Slalom numbers and Cohen reals

$E \subseteq {}^{\omega}\mathcal{P}(\omega)$ $\quad J \subseteq \mathcal{P}(\omega)$ ideal

$([\omega]^{<\aleph_0} \subseteq J)$

$\mathrm{Lc}_J(E) = \langle {}^{\omega}\omega, E, \in^{J^d} \rangle$

J^d: dual filter of J

$x \in^A y \Leftrightarrow \{i<\omega : x(i) \in y(i)\} \in A$

R: relation, $x, y \in {}^{\omega}\omega$

$\|x R y\| := \{i<\omega : x(i) R y(i)\}$.

$x R^A y \Leftrightarrow \|x R y\| \in A$

$x \in^{J^d} y \Leftrightarrow \|x \in y\| \in J^d$

$E = \prod_{n<\omega} A_n \quad A_n \subseteq \mathcal{P}(\omega)$

$E = {}^{\omega}I$, I: ideal

$E = \prod_{n<\omega} [\omega]^{\leq h(n)}$, $h \in {}^{\omega}\omega$

$$\mathrm{sl}_t(E,J) = \mathfrak{d}(\mathrm{Lc}_J(E))$$

$$= \min\{|S| : S \subseteq E,\ \forall x \in {}^\omega\omega\ \exists y \in S : x \in^{J^d} y\}$$

$$\mathrm{sl}_t(E,J)^\perp = \mathfrak{b}(\mathrm{Lc}_J(E))$$

$$= \min\{|F| : F \subseteq {}^\omega\omega,\ \neg\exists y \in E\ \forall x \in F : x \in^{J^d} y\}$$

$\boxed{E = {}^{\omega}I}$ $\quad Lc_J(E) = Lc_J(I)$

$$s_t(I, J) \qquad s_t^{\perp}(I; J)$$

$\boxed{E = \prod_{n<\omega} [\omega]^{\leq h(n)}}$ $\quad Lc_J(E) = Lc_J(h)$

$$s_t(h, J) \qquad s_t^{\perp}(h, J)$$

$$pLc_J(E) = \langle {}^{\omega}\omega, E, \in^{J^+} \rangle$$

$$[J^+ = \mathcal{P}(\omega) \setminus J]$$

$$s_e(E, J) := \mathfrak{d}(pLc_J(E))$$

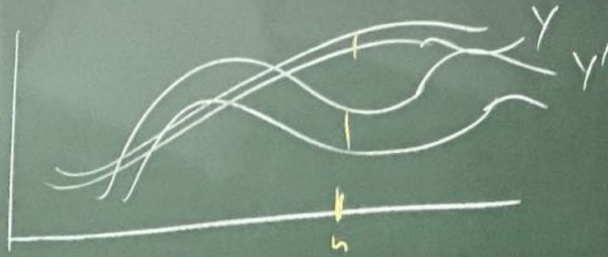
$$s_e^{\perp}(E, J) := \mathfrak{b}(pLc_J(E))$$

Notation:

$C \subseteq \mathcal{P}(\omega)$ is an <u>ideal base</u> if $\exists \underset{\text{ideal}}{J \subseteq \mathcal{P}(\omega)} : C \subseteq J$.

$S \subseteq {}^{\omega}\mathcal{P}(\omega)$ has fupc (finite union property coordinate-wise)

if $\forall n < \omega : \{ Y(n) : Y \in S \}$ is an ideal base.



$$\mathfrak{sl}_t(*,\mathcal{J}) = \min\{|S| : S \subseteq {}^{\omega}\mathcal{P}(\omega) \text{ fupc } \forall x \in {}^{\omega}\omega\ \exists y \in S : x \in^{\mathcal{J}^d} y\}$$

$$\mathfrak{sl}_e(*,\mathcal{J}) = \min\{|S| : S \subseteq {}^{\omega}\mathcal{P}(\omega) \text{ fupc } \forall x \in {}^{\omega}\omega\ \exists y \in S : x \in^{\mathcal{J}^+} y\}$$

Joint w/ M. Cardona, V. Gavalová, J. Šupina, M. Repický.

$$\mathfrak{sl}_{\substack{t\\e}}(*,\mathcal{J}) = \min\Big\{\mathfrak{sl}_{\substack{t\\e}}\big(\textstyle\prod_{n\in\omega} \mathcal{I}_n, \mathcal{J}\big) : \mathcal{I}_n : \text{ideal } (n<\omega)\Big\}$$

<u>Thm</u>: $\mathfrak{sl}_{\substack{t\\e}}(*,\mathcal{J}) = \min\limits_{\mathcal{I}:\text{ideal}} \mathfrak{sl}_{\substack{t\\e}}(\mathcal{I},\mathcal{J})$

Classical

$h \to \infty$: $\quad \mathrm{sl}_t(h, \mathrm{Fin}) = \mathrm{cof}(\mathcal{N})$, $\quad \mathrm{sl}_t^{\perp}(h, \mathrm{Fin}) = \mathrm{add}(\mathcal{N})$

$h \geq^* 1$: $\quad \mathrm{sl}_e(h, \mathrm{Fin}) = \mathrm{non}(\mathcal{M})$, $\quad \mathrm{sl}_e(h, \mathrm{Fin}) = \mathrm{cov}(\mathcal{M})$

$$\mathrm{sl}_t(\mathrm{Fin}, \mathrm{Fin}) = \mathfrak{d} = \mathrm{sl}_e^{\perp}(\mathrm{Fin}, \mathrm{Fin})$$

$$\mathrm{sl}_t^{\perp}(\mathrm{Fin}, \mathrm{Fin}) = \mathfrak{b} = \mathrm{sl}_e(\mathrm{Fin}, \mathrm{Fin})$$

$$\mathrm{sl}_e(I,J) \leq \mathrm{sl}_t(I,J)$$

$$\mathrm{sl}_e(I, F_{in}) = \min\{\mathfrak{b}, \mathrm{cov}^*(I)\}$$

$$\mathrm{sl}_e(*, F_{in}) = \mathfrak{p}$$

(Šupina) $\mathrm{sl}_t(*, F_{in}) = \mathrm{cov}(\mathcal{M})$

$$\mathfrak{b}_J = \mathfrak{b}({}^\omega\omega, \leq^{J^d})$$

$$\mathfrak{d}_J = \mathfrak{d}({}^\omega\omega, \leq^{J^d})$$

$$\mathrm{sl}_t(F_{in}, J) = \mathfrak{d}_J = \mathrm{sl}_e^{\perp}(F_{in}, J)$$

$$\mathrm{sl}_e(F_{in}, J) = \mathfrak{b}_J = \mathrm{sl}_t^{\perp}(F_{in}, J)$$

J BP $\Rightarrow \mathrm{sl}_t(I,J) = \mathrm{sl}_t(I, F_{in})$

sl_e?

non$(\mathcal{M})$ $\quad h \geq^* 1$ $\qquad\qquad\qquad\qquad$ $\mathrm{cof}(\mathcal{N})$ $\quad h \to \infty$

$$
\begin{tikzcd}
\boxed{\mathrm{sl}_e(h,\mathrm{Fin})} \arrow[r] & \mathrm{sl}_e(h,\mathcal{J}) \arrow[r] & \mathrm{sl}_t(h,\mathcal{J}) \arrow[r] & \boxed{\mathrm{sl}_t(h,\mathrm{Fin})} \\
\mathfrak{b} = \mathrm{sl}_e(\mathrm{Fin},\mathrm{Fin}) \arrow[r] \arrow[u] & \mathfrak{b}_{\mathcal{J}} = \mathrm{sl}_e(\mathrm{Fin},\mathcal{J}) \arrow[r] \arrow[u] & \mathrm{sl}_t(\mathrm{Fin},\mathcal{J}) = \mathfrak{d}_{\mathcal{J}} \arrow[r] \arrow[u] & \mathfrak{d} = \mathrm{sl}_t(\mathrm{Fin},\mathrm{Fin}) \arrow[u] \\
\mathrm{sl}_e(\mathcal{I},\mathrm{Fin}) \arrow[r] \arrow[u] & \mathrm{sl}_e(\mathcal{I},\mathcal{J}) \arrow[r] \arrow[u] & \mathrm{sl}_t(\mathcal{I},\mathcal{J}) \arrow[r] \arrow[u] & \mathrm{sl}_t(\mathcal{I},\mathrm{Fin}) \arrow[u] \\
\mathfrak{p} = \mathrm{sl}_e(*,\mathrm{Fin}) \arrow[r] \arrow[u] & \mathrm{sl}_e(*,\mathcal{J}) \arrow[r] \arrow[u] & \mathrm{sl}_t(*,\mathcal{J}) \arrow[r] \arrow[u] & \mathrm{sl}_t(*,\mathrm{Fin}) = \mathrm{cov}(\mathcal{M}) \arrow[u]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{sl}_e(h, F_{in}) \arrow[r] & \mathrm{sl}_e(h, J) \arrow[r] & \mathrm{sl}_t(h, J) \arrow[r] & \mathrm{sl}_t(h, F_{in}) \\
\mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{b}_J \arrow[u] \arrow[r] & \mathfrak{d}_J \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] \\
\boxed{\mathrm{sl}_t^{\perp}(h, F_{in})} \arrow[u] \arrow[r] & \mathrm{sl}_t^{\perp}(h, J) \arrow[u] \arrow[r] & \mathrm{sl}_e^{\perp}(h, J) \arrow[u] \arrow[r] & \boxed{\mathrm{sl}_e^{\perp}(h, F_{in})} \arrow[u] \\
\mathrm{add}(\mathcal{N}) & & & \mathrm{cov}(\mathcal{M})
\end{tikzcd}
$$

$\boxed{\text{Canjar 1988}}$ $\partial_{\mathcal{J}}$

<u>Lemma (Canjar 1988)</u> In V, $\mathcal{J}$: ideal on ω.

Then $\Vdash_{\mathbb{C}} \mathcal{J} \cup \{\{i<\omega : \dot{c}(i) < x(i)\} : x \in {}^{\omega}\omega \cap V\}$

is an ideal base, generates $\mathcal{J}'$ | $\mathcal{J}' \cap V = \mathcal{J}$.

$\rightarrow$ $\forall x \in {}^{\omega}\omega \cap V : x \leq^{\mathcal{J}'\,d} c$

$\boxed{\mathbb{C}_\lambda} \cong \mathbb{C}_{\lambda+\mu}$ $\overset{\text{regular}}{\mu} \leq \lambda$

$\Vdash \exists J_\mu$ $\mathfrak{b}_{J_\mu} = \mathfrak{d}_{J_\mu} = \mu$

$\boxed{\mathrm{sl}_t(h, J)}$

$\uparrow$

$\mathfrak{d}_J$

Notation.

$h \xrightarrow[J]{} \infty \Leftrightarrow \forall m \cdot \|h \leq m\| \in J$

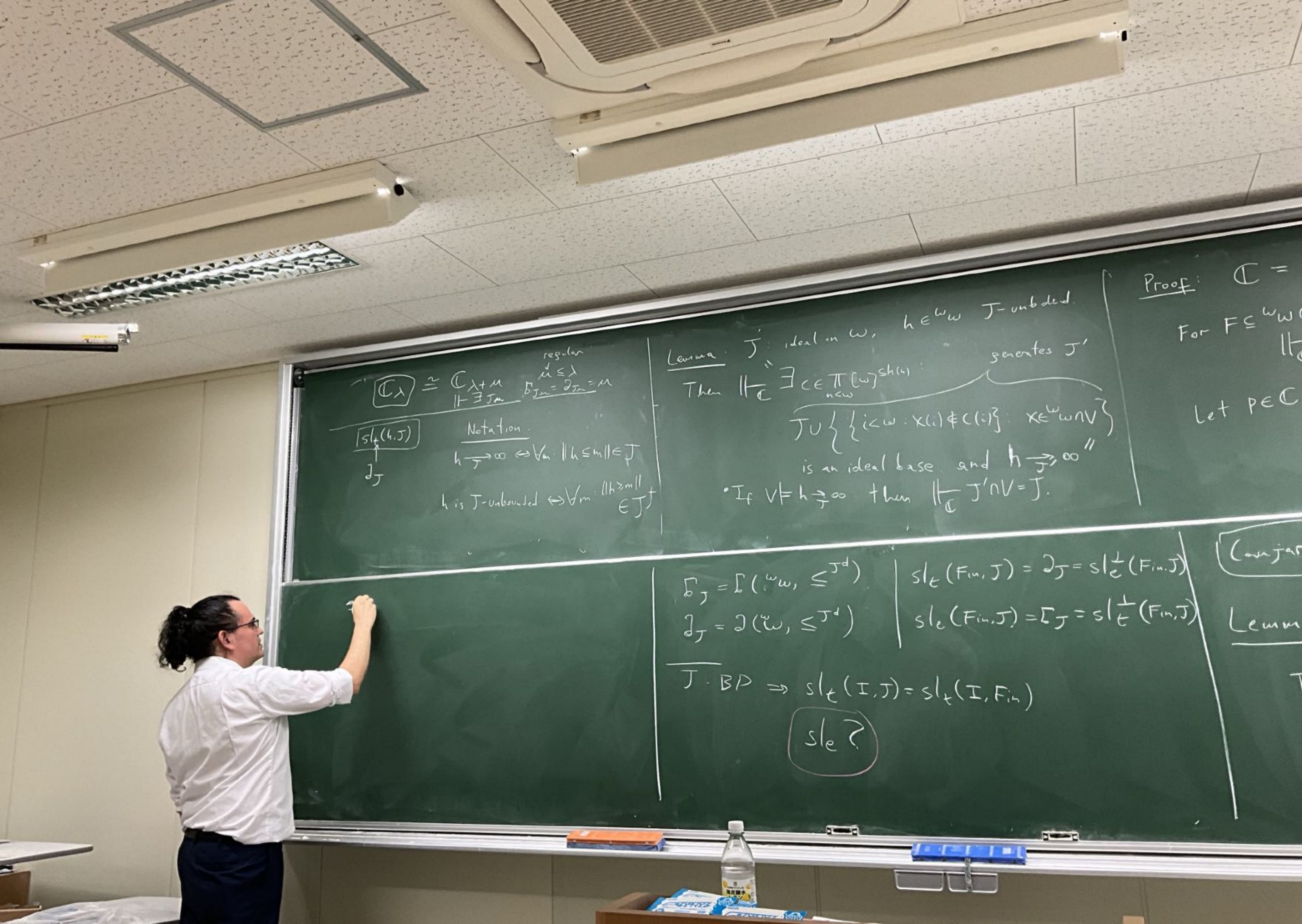
h is J-unbounded $\Leftrightarrow \forall m : \|h \geq m\| \in J^+$

Lemma. J: ideal on ω, $h \in {}^\omega\omega$ J-unbdd.

Then $\Vdash_{\mathbb{C}}$ " $\exists_{c \in \prod_{n<\omega} [\omega]^{\leq h(n)}}$: $\overbrace{J \cup \{\{i<\omega : x(i) \notin c(i)\} : x \in {}^\omega\omega \cap V\}}^{\text{generates } J'}$

is an ideal base and $h \xrightarrow[J']{} \infty$ "

- If $V \models h \xrightarrow[J]{} \infty$ then $\Vdash_{\mathbb{C}} J' \cap V = J$.


$\boxed{\mathbb{C}_\lambda} \simeq \mathbb{C}_{\lambda+\mu}$ $\Vdash \exists J_m$
regular
$\mu \le \lambda$
$\mathfrak{b}_{J_m} = \mathfrak{d}_{J_m} = \mu$
$\boxed{\mathrm{sl}_t(h, J)}$
$\mathfrak{d}_J$
Notation.
$h \xrightarrow[J]{} \infty \Leftrightarrow \forall m \colon \|h \le m\| \in J$
h is J-unbounded $\Leftrightarrow \forall m \colon \|h \ge m\| \in J^+$
Lemma: J: ideal on ω, $h \in {}^\omega\omega$ J-unbdd.
Then $\Vdash_{\mathbb{C}}$ "$\exists c \in \prod_{n<\omega} [\omega]^{\le h(n)}$:
$J \cup \{\{i<\omega : x(i) \notin c(i)\} : x \in {}^\omega\omega \cap V\}$ generates J'
is an ideal base and $h \xrightarrow[J']{} \infty$"
• If $V \models h \xrightarrow[J]{} \infty$ then $\Vdash_{\mathbb{C}} J' \cap V = J$.
Proof: $\mathbb{C} =$
For $F \subseteq {}^\omega\omega$
Let $p \in \mathbb{C}$
$\mathfrak{b}_J = \mathfrak{b}({}^\omega\omega, \le^{J^d})$
$\mathfrak{d}_J = \mathfrak{d}({}^\omega\omega, \le^{J^d})$
$\mathrm{sl}_t(\mathrm{Fin}, J) = \mathfrak{d}_J = \mathrm{sl}_e^\perp(\mathrm{Fin}, J)$
$\mathrm{sl}_e(\mathrm{Fin}, J) = \mathfrak{b}_J = \mathrm{sl}_t^\perp(\mathrm{Fin}, J)$
Conjec
Lemm
J BP $\Rightarrow \mathrm{sl}_t(I, J) = \mathrm{sl}_t(I, \mathrm{Fin})$
sl_e?

Proof: $\mathbb{C} = \prod^{\text{fin}}_{n<\omega} [\omega]^{\leq h(n)}$.

For $F \subseteq {}^\omega\omega \cap V$ and $a \in J$, we show

$$\Vdash_{\mathbb{C}} a \cup \bigcup_{x \in F} \| x \notin C \| \text{ is not cofinite.}$$

Let $p \in \mathbb{C}$ and $m < \omega$.

$$\left.\begin{array}{r} a \cup \operatorname{dom} p \cup m \in J \\ \| h \geq m + |F| \| \in J^+ \end{array}\right\} \| h \geq m + |F| \| \setminus (a \cup \operatorname{dom} p \cup m) \in J^+$$

Pick some i in this set.

$$q := p \cup \{(i, \overbrace{\{x(i) : x \in F\}}^{c(i)})\} \in \mathbb{C}$$

$$\le |F| \le h(i)$$

$$m \le h(i)$$

$$q \Vdash i \ge m \text{ and } i \notin a \cup \bigcup_{x \in F} \| x \notin c \| \cup \underbrace{\| m > h \|}$$

$$h \xrightarrow[J']{} \infty$$

- In the case $V \models h \xrightarrow[J]{} \infty$

$$b \in J^+ \longrightarrow \Vdash b \in (J')^+$$

$\boxtimes$

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$$b \in J^+ \longrightarrow \Vdash b \in (J')^+$$

$\boxtimes$

Lemma In V, J: ideal on ω

$S \subseteq {}^{\omega}\mathcal{P}(\omega)$ with the fupc.

Then $\Vdash_{\mathbb{C}} J \cup \{\|c \in y\| : y \in S\}$ generates an ideal J'

and $J' \cap V = J$.

After adding Cohen reals

- different values of

$$sl_t^{\perp}(h,J) = sl_t(h,J)$$

$$sl_e(*,J) = sl_t(h,J) = \kappa$$

requires

$V \models \lambda^{<\kappa} = \lambda$

expected continuum.

$\mathbb{P}_S$

$\mathbb{P}_A$

$$J_0 \oplus J_1 = \{a_0 \times \{0\} \cup a_1 \times \{1\} : a_0 \in J_0,\ a_1 \in J_1\}$$

ideal on $\omega \times \{0,1\}$

$\boxed{\text{mostly}}$

$$\underset{e}{sl_t}(E_0 \oplus E_1, J_0 \oplus J_1) = \max\{sl_t(E_0, J_0),\ sl_t(E_1, J_1)\}$$

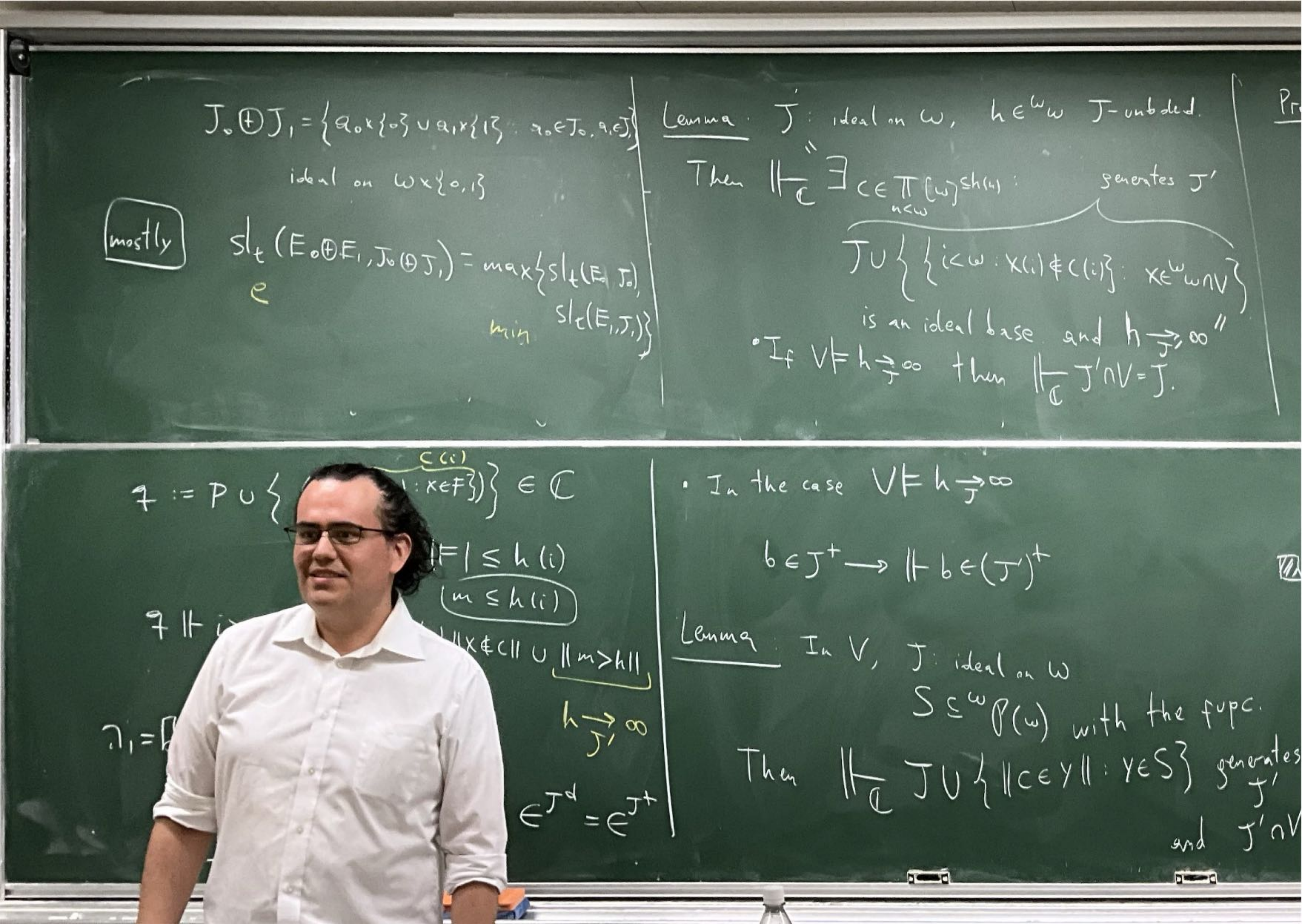
min

<u>Lemma</u>: J: ideal on ω, $h \in {}^\omega\omega$ J-unbdd.

Then $\Vdash_{\mathbb{C}}$ "$\exists_{c \in \prod_{n<\omega} [\omega]^{\leq h(n)}}$: $\overbrace{J \cup \{\{i<\omega : x(i) \notin c(i)\} : x \in {}^\omega\omega \cap V\}}^{\text{generates } J'}$

is an ideal base and $h \xrightarrow[J']{} \infty$"

- If $V \models h \xrightarrow[J]{} \infty$ then $\Vdash_{\mathbb{C}} J' \cap V = J$.



$J_0 \oplus J_1 = \{a_0\times\{0\}\cup a_1\times\{1\} : a_0\in J_0, a_1\in J_1\}$
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$sl_t(E_0\oplus E_1, J_0\oplus J_1) = \max\{sl_t(E_0, J_0), sl_t(E_1, J_1)\}$
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Lemma: J: ideal on ω, $h\in{}^\omega\omega$ J-unbdd.
Then $\Vdash_{\mathbb{C}}$ "$\exists c\in\prod_{n<\omega}[\omega]^{\le h(n)}$: $J\cup\{\{i<\omega : x(i)\notin c(i)\} : x\in{}^\omega\omega\cap V\}$ generates J'
is an ideal base and $h\xrightarrow[J']{}\infty$"
• If $V\models h\xrightarrow[J]{}\infty$ then $\Vdash_{\mathbb{C}} J'\cap V = J$.
Pr
$q := p\cup\{ \ldots : x\in F\})\} \in \mathbb{C}$
$c(i)$
$|F|\le h(i)$
$m\le h(i)$
$q\Vdash i \ldots \|x\notin c\|\cup\|m>h\|$
$h\xrightarrow[J']{}\infty$
$\neg_1 = [$
$\in^{J^d} = \in^{J^+}$
• In the case $V\models h\xrightarrow[J]{}\infty$
$b\in J^+ \longrightarrow \Vdash b\in(J')^+$
Lemma: In V, J: ideal on ω
$S\subseteq{}^\omega\mathcal{P}(\omega)$ with the fupc.
Then $\Vdash_{\mathbb{C}} J\cup\{\|c\in y\| : y\in S\}$ generates J'
and $J'\cap V$

After adding Cohen reals $\mathbb{C}_\lambda$ $\lambda = \beth_\lambda$

- different values of

$$\mathfrak{sl}_t^{\perp}(h, J) = \mathfrak{sl}_t(h, J)$$

$$\mathfrak{sl}_e(x, J) = \mathfrak{sl}_t(h, J) = \kappa$$

$\lambda^{<\lambda} = \lambda$

requires

$V \models \lambda^{<\kappa} = \lambda$

expected continuum.

$\mathbb{P}_\delta$

$\mathbb{P}_A$