Preservation of positive Hausdorff measure

Thm. (Woodin, Judah-Shelah, Pawlikowski)

Laver forcing preserves Lebesgue outer measure

i.e. If $A \subseteq 2^\omega$ then $\mathbb{L} \Vdash \mu^*(A) = \mu^*(A)^V$.

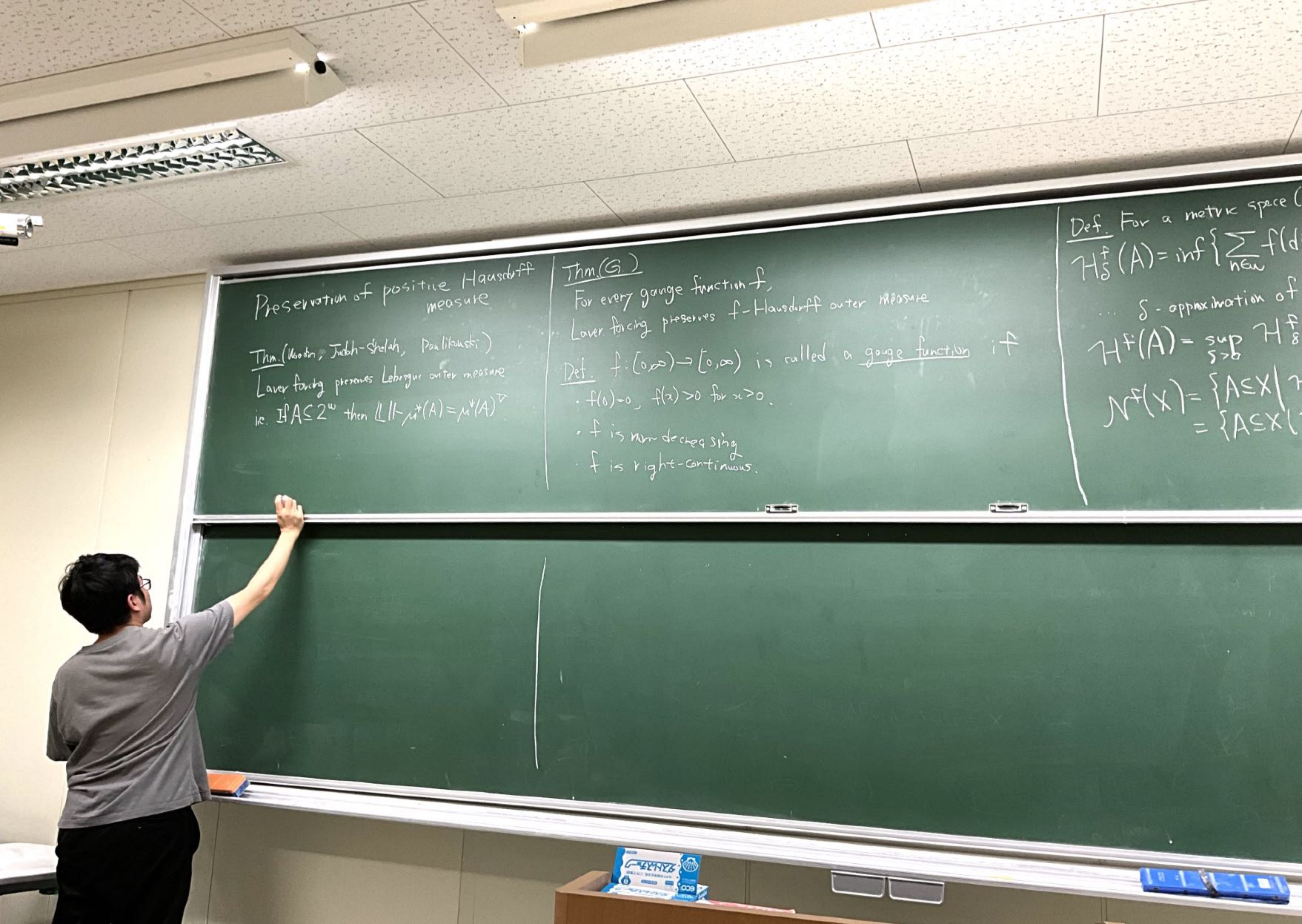
Thm. (G.)

For every gauge function f,

Laver forcing preserves f-Hausdorff outer measure

Def. $f : (0,\infty) \to [0,\infty)$ is called a gauge function if

- $f(0)=0$, $f(x)>0$ for $x>0$.
- f is non-decreasing
- f is right-continuous.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Def. For a metric space (X,d), $A \subseteq X$, f: gauge, $\delta \in (0,\infty]$, let

$$\mathcal{H}^f_\delta(A) = \inf\left\{ \sum_{n\in\omega} f(\mathrm{diam}(C_n)) \;\middle|\; C_n \subseteq X,\ A \subseteq \bigcup_n C_n,\ \mathrm{diam}(C_n) \leq \delta \right\}.$$

… δ-approximation of f-Hausdorff measure.

$$\mathcal{H}^f(A) = \sup_{\delta>0} \mathcal{H}^f_\delta(A).$$ … f-Hausdorff (outer) measure

$$\mathcal{N}^f(X) = \{A \subseteq X \mid \mathcal{H}^f(A) = 0\}$$
$$= \{A \subseteq X \mid \mathcal{H}^f_\infty(A) = 0\}.$$

Metrize the Cantor space 2^ω as follows:

$$d(x,y) = 2^{-\min\{n \mid x(n) \neq y(n)\}} \quad \text{for } x \neq y \text{ in } 2^\omega.$$

When the space is $(2^\omega, d)$, write $\mathcal{N}^f := \mathcal{N}^f(2^\omega, d)$

If $f(x) = x$ then $\mathcal{N}^f = \mathcal{N}$.

If $f(x) = x^{1.001} \Rightarrow \mathcal{N}^f = \mathcal{P}(2^\omega)$

$f(x) = x^r$ $(0 < r < 1)$.

Def.

An outer measure μ^* on a space X satisfies increasing sets lemma if

$$\mu^*\Big(\bigcup_{n\in\omega} A_n\Big) = \sup_{n\in\omega} \mu^*(A_n)$$

for increasing sequence $(A_n \mid n\in\omega) \subseteq \mathcal{P}(X)$.

Thm. (Davies)

$\mathcal{H}^f_\delta$ satisfies the increasing sets lemma

if the space X is compact and f is continuous.

Def.

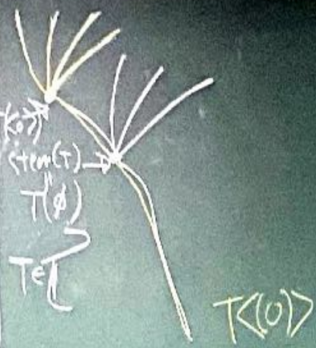
$\mathbb{L} = \{ T \mid T \subseteq \omega^{<\omega}$ perfect subtree

& $\forall t \supseteq \mathrm{stem}(T)$ in T $|\mathrm{succ}_T(t)| = \aleph_0 \}$

$T' \leq T \iff T' \subseteq T$

For $T \in \mathbb{L}$ and $t \in T$, $T_t = \{ s \in T \mid s \subseteq t$ or $t \subseteq s \}$

For $T \in \mathbb{L}$ and $\tau \in \omega^{<\omega}$, $T(\tau)$ denotes the image of τ under the cannonical isomorphism from $\omega^{<\omega}$ into T.



For $T \in \mathbb{L}$ and $\tau \in \omega^{<\omega}$,

$T\langle \tau \rangle = T_{T(\tau)}$.

For $S, T \in \mathbb{L}$ and $n \in \omega$,

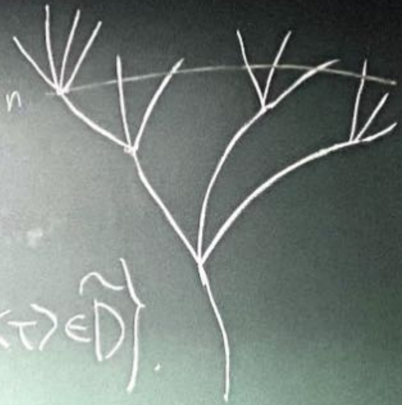
$S \leq_n T :\iff S \leq T$ & $\forall \tau \in \omega^n \; S(\tau) = T(\tau)$.

Def.

For an open subset D of $\mathbb{L}$:

$$\tilde{D} = \{T \in \mathbb{L} \mid \forall S \le_0 T \ S \notin D\} \cup D$$

$$D^* = \{T \in \mathbb{L} \mid \exists n \in \omega \ \forall \tau \in \omega^{<\omega} \ |\tau| > n \to T\langle\tau\rangle \in \tilde{D}\}.$$



Lem. 1

If $D \subseteq \mathbb{L}$ is open and $\mathcal{S} \le T \in D^*$ and D is nonempty below $\mathcal{S}$ then there is $s \in \mathcal{S}$ s.t. $T_s \in D$.

pf. Fix $R \le \mathcal{S}$ s.t. $R \in D$.

Fix $n \in \omega$ s.t. $\forall_{\tau \in \omega^{<\omega}}\ |\tau| > n \Rightarrow T\langle\tau\rangle \in \widehat{D}$.

Taking a sufficient long $s \in R$, we have

$R_s \le T_s$ & $R_s \in D$ $\therefore T_s \in D$. $\square$

Lem. 2

(1) If $D \in \mathbb{L}$ is open, $n \in \omega$ and $T \in \mathbb{L}$, then there is $\mathcal{S} \le_n T$ s.t. $\mathcal{S} \in D^*$

(2) If $D_i \in \mathbb{L}$ are open ($i \in \omega$), $n \in \omega$, $T \in \mathbb{L}$ then there is $\mathcal{S} \le_n T$ s.t. $\mathcal{S} \in \bigcap_i D_i^*$.

Note. That $\mathbb{L}$ is proper can be deduced by these lemmas.

Lem. 3 Let μ^* be an outer measure on 2^ω s.t. the increasing sets lemma holds for μ^*. Let $a \in (0, \infty)$.

Let $(A_\sigma \mid \sigma \in \omega^{<\omega}) \subseteq \mathcal{P}(2^\omega)$ s.t.

(1) $\mu^*(A_\sigma) \leq a$

(2) $A_\sigma \subseteq \liminf_{n \in \omega} A_{\sigma^\frown n} = \bigcup_{m \in \omega} \bigcap_{n \geq m} A_{\sigma^\frown n}$.

Then $\mu^*\left(\bigcap_{T\in\mathbb{L}}\bigcup_{\sigma\in T} A_\sigma\right) \le a$.

pf. Define $\langle A^\alpha_\sigma \mid \sigma\in\omega^{<\omega}, \alpha\in\omega_1\rangle$.

$$A^0_\sigma = A_\sigma$$

$$A^{\alpha+1}_\sigma = \liminf_{n\in\omega} A^\alpha_{\sigma^\frown n}$$

$$A^\lambda_\sigma = \bigcup_{\alpha<\lambda} A^\alpha_\sigma \quad (\lambda:\text{limit})$$

$\langle A^\alpha_\sigma \mid \alpha\in\omega_1\rangle$: increasing for every $\sigma\in\omega^{<\omega}$.

$\mu^*(A^\alpha_\sigma) \le a$ ← increasing sets lemma.

$\exists\alpha_\sigma\in\omega_1$ $\mu^*(A^\beta_\sigma)=\mu^*(A^\alpha_\sigma)$ for $\beta,\alpha\ge\alpha_\sigma$.

$\exists\alpha\in\omega_1$ ——— // ——— for $\sigma\in\omega^{<\omega}$.

We have $\mu^*(A^{\alpha+1}_\sigma\setminus A^\alpha_\sigma)=0$.

$$\therefore \mu^*\Big(A_\phi^{\alpha+1} \cup \bigcup_{\sigma \in \omega^{<\omega}} (A_\sigma^{\alpha+1} \setminus A_\sigma^\alpha)\Big) \le a.$$

<u>claim</u> $$\bigcap_{T \in \mathcal{L}} \bigcup_{\sigma \in T} A_\sigma \subseteq A_\phi^{\alpha+1} \cup \bigcup_{\sigma \in \omega^{<\omega}} (A_\sigma^{\alpha+1} \setminus A_\sigma^\alpha)$$

$\because$ $x \notin A_\phi^{\alpha+1} \cup \bigcup_{\sigma \in \omega^{<\omega}} (A_\sigma^{\alpha+1} \setminus A_\sigma^\alpha)$.

$x \notin A_\phi^{\alpha+1} = \liminf_{n \in \omega} A_{\langle n \rangle}^\alpha$. $\exists^\infty n \; x \notin A_{\langle n \rangle}^\alpha$.

$\therefore \exists^\infty n \; x \notin A_{\langle n \rangle}^{\alpha+1}$

If $x \notin A^{\alpha+1}_{\langle n \rangle}$,

$\exists^\infty m \; x \notin A^{\alpha}_{\langle n, m \rangle}$. $\therefore \exists^\infty n \exists^\infty m \; x \notin A^{\alpha}_{\langle n, m \rangle}$.

In this way, we construct $T \in \mathbb{L}$ s.t.

$x \notin \bigcup_{\sigma \in T} A^{\alpha+1}_{\sigma}$. $\therefore x \notin \bigcup_{\sigma \in T} A^{\alpha}_{\sigma}$.

$\square$

Lem. 2

(1) If $D \in \mathbb{L}$ is open $n \in \omega$ and $T \in \mathbb{L}$.

then there is $S \leq_n T$ s.t. $S \in D^*$.

(2) If $D_i \in \mathbb{L}$ are open $(i \in \omega)$, $n \in \omega$, $T \in \mathbb{L}$

then there is $S \leq_n T$ s.t. $S \in \bigcap_i D_i^*$.