ccc ideals and approximation of Solovay sets by Borel sets

<u>Def</u> Let I be a σ-ideal on $\mathrm{Borel}(\mathbb{R})$.
Let $\mathbb{P}_I := \mathrm{Borel}(\mathbb{R})/I$.
We say I is ccc if $\mathbb{P}_I$ is ccc.

Def. $\mathrm{Gen}(\mathcal{I}, N) = \{x \in \mathbb{R} \mid x$ is a $\mathcal{I}$- generic real over $N\}$

$= \{x \mid \forall\, \mathcal{I}$-null Borel set B coded in N $x \notin B\}$.

Def. Let N be a model of ZFC*.

A set $S \subseteq \mathbb{R}$ is called a Solovay set over N if there is a formula φ s.t. with parameters in N

$\forall x \in \mathbb{R}$ with $N[x] \models ZFC^*$ (generic over N)

$x \in S \iff N[x] \models \varphi(x)$

Lem (Solovay)

Let I be a ccc σ-ideal.

Let $N \prec H_\theta$ be a countable model with $I \in N$.

Let S be a Solovay set over N. Then there is a Borel set $B \in N$ s.t.

$\forall x \in \mathrm{Gen}(I, N)\quad x \in S \iff x \in B.$

pf. Take a formula φ defining the Solovay set S.

Let $\dot{x}_{gen}$ be the standard name of I-generic real

$$\underset{V}{\uparrow}^{\mathbb{P}_I}$$

Put $B = [\![\varphi(\dot{x}_{gen})]\!]_I$. $B \in N$.

Fix $x \in Gen(I, N)$.

$$x \in S \iff N[x] \models \varphi(x)$$

$$\iff N[G] \models \varphi(\dot{x}_{gen}[G])$$, where G: $\mathbb{B}_I$-gen. filter corresponding to x

$$\iff [\![\varphi(\dot{x}_{gen})]\!] \in G$$

$$\iff x \in [\![\varphi(\dot{x}_{gen})]\!] = B$$

$\square$

Def.
I is nowhere ccc $\iff$ there is a perfect subset $A \subseteq K(\mathbb{R})$ consiting of I-positive sets.

pairwise disjoint

the space consisting of all compact subsets of $\mathbb{R}$

Prop. Let I be a nowhere ccc σ-ideal.
Let $N \prec H_\theta$ be countable, $I \in N$.
Then there is a S be a Solovay set over N. s.t.

no Borel set $B \in N$ satisfies:

$$\forall x \in \mathrm{Gen}(\mathbb{I}, N) \quad x \in \mathcal{S} \iff x \in B.$$

<u>pf.</u> Let $\mathcal{A} \subseteq K(\mathbb{R})$ be a witness of nowhere ccc, $\mathcal{A} \in N$. Let $\mathcal{S} = \bigcup(\mathcal{A} \cap N)$.

claim $\mathcal{S}$ is a Solovay set over N.

Fix $x \in \mathbb{R}$ s.t. $N[x] \models ZFC^*$. Let G be a $\mathrm{Coll}(\aleph_0, 2^{\aleph_0})$-generic filter over $N[x]$.

$\Theta \cap N \in N[x][G]$ & $N[x][G] \models \Theta \cap N$: ctble

$x \in \mathcal{S} \iff (\exists K \in \Theta \cap N)(x \in K)$ Σ^1_1 as inputs are x and K.

$\iff N[x][G] \models (\exists K \in \Theta \cap N)(x \in K)$

$\iff N[x] \models$ "$\mathbb{1} \Vdash_{\mathrm{Coll}(\aleph_0, 2^{\aleph_0})} (\exists K \in \Theta \cap N)(x \in K)$".

G: $\mathrm{Coll}(\aleph_0, 2^{\aleph_0})$-gen fil over N

$\Rightarrow N[G] \models N \cap \mathbb{R}$ is countable.

Let $B \in N$ be a Borel set.

Suppose $B \triangle \dot{S} \in I$.

$$\mathcal{A} \cap N = \{A \in \mathcal{A} \mid A \cap B \in I^+\},$$

$$\overset{\mathcal{A}}{\underset{N}{\in}} \qquad \overset{\cap}{N}.$$

$\therefore B \triangle \dot{S} \in I^+$. $\mathrm{Gen}(I, N) \cap (\dot{S} \triangle B) \neq \emptyset$

$\square$

$\mathcal{B} := \mathcal{A} \cap N$.

Suppose $\mathcal{B} \in N$

$\exists K \in \mathcal{A}\ K \notin \mathcal{B}$.

$N \models \exists K \in \mathcal{A}\ K \notin \mathcal{B}$.

Note.

(1-step) Laver forcing preserves Hausdorff (outer) measures.

(1-step) Laver forcing satisfies $(\star)^f$ which is a stronger property than preserving Hausdorff measure.

$\mathbb{P}_{\mathcal{N}^f}$ ... nowhere ccc

$(\star)^f$ is preserved by csi.

$\mathrm{Con}(\mathrm{GP}(\mathcal{N}^f))$

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