x) \rightarrow \tau(x, \ulcorner\rho\urcorner) \leftrightarrow \rho(x))$ by (**)

In particular by (✡) + (✡✡) we have

$$T \vdash \tau(\ulcorner\rho\urcorner, \ulcorner\rho\urcorner) \leftrightarrow \rho(\ulcorner\rho\urcorner)$$

On the other hand by the def. of ρ

$$T \vdash \rho(\ulcorner\rho\urcorner) \Leftrightarrow \neg \tau(\ulcorner\rho\urcorner, \ulcorner\rho\urcorner)$$

Thus T is inconsistent ↯ $\boxtimes$

An application:

"$V_{\underline{\kappa}} \prec V$" is not formalizable in ZFC.

$$V_\kappa \prec V_{\underline{\lambda}}$$

concretely given in metamathematics T?

Thm 3 For any specific extension of (a large enough fragment of $\underline{\underline{ZFC}}$) in $\mathcal{L}_{\{\in\}}$ then there is no $\mathcal{L}_{\{\in\}}$ formula $E(\underline{\kappa})$ s.t.

(a) $T \vdash \forall \underline{\kappa}\, (E(\underline{\kappa}) \rightarrow \underline{\kappa}$ is an infinite cardinal$)$

(b) $T \vdash \forall \underline{\kappa}\, (E(\underline{\kappa}) \rightarrow V_{\underline{\kappa}} \prec_{\Sigma_n} V)$ for all $n \in \mathbb{N}$

(c) $T + \exists \underline{\kappa}\, E(\underline{\kappa})$ is consistent.

$\varphi(\underline{x}, \ulcorner\psi\urcorner) \leftrightarrow \psi$ is absolute between V_κ and V

Assume that there would be E as above

$\forall \psi \in \mathrm{Fml}_{\Sigma_n} \, (\varphi(x, \ulcorner\psi\urcorner))$

Proof Let $\varphi \mapsto \ulcorner\varphi\urcorner$ be the "real" Gödel numbering

$\omega(x) := x \in \underline{\omega}$ and

$\tau(\underline{a}, \underline{\varphi}) :\leftrightarrow \forall \kappa \, (E(\underline{\kappa}) \to (V_{\underline{\kappa}} \models \underline{\varphi}(\underline{a})))$

Then $\tau(\underline{a}, \underline{\varphi})$ together with $\omega(\underline{x})$ is a truth definition

in the sense of Thm ?.

For any formula $\eta = \eta(\underline{x})$

$\underline{\eta}$ is Σ_n for some large enough n

So $V_{\underline{\kappa}} \models \varphi(\underline{a})$ for $\underline{\kappa}$ with

$E(\underline{\kappa})$ implies $\varphi(\underline{a})$ by (1).

$\boxtimes$

Lemma 4 (1) "$\underline{M}$ is transitive $\wedge$ $\underline{\omega} \in \underline{M}$ $\wedge$ $\underline{M}$ closed wrt $\{\cdot,\cdot\}$ $\wedge$ $\underline{M} \models \underline{\varphi}(\underline{a})$" is a Δ_1 statement

(2) For each number $n \in \mathbb{N}$, there is a Γ-formula $\varphi_n = \varphi_n(\underline{\psi}, \underline{x})$ where

$$\Gamma = \begin{cases} \Delta_1 & \text{if } n = 0 \\ \Sigma_n & n > 0 \end{cases}$$

s.t. φ_n expresses "$\underline{\psi} \in Fml_{\Sigma_n}$ and $\underline{\psi}$ holds" and for any Σ_n formula ψ

$$ZFC_0 \vdash \forall x\,(\varphi_n(\ulcorner\psi\urcorner, \vec{x}) \Leftrightarrow \psi(\vec{x}))$$

(3) For any $n \in \mathbb{N}$ "$\underline{M}$ is transitive $\wedge$ $\underline{\omega} \in \underline{M}$ $\wedge$ $\underline{M}$ is closed wrt. $\{\cdot,\cdot\}$ $\wedge$ $\underline{M} \prec_{\Sigma_n} V$" is Π_n

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Remark $j : V \xrightarrow{\prec}_{\Sigma_n} M$ is definable !!! for definable

inner model M

(being a inner model is expressible by a single formula)

Lemma Thm 13.9 (Kanamori §0)

$j : V \to_x M$ (transitive) is Σ_1-elementary implies the full elementarity

(A Theorem by Gaifman)

Further details:
https://fuchino.ddo.jp/notes/math-notes-11.pdf#page50