Coherent systems of FS iterations and applications (I)

$\boxed{2D}$ Blass & Shelah (1989) $\mathfrak{u} < \mathfrak{d}$ (large cont.)

$(\kappa+1)\times(\pi+1)$

Brendle & Fischer (2010)

$\mathfrak{b} = \mathfrak{a} < \mathfrak{s}$ (large cont.)

$\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$

uses measurable

(M. 2013) Applications in Cichoń's diagram

(Dow & Shelah 2016) Con(ZFC + $\mathfrak{s}$ singular)

(Fischer, Friedman, M. & Montoya 2018)

→ Define coherent systems

→ 2D

→ 3D → [CGMRŠ] Applications to Slalom numbers

(M. 2019) Beyond 3D → w. Cardona: New models for cof(SN)

Definition:

M: transitive model of ZFC

$\mathbb{P} \in M$, $\mathbb{Q}$ posets

$\mathbb{P} \subseteq_M \mathbb{Q}$: $\mathbb{P} \subseteq \mathbb{Q}$, maximal antich. in $\mathbb{P}$ from M are preserved.

$$\begin{tikzcd}
\mathbb{Q} \in N \arrow[r] & N[G] \\
\mathbb{P} \in M \arrow[u, "\subseteq"] \arrow[r] & M[G \cap \mathbb{P}] \arrow[u, "\subseteq"]
\end{tikzcd}$$

If $\mathbb{P} \subseteq_M \mathbb{Q}$ and $G \subseteq \mathbb{Q}$ gen $/N$ then $\mathbb{P} \cap G$ is $\mathbb{P}$-gen$/M$ and $M[\mathbb{P} \cap G] \subseteq N[G]$.

Ex. (1) $\mathbb{Q} \subseteq_M \mathbb{Q}$ $(\mathbb{Q} \in M)$

(2) $\mathbb{1} \subseteq_M \mathbb{Q}$

(3) $\mathbb{S}$ is Suslin ccc coded in M, and $M \subseteq N$, then $\mathbb{S}^M \subseteq_M \mathbb{S}^N$.

⋮

Def (Coherent System)

A coherent system $\mathbb{S}$ is composed by

(1) $\langle I, \leq_I \rangle$ partial order ← base

(2) For $i \in I$, $\langle \mathbb{P}_{i,\xi}, \dot{\mathbb{Q}}_{i,\xi} : \xi < \pi \rangle$ a FS iteration

π: length of the system, ordinal — same for all $i \in I$.

(3) (Coherence) if $i \leq_I j$, $\xi < \pi$ and $\mathbb{P}_{i,\xi} \subseteq \mathbb{P}_{j,\xi}$ then $V_{j,\xi} \models \mathbb{Q}_{i,\xi} \subseteq_{V_{i,\xi}} \mathbb{Q}_{j,\xi}$

$$\boxed{V_{i,\xi} = V^{\mathbb{P}_{i,\xi}}}$$

$$\begin{tikzcd}
V_{j,\xi} \arrow[r, "Q_{j,\xi}"'] & V_{j,\xi+1} \\
V_{i,\xi} \arrow[u, phantom, "\rotatebox{90}{$\subseteq$}"] \arrow[r, "Q_{i,\xi}"'] & V_{i,\xi+1} \arrow[u, phantom, "\rotatebox{90}{$\subseteq$}"]
\end{tikzcd}$$

<u>Note 1</u>: (3) implies

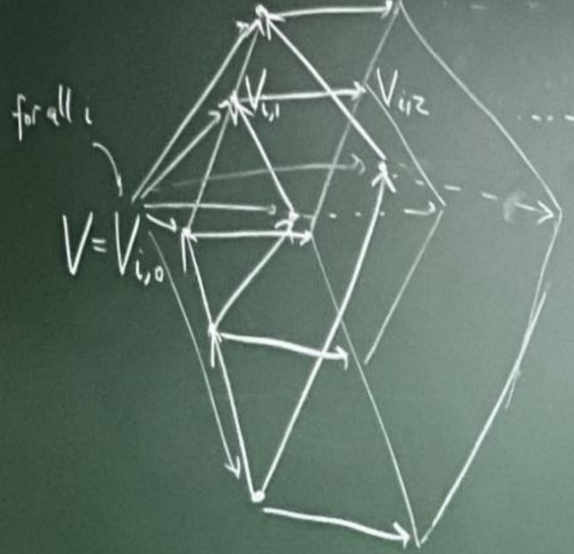
$$\mathbb{P}_{i,\xi} \subseteq \mathbb{P}_{j,\xi} \Rightarrow \mathbb{P}_{i,\xi+1} \subseteq \mathbb{P}_{j,\xi+1}$$

<u>Note 2</u>: (by induction)

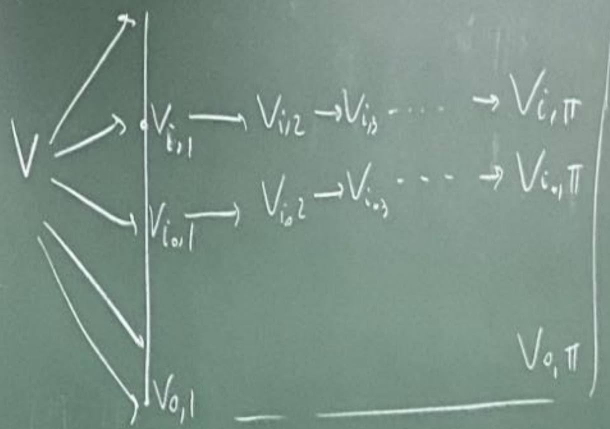
$$\forall i \leq_I j \; \forall \xi \leq \pi : \mathbb{P}_{i,\xi} \subseteq \mathbb{P}_{j,\xi}.$$

[Always $\mathbb{P}_{i,0} = \{\emptyset\}$]

$\langle I, \leq_I \rangle$



$\boxed{I = \gamma \text{ ordinal}}$

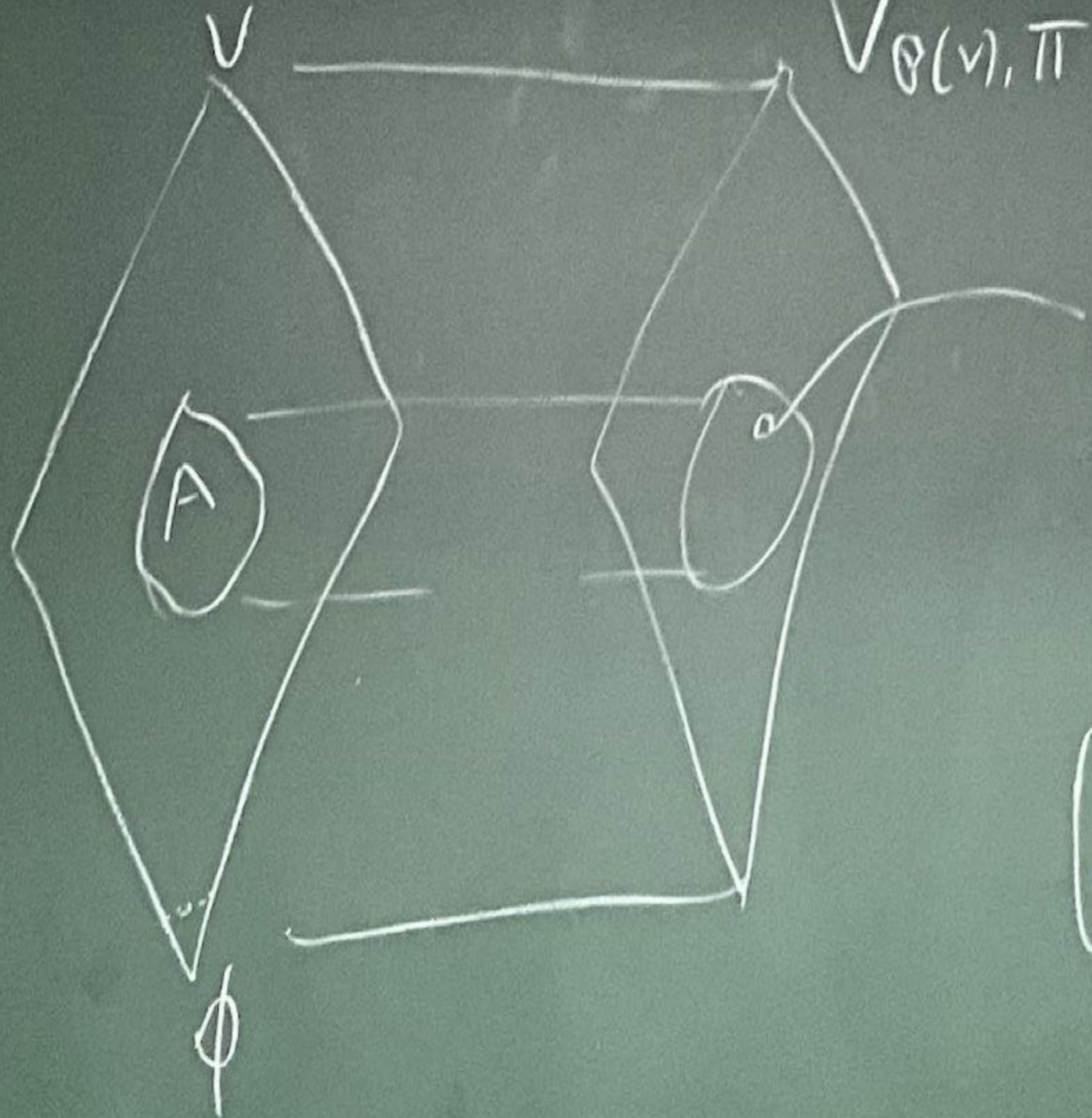


$\boxed{I = \gamma \times \delta}$

3D

$\boxed{\text{M. 2019}}$ $I = \mathcal{B}(V)$

$\mathbb{C}_\lambda$, $A \leq \lambda$, $\mathbb{C}_A \lessdot \mathbb{C}_\lambda$



$\mathbb{P}_{A,\pi} \lessdot \mathbb{P}_{V,\pi}$

$\top$

$\mathbb{P}_{V,\pi} |_A$

$\boxed{\text{Vertical restictions}}$

Ex. (1) $\mathbb{Q} \subseteq_M \mathbb{Q}$ $(\mathbb{Q} \in M)$

(2) $\mathbb{1} \subseteq_M \mathbb{Q}$

(3) $\mathbb{S}$ is Suslin ccc coded in M, and $M \subseteq N$, then $\mathbb{S}^M \subseteq_M \mathbb{S}^N$.

$\vdots$

How to construct an $\mathbb{S}$ on $\langle I, \leq_I \rangle$?

1) Define $\langle \mathbb{P}_{i,1} : i \in I \rangle$ s.t. $i \leq_I j \Rightarrow \mathbb{P}_{i,1} \subseteq \mathbb{P}_{j,1}$

$\left(\text{e.g. } I = (\mathcal{P}(\nu), \subseteq) \right)$, $\mathbb{P}_{A,1} := \mathbb{C}_A = \mathbb{Q}_{A,0}$

2) Proceed by recursion on $\xi \leq \pi$.

successor step • Having $\langle \mathbb{P}_{i,\xi} : i \in I \rangle$,

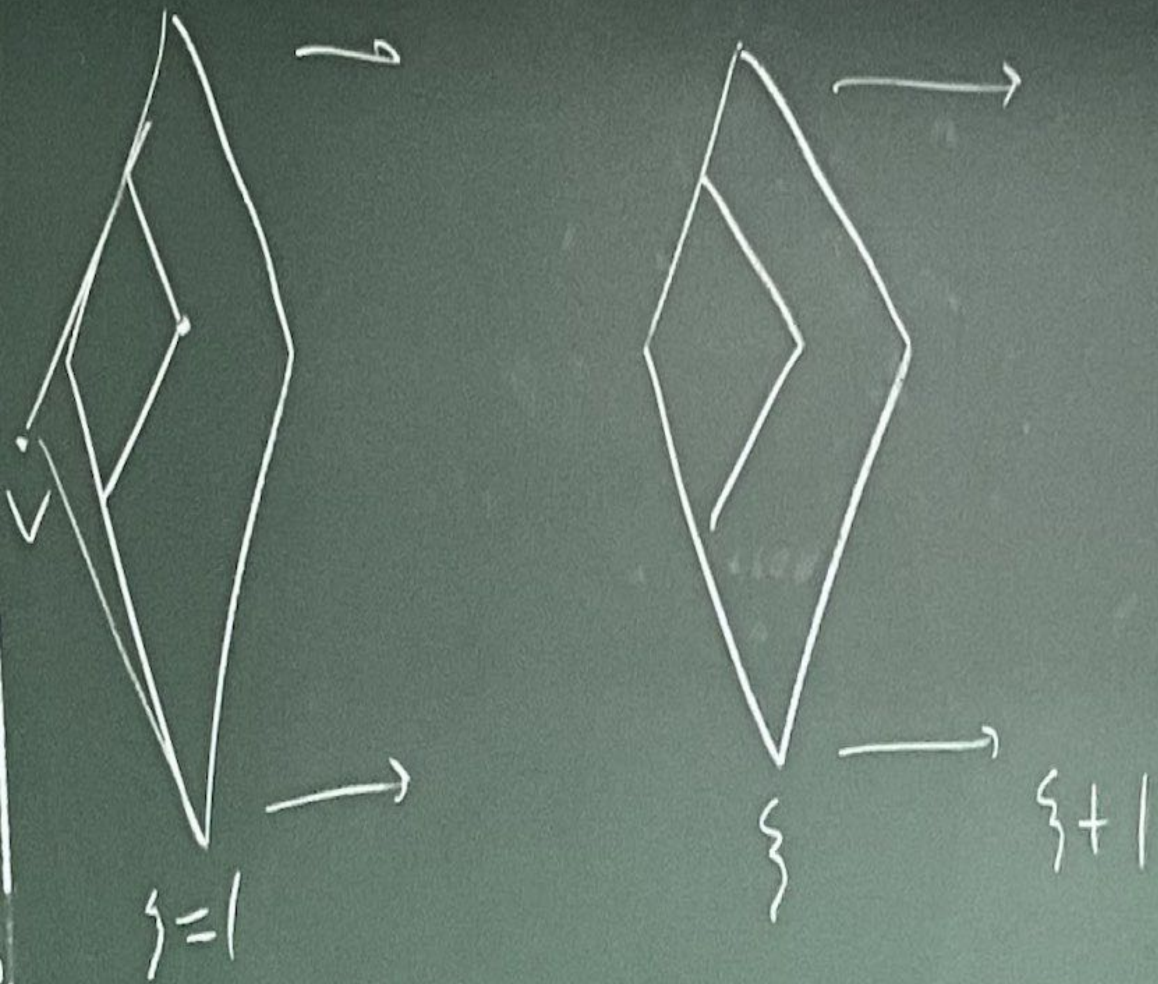
pick $\langle \dot{\mathbb{Q}}_{i,\xi} : i \in I \rangle$ s.t.

$\boxed{i \leq j \to V_{j,\xi} \models \mathbb{Q}_{i,\xi} \subseteq_{V_{i,\xi}} \mathbb{Q}_{j,\xi}}$

$\mathbb{P}_{i,0} = \{\emptyset\}$

$\mathbb{P}_{i,1} = \mathbb{P}_{i,0} * \mathbb{Q}_{i,0}$

$= \mathbb{Q}_{i,0}$



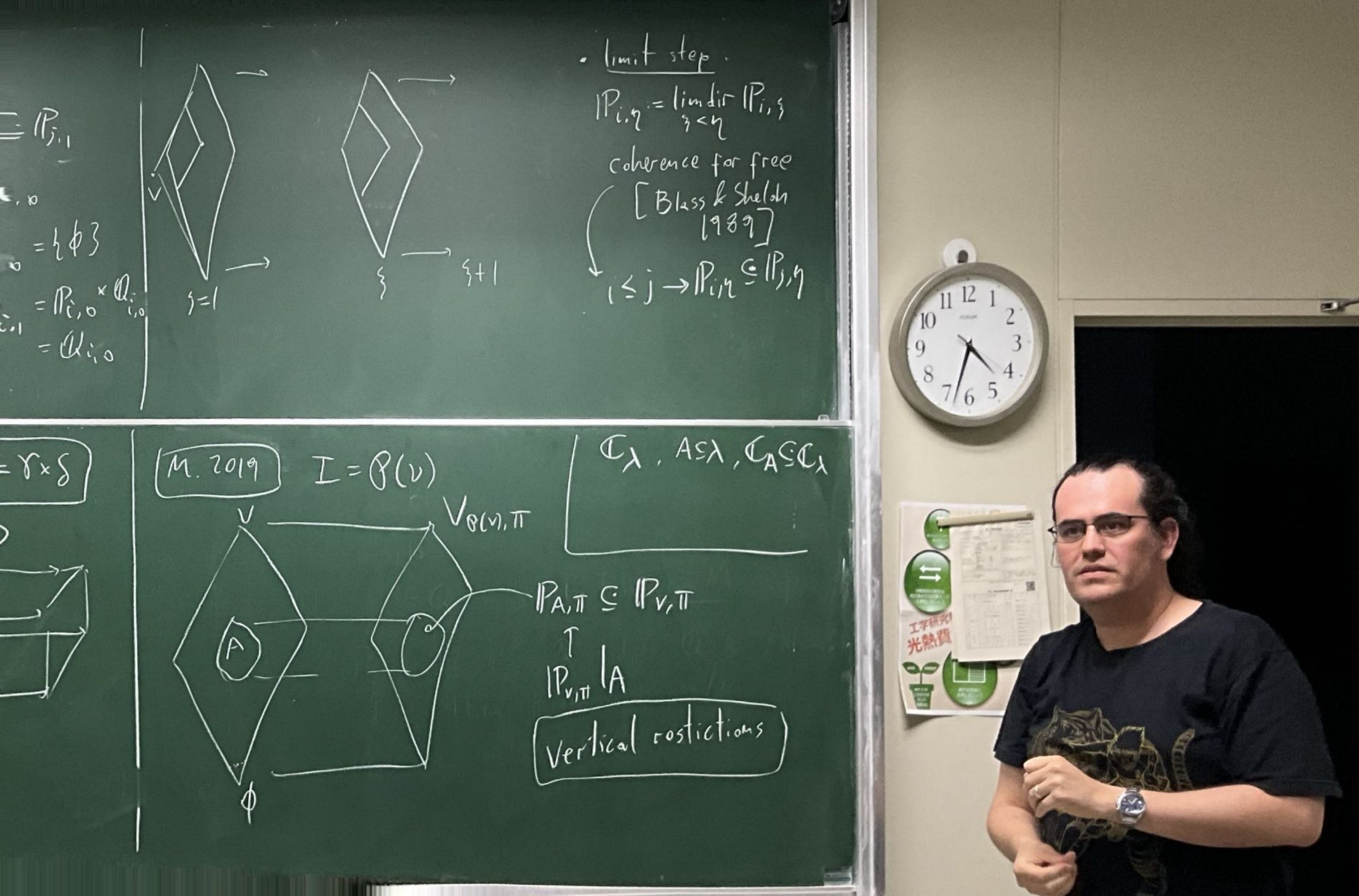
- <u>limit step</u>:

$$\mathbb{P}_{i,\eta} := \lim_{\xi<\eta} \mathrm{dir}\, \mathbb{P}_{i,\xi}$$

coherence for free

[Blass & Shelah 1989]

$$i \leq j \to \mathbb{P}_{i,\eta} \subseteq \mathbb{P}_{j,\eta}$$


limit step.
$\mathbb{P}_{i,\eta} = \lim\mathrm{dir}_{\xi<\eta} \mathbb{P}_{i,\xi}$
coherence for free
[Blass & Shelah 1989]
$i \leq j \to \mathbb{P}_{i,\eta} \subseteq \mathbb{P}_{j,\eta}$
$\xi = 1$
ξ
$\xi + 1$
$= \mathbb{P}_{j,1}$
$= \{\emptyset\}$
$= \mathbb{P}_{i,0} \times \mathbb{Q}_{i,0}$
$= \mathbb{Q}_{i,0}$
$= \gamma \times \delta$
M. 2019
$I = \mathcal{P}(V)$
V
$V_{\mathcal{P}(V),\pi}$
A
$\emptyset$
$\mathbb{C}_\lambda,\ A \subseteq \lambda,\ \mathbb{C}_A \subseteq \mathbb{C}_\lambda$
$\mathbb{P}_{A,\pi} \subseteq \mathbb{P}_{V,\pi}$
$\mathbb{P}_{V,\pi} \upharpoonright A$
Vertical restictions
11 12 1 2 3 4 5 6 7 8 9 10

Examples of extending $V_{i,\xi} \to V_{i,\xi+1}$ ($\xi \geq 1$)

(1) Adding a full generic

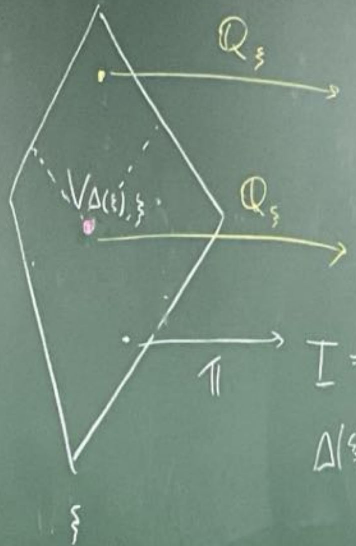
Pick Suslin ccc $\mathbb{S}$ (coded in V)

ξ

$\mathbb{Q}_{i,\xi} := \mathbb{S}^{V_{i,\xi}}$

$i \leq j \to \Vdash_{j,\xi} \mathbb{S}^{V_{i,\xi}} \subseteq_{V_{i,\xi}} \mathbb{S}^{V_{j,\xi}}$

Restricted generic. Pick $\Delta(\xi) \in \mathcal{I}$



$\mathbb{Q}_\xi \in V_{\Delta(\xi),\xi}$

$$\mathbb{Q}_{i,\xi} = \begin{cases} \mathbb{Q}_\xi & \text{if } i \supseteq \Delta(\xi) \\ \mathbb{1} & \text{if } i \not\supseteq \Delta(\xi) \end{cases}$$

$\mathcal{I} = \mathcal{P}(\nu)$

$\Delta(\xi) \in [\nu]^{<\kappa}$

e.g. $\mathbb{Q}_\xi = \mathbb{D}^{V_{\Delta(\xi),\xi}}$

$= \mathbb{D}^{N_\xi}$ $\quad N_\xi \in V_{\Delta(\xi),\xi}$

Lemma: Assume that $\mathbb{S}$ is a <u>ccc</u> coherent system.

Fix $j \in I$.

Assume

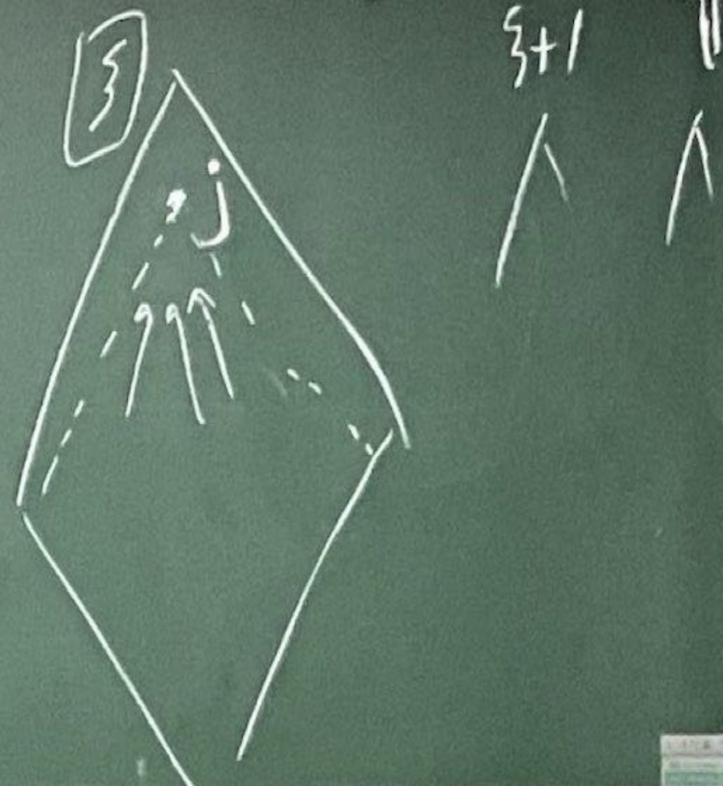
(i) $\mathfrak{b}(I_{<j}) \geq \aleph_1$,

(ii) for $\xi < \pi$: $\mathbb{P}_{j,\xi} = \operatorname{limdir}_{i<j} \mathbb{P}_{i,\xi} \Rightarrow \Vdash_{j,\xi} \dot{\mathbb{Q}}_{j,\xi} = \bigcup_{i<j} \dot{\mathbb{Q}}_{i,\xi}$

Then, $\forall \xi \leq \pi$

(a) $\mathbb{P}_{j,\xi} = \lim_{i<j} \mathbb{P}_{i,\xi}$

(b) $\mathbb{R} \cap V_{j,\xi} = \bigcup_{i<j} \mathbb{R} \cap V_{i,\xi}$



$\boxed{\mathbb{I} = \mathcal{P}(\nu)}$ Full + restricted

ensure $\Delta(\xi) \in [\nu]^{<\kappa}$

Lemma holds for $j \in [\nu]^{\geq \kappa}$

i.e., $A \subseteq \nu$, $|A| \geq \kappa$ then

$$\mathbb{P}_{A,\xi} = \lim_{\substack{X \subseteq A \\ |X| < \kappa}} \mathbb{P}_{X,\xi}$$

$\mathbb{I} = \gamma + 1$

1) $\Delta(\xi)$ successor

Lemma holds for j of unctble cof.

$v_{i,\xi}$

$\mathbb{I} = \gamma + 1$

1) ! $\underline{\Delta(\xi) \text{ successsor}}$.

Lemma holds for j of unctble cof.

M. 2019 Matrix ... vertical support restrictions.

$v_{i,\xi}$