A Journey guided by the Stars
Part II

Forcing "NS_{ω_1} is ω_1-dense" from Large Cardinals

Andreas Lietz

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Recap, Dense Ideals

An ideal I on ω_1 is a collection of "small subsets" of ω_1.

- $\emptyset \in I$, $\omega_1 \notin I$.
- $X \subseteq Y \in I \Rightarrow X \in I$
- in this talk: if $X_n \in I$ for $n < \omega$ then $\bigcup_{n<\omega} X_n \in I$
- in this talk: all countable subsets of ω_1 are in I and I is normal.

I induces equivalence relation on $\mathcal{P}(\omega_1)$, $X \sim_I Y$ if $X \triangle Y \in I$. $\mathbb{P}_I = (\mathcal{P}(\omega_1)/\sim_I)^+$ with order induced by $\subseteq$.

ω_1-dense $\Rightarrow$ saturated $\Rightarrow$ precipitous.

Definition

An ideal I on ω_1 is ω_1-dense if $\mathbb{P}_I$ has a dense subset of size ω_1.

Special focus on NS_{ω_1}, the nonstationary ideal on ω_1. NS_{ω_1} is ω_1-dense if $\exists \langle S_i \mid i < \omega_1 \rangle$ sequence of stationary sets so that for all stationary $T \subseteq \omega_1$, $\exists i < \omega \exists C \subseteq \omega_1$ club $S_i \cap C \subseteq T$.

Recap, Woodin's Results

Theorem (Woodin, late 70s)

Assume $V \models \mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} +$ "Θ is regular". Then in a forcing extension $\mathsf{ZFC} + \mathsf{CH} +$ "there is a dense ideal on ω_1" holds.

Theorem (Woodin)

Suppose there is an almost huge cardinal. Then in a forcing extension there is a dense ideal on ω_1 and CH hold.

Theorem (Woodin)

Assume $L(\mathbb{R}) \models \mathsf{AD}$. Then in a forcing extension of $L(\mathbb{R})$, $\mathsf{ZFC} +$ "NS_{ω_1} is ω_1-dense" holds.
In fact, if there are a proper class of Woodin cardinals then "NS_{ω_1} is ω_1-dense" is Ω-consistent, i.e. $\mathsf{ZFC} \not\vdash_\Omega \neg($"$\mathsf{NS}_{\omega_1}$ is ω_1-dense"$)$.

The Ω-Conjecture suggests that "NS_{ω_1} is ω_1-dense" can be forced from large cardinals.

Recap, Main Result

Question (Woodin, late 90's)

Assume some large cardinal. Is there a stationary set preserving partial order $\mathbb{P}$ so that

$$V^{\mathbb{P}} \models \text{“}\mathsf{NS}_{\omega_1} \text{ is } \omega_1\text{-dense”} \ ?$$

Theorem (L.)

Suppose there is an inaccessible cardinal κ which is a limit of $<\kappa$-supercompact cardinals. Then there is a stationary set preserving forcing $\mathbb{P}$ with

$$V^{\mathbb{P}} \models \text{“}\mathsf{NS}_{\omega_1} \textit{ is } \omega_1\textit{-dense”}.$$

Recap, Strategy

Idea: By Asperó-Schindler $\mathsf{MM}^{++} \Rightarrow (*)$ suggests to solve for x:

$$\frac{\mathsf{MM}^{++}}{(*)} = \frac{x}{\mathbb{Q}_{\mathsf{max}}\text{-}(*)}$$

Theorem (L.)

There is a forcing axiom QM *so that*

- QM *can be forced by stationary set preserving forcing from a supercompact limit of supercompact cardinals.*
- QM *implies* $\mathbb{Q}_{\mathsf{max}}$-$(*)$ *(in particular "*NS_{ω_1} *is* ω_1*-dense").*

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NS_{ω_1} is ω_1-dense $\Leftrightarrow \exists \pi\colon \mathrm{Col}(\omega, \omega_1) \to \mathbb{P}_{\mathrm{NS}_{\omega_1}}$ dense embedding.

Definition

Suppose $\mathbb{B}$ is a forcing of size $\leqslant \omega_1$. $\diamondsuit(\mathbb{B})$ holds if there is an embedding $\pi\colon \mathbb{B} \to \mathcal{P}(\omega_1) \setminus \mathrm{NS}_{\omega_1}$ so that $\forall p \in \mathbb{B}$ there are stationarily many countable $X \prec H_{\omega_2}$ with

$$p \in \{q \in \mathbb{B} \cap X \mid \omega_1 \cap X \in \pi(q)\} \text{ is a filter generic over } X.$$

$\diamondsuit(\omega_1^{<\omega})$ is $\diamondsuit(\mathrm{Col}(\omega, \omega_1))$.

Definition

QM holds if $\exists \pi$ witnessing $\diamondsuit(\omega_1^{<\omega})$ and $\mathrm{FA}_{\omega_1}(\{\mathbb{P} \mid \mathbb{P} \text{ preserves } \pi\})$ holds, i.e. whenever $V^{\mathbb{P}} \models$ "π witnesses $\diamondsuit(\omega_1^{<\omega})$" and $\langle D_i \mid i < \omega_1 \rangle$ are dense subsets of $\mathbb{P}$, there is a $\mathbb{P}$-filter meeting all D_i.

Recap, Q-Iterations

Definition

Suppose π witnesses $\diamondsuit(\omega_1^{<\omega})$. A Q-iteration is a nice iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ of π-preserving forcing so that

- For $\alpha < \gamma$ successor

$$V^{\mathbb{P}_\alpha} \models \text{“}\dot{\mathbb{Q}}_\alpha \text{ forces SRP and that } \mathrm{ran}(\pi) \text{ is dense for old sets”}$$

- For $\alpha < \gamma$ limit no further requirement on $\dot{\mathbb{Q}}_\alpha$.

Work-Life-Balance Theorem (L.)

Q-iterations preserve π.

Today's Goal

Lemma

Suppose π witnesses $\diamondsuit(\mathbb{B})$ and there is a supercompact cardinal. Then SRP *holds in an π-preserving extension.*

Lemma

Suppose π witnesses $\diamondsuit(\omega_1^{<\omega})$ and there are two Woodin cardinals with a measurable above. The there is an π-preserving forcing $\mathbb{P}$ which makes "π dense for old sets".

π-Stationary Sets

Proposition (Folklore)

Suppose $S \subseteq \omega_1$. The following are equivalent:

1. *S is stationary.*
2. *For any club $\mathcal{C} \subseteq [H_{\omega_2}]^\omega$, there is $X \prec H_{\omega_2}$ with $X \in \mathcal{C}$ and $X \cap \omega_1 \in S$.*

Definition

Suppose π witnesses $\diamondsuit(\mathbb{B})$. A countable $X \prec H_\theta$ is π-**slim** if $\{q \in \mathbb{B} \mid \omega_1 \cap X \in \pi(q)\}$ is a filter generic over X. A subset $S \subseteq \omega_1$ is π-**stationary** if for any club $\mathcal{C} \subset [H_{\omega_2}]^\omega$ there is a π-slim $X \prec H_{\omega_2}$ with $X \in \mathcal{C}$ and $\omega_1 \cap X \in S$.

NS_π is the ideal of π-nonstationary sets.

$\diamondsuit$-Forcing

$$\begin{tikzcd}
\text{Classical} & \diamondsuit\text{-Forcing} \\
\text{complete}(\approx \sigma\text{-closed}) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \pi\text{-complete} \arrow[d, Rightarrow] \\
\text{proper} \arrow[d, Rightarrow] & \pi\text{-proper} \arrow[d, Rightarrow] \\
\text{semiproper} \arrow[d, Rightarrow] & \pi\text{-semiproper} \arrow[d, Rightarrow] \\
\text{stationary set preserving} \arrow[d, Rightarrow] & \pi\text{-stationary set preserving} \arrow[d, Rightarrow] \\
\omega_1\text{-preserving} & \pi\text{-preserving} \arrow[l, Rightarrow]
\end{tikzcd}$$

σ-Closed Forcing preserves π

Proposition

Suppose π witnesses $\diamondsuit(\mathbb{B})$. Every σ-closed forcing preserves π.

Proof.

Suppose θ is sufficiently large, regular. Let $X \prec H_\theta$ be π-slim, i.e. countable and

$$g := \{p \in \mathbb{B} \cap X \mid \delta^X \in \pi(p)\} \text{ is a filter and generic over } X.$$

Suppose $\mathbb{B}, \mathbb{P} \in X$ and $p \in \mathbb{P} \cap X$. We will find $q \leqslant p$ so that

$$q \Vdash \text{“}\check{X} \sqsubseteq \check{X}[\dot{G}] \wedge \check{X}[\dot{G}] \text{ is } \pi\text{-slim”}.$$

Let $\rho \colon M_X \to X$ be the anticollapse. Let $\bar{g} = \rho^{-1}[g]$. Then $\bar{g}$ is generic over M_X. If $a \in X$ then $\bar{a} = \rho^{-1}(a)$.

σ-Closed Forcing preserves π

Proof (continued).

Let $\langle q_n \mid n < \omega \rangle$ be a descending sequence in $\bar{\mathbb{B}}$ with

- $\rho(q_0) = p$ and
- for any dense $D \subseteq \bar{\mathbb{B}}$ with $D \in M_X[\bar{g}]$, have some $q_n \in D$.

As $\mathbb{P}$ is σ-closed, can find some $q \leqslant \rho(q_n)$, all $n < \omega$.
Now suppose G is $\mathbb{P}$-generic with $q \in G$. Then $\bar{G} = \rho^{-1}[G]$ is generic over $M_X[\bar{g}]$, in particular over M_X. So $X[G] \cap V = X$.
By the product lemma, $\bar{g}$ is generic over $M_X[\bar{G}] = M_{X[G]}$. Hence g is generic over $X[G]$, so $X[G]$ is π-slim. $\square$

Isolating the properties of q leads to π-semiproper forcing.

π-Semiproper Forcing

Definition

Suppose π witnesses $\diamondsuit(\mathbb{B})$. A forcing $\mathbb{P}$ is π-semiproper if for all large enough regular θ and all π-slim $X \prec H_\theta$ with $\mathbb{P} \in X$: If $p \in \mathbb{P} \cap X$ then there is $q \leqslant p$ with

$$q \Vdash \text{“}\check{X} \sqsubseteq \check{X}[\dot{G}] \wedge \check{X}[\dot{G}] \text{ is } \check{\pi}\text{-slim”}.$$

Have σ-closed$\Rightarrow$ π-semiproper$\Rightarrow$ π-preserving.

Theorem (L.)

Nice iterations of π-semiproper forcing are π-semiproper.

Forcing SRP

Suppose π witnesses $\diamondsuit(\mathbb{B})$.

- There is a standard forcing to force "an instance of SRP".
- The same argument which shows this is semiproper shows it is π-semiproper.
- Iterating these up to a supercompact gives SRP.

In fact can get a forcing axiom this way:

Definition

$\mathsf{MM}(\pi)$ is $\mathsf{FA}_{\omega_1}(\{\mathbb{P} \mid \mathbb{P}$ is π-semiproper$\})$.

Corollary

Suppose there is a supercompact cardinal. Then $\mathsf{MM}(\pi)$ *holds in a forcing extension by* π*-semiproper forcing.*

Unintended Consequences

The iteration theorem for π-semiproper forcing generates known iteration theorems.
If $\mathbb{B} = \{0\}$ is the trivial forcing and $\pi(0) = \omega_1$, then get:

Theorem (Miyamoto)

Nice iterations of semiproper forcing are semiproper.

If $\mathbb{B}$ is a Suslin tree then (with a small trick) get:

Theorem (Miyamoto)

Suppose T is a Suslin tree. Then nice iterations of semiproper T-preserving forcings preserve T.

Generic Iterations

Definition

Suppose

- M is a countable transitive model of (sufficiently much of) ZFC.
- $M \models$ "I is an ideal on ω_1"

A generic iteration of (M, I) is a sequence $\langle (M_\alpha, I_\alpha), \mu_{\alpha,\beta} \mid \alpha \leqslant \beta \leqslant \gamma \rangle$ with

- $(M_0, I_0) = (M, I)$
- $\mu_{\alpha,\alpha+1} : M_\alpha \to M_{\alpha+1}$ is a generic ultrapower of M_α w.r.t I_α
- If $\alpha \in \mathrm{Lim}$ then

$$\langle M_\alpha, \mu_{\beta,\alpha} \mid \beta < \alpha \rangle = \varinjlim \langle M_\beta, M_{\beta,\xi} \mid \beta \leqslant \xi < \alpha \rangle$$

(M, I) is generically iterable if all (countable) generic iterations of (M, I) produce wellfounded models.

$\mathbb{P}_{\max}$-Variations

Assume $\mathbb{V}_{\max}$ is a $\mathbb{P}_{\max}$-variation say

- Conditions are of the form $p = (M, I, a)$
- (M, I) is generically iterable.
- $a \in M$ has some first order property in (M, I).

and $q = (N, J, b) <_{\mathbb{V}_{\max}} (M, I, a) = p$ if $p \in N$ is countable in N and there is a generic iteration (map)

$$j : p \to p^* = (M^*, I^*, a^*)$$

in q, so that some first order formula is absolute between $(M^*; \in, I^*, a^*)$ and $(N; \in, J, b)$, e.g.:

- $\omega_1^{M^*} = \omega_1^N$
- $a^* = b$
- Maybe $I^* = J \cap M^*$

Also assume $\mathbb{V}_{\max}, \leqslant_{\mathbb{V}_{\max}}$ are projective.

$(*)$-Forcing

Assume that

- NS_{ω_1} is saturated.
- $\mathcal{H} = (H_{\omega_2}, \mathsf{NS}_{\omega_1}, A)$ is "almost a $\mathbb{V}_{\mathsf{max}}$-condition", i.e.

$$V^{\mathsf{Col}(\omega, 2^{\omega_1})} \models \mathcal{H} \in \mathbb{V}_{\mathsf{max}}$$

- $D \in L(\mathbb{R})$ is dense in $\mathbb{V}_{\mathsf{max}}$ and universally Baire.

Let $g = \{p \in \mathbb{V}_{\mathsf{max}} \mid V^{\mathsf{Col}(\omega, 2^{\omega_1})} \models \mathcal{H} <_{\mathbb{V}_{\mathsf{max}}} p\}$. Goal: Show that g witnesses $\mathbb{V}_{\mathsf{max}}$-$(*)$ from appropriate forcing axiom.

Asperó-Schindler constructed a forcing $\mathbb{P}$ so that in $V^{\mathbb{P}}$ the following picture exists:

$$\begin{tikzcd}
& D^* & \\
& q_0 \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, "\sigma_{0,\omega_1}"] & q_{\omega_1} = (N^*, I^*, b^*) \\
p_0 \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] \arrow[r, "\mu_{0,\omega_1^N}"] & p_{\omega_1^N} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, "\mu_{\omega_1^N,\omega_1}"] & p_{\omega_1} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[d, equal] \\
\mathbb{V}_{\mathrm{max}} & & ((H_{\omega_2})^V, (\mathrm{NS}_{\omega_1})^V, A) = \mathcal{H}
\end{tikzcd}$$

- μ_{0,ω_1^N} witnesses $q_0 <_{\mathbb{V}_{\mathrm{max}}} p_0$ and $\mu_{0,\omega_1} = \sigma_{0,\omega_1}(\mu_{0,\omega_1^N})$.
- The top iteration $q_0 \to q_{\omega_1}$ is *correct* in $V^{\mathbb{P}}$, i.e. $I^* = (\mathrm{NS}_{\omega_1})^{V^{\mathbb{P}}} \cap N^*$.

Definition

A generic iteration $\langle (M_\alpha, I_\alpha), \mu_{\alpha,\beta} \mid \alpha \leqslant \beta \leqslant \omega_1 \rangle$ is a $\diamondsuit$-iteration if: For any sequence $\langle D_i \mid i < \omega_1 \rangle$ of dense subsets of $(\mathbb{P}_{I_{\omega_1}})^{M_{\omega_1}}$ and any $S \in I^+_{\omega_1} \cap M_{\omega_1}$ have

$$\{\alpha \in S \mid \forall i < \alpha \; G_\alpha \cap \mu^{-1}_{\alpha,\omega_1}[D_i] \neq \emptyset\} \in \mathrm{NS}^+_{\omega_1}$$

where G_α is the $\mathbb{P}_{I_\alpha}$-generic filter over M_α used in this iteration.

Theorem (L.)

Can modify Asperó-Schindler's $\mathbb{P}$ to $\mathbb{P}_\diamondsuit$ so that in $V^{\mathbb{P}_\diamondsuit}$ the same picture as before exists and top iteration $q_0 \to q_{\omega_1}$ is a $\diamondsuit$-iteration in $V^{\mathbb{P}_\diamondsuit}$.

Lemma

Suppose $\langle (M_\alpha, (\mathsf{NS}_{\omega_1})^{M_\alpha}), \mu_{\alpha,\beta} \mid \alpha \leqslant \beta \leqslant \omega_1 \rangle$ *is a* $\diamondsuit$*-iteration. If*

$$M_{\omega_1} \models \text{“}\pi \text{ witnesses } \diamondsuit(\omega_1^{<\omega}) \text{ and } [\cdot]_{\mathsf{NS}_{\omega_1}} \circ \pi \text{ is a complete embedding (say } \pi \text{ witnesses } \diamondsuit^+(\omega_1^{<\omega}))\text{”}$$

then $(\mathsf{NS}_{\omega_1})^{M_{\omega_1}} = \mathsf{NS}_\pi \cap M_{\omega_1}$*. In particular* π *witnesses* $\diamondsuit(\omega_1^{<\omega})$.

Proof.

- Let $S \in (\mathsf{NS}^+_{\omega_1})^{M_{\omega_1}}$. Have to show that S is π-stationary.
- Let $\mathcal{C} \subseteq [H_{\omega_2}]^\omega$ club. Find $\langle X_i \mid i < \omega_1 \rangle$ continuous increasing sequence of countable elementary substructures of H_{ω_2}, all in $\mathcal{C}$.
- Let $\langle D_i \mid i < \omega_1 \rangle$ enumerate all dense subsets of $\mathsf{Col}(\omega, \omega_1)$ appearing in the X_i.
- By assumption, $\hat{\pi} := [\cdot]_{\mathsf{NS}^{M_{\omega_1}}_{\omega_1}} \circ \pi : \mathsf{Col}(\omega, \omega_1) \to ((\mathcal{P}(\omega_1)^{M_{\omega_1}}/\mathsf{NS}_{\omega_1})^+)^{M_{\omega_1}}$ is a complete embedding.

Proof continued.

- $E_i = \hat{\pi}[D_i] \downarrow$ is dense.
- $\Rightarrow T = \{\alpha \in S \mid \forall i < \alpha \; G_\alpha \cap \mu^{-1}_{\alpha,\omega_1}[E_i] \neq \emptyset\} \in \mathsf{NS}^+_{\omega_1}$
- Find $\alpha \in T$ with
  - $\omega_1^{M_\alpha} = \alpha = \omega_1 \cap X_\alpha$
  - $\langle D_i \mid i < \alpha \rangle$ enumerates all dense subsets of $\mathrm{Col}(\omega, \omega_1)$ in X_α
  - $\pi = \mu_{\alpha,\omega_1}(\bar{\pi})$, some $\bar{\pi} \in M_\alpha$.
- For $p \in \mathrm{Col}(\omega, \omega_1^{M_\alpha})$, have $\omega_1^{M_\alpha} \in \pi(p)$ iff $[\bar{\pi}(p)]_{\mathsf{NS}^{M_\alpha}_{\omega_1}} \in G_\alpha$ (generic used in step α of iteration).
- Hence $\{p \in \mathrm{Col}(\omega, \omega_1) \cap X_\alpha \mid \alpha \in \pi(p)\}$ meets all D_i with $i < \alpha$, so is generic over X_α.

$\square$

This suggests taking $\mathbb{V}_{\max}$ conditions are of the form (M, I, π) and

$$M \models \text{“}\pi \text{ witnesses } \diamondsuit^+(\omega_1^{<\omega})\text{”}$$

Definition

$\mathbb{Q}_{\max}^{-}$-conditions are of the form $(M, \mathsf{NS}_{\omega_1}^M, \pi)$ with:

- $(M, \mathsf{NS}_{\omega_1}^M)$ is generically iterable.
- $M \models$"π witnesses $\diamondsuit^+(\omega_1^{<\omega})$"

$q = (N, \mathsf{NS}_{\omega_1}^N, \pi) <_{\mathbb{Q}_{\max}^{-}} (M, \mathsf{NS}_{\omega_1}^M, \mu) = p$ iff $p \in N$ is countable in N and there is a generic iteration (map)

$$j : p \to p^* = (M^*, \mathsf{NS}_{\omega_1}^{M^*}, \mu^*)$$

such that:

- $\mu^* = \pi$
- π is dense for sets in M^*: If $S \in \mathcal{P}(\omega_1)^{M^*} \setminus \mathsf{NS}_{\omega_1}^N$ then have

$$\pi(p) \subseteq S \mod \mathsf{NS}_{\omega_1}^N$$

  for some $p \in \mathrm{Col}(\omega, \omega_1)^N$.

Do not need whole $\mathbb{P}_{\max}$-machinery/theory.
Only crucial property of $\mathbb{Q}_{\max}^{-}$ for us is:

Lemma

Assume there is a Woodin cardinal and a measurable above. Then $\mathbb{Q}_{\max}^{-} \neq \emptyset$ and for all $p \in \mathbb{Q}_{\max}^{-}$ there is $q \in \mathbb{Q}_{\max}^{-}$ with $q < p$.

$\mathbb{P}_\diamondsuit$ for $\mathbb{Q}_{\max}^-$

Assume NS_{ω_1} is saturated and $\mathcal{H} = (H_{\omega_2}, \mathrm{NS}_{\omega_1}, \pi)$ is almost in $\mathbb{Q}_{\max}^-$. In $V^{\mathbb{P}_\diamondsuit}$:

$$\begin{tikzcd}
 & \rotatebox[origin=c]{90}{$\in$} & & \\
 & q_0 \arrow[rr, "\sigma_{0,\omega_1}"] & & q_{\omega_1} = (N^*, \mathrm{NS}_{\omega_1}^{N^*}, \nu^*) \\
 & \rotatebox[origin=c]{90}{$\in$} & & \rotatebox[origin=c]{90}{$\in$} \\
p_0 \arrow[r, "\mu_{0,\omega_1^N}"] & p_{\omega_1^N} \arrow[rr, "\mu_{\omega_1^N,\omega_1}"] & & p_{\omega_1} \\
 & & & \parallel \\
\mathbb{Q}_{\max}^- & & & ((H_{\omega_2})^V, (\mathrm{NS}_{\omega_1})^V, \pi) = \mathcal{H}
\end{tikzcd}$$

- μ_{0,ω_1^N} witnesses $q_0 <_{\mathbb{Q}_{\max}^-} p_0$ and $\mu_{0,\omega_1} = \sigma_{0,\omega_1}(\mu_{0,\omega_1^N})$.
- The top iteration $q_0 \to q_{\omega_1}$ is a $\diamondsuit$-iteration in $V^{\mathbb{P}}$.

$\Rightarrow \pi = \nu^*$ still witnesses $\diamondsuit(\omega_1^{<\omega})$ in $V^{\mathbb{P}_\diamondsuit}$.
By definition of order of $\mathbb{Q}_{\max}^-$, $\mathbb{P}_\diamondsuit$ makes π "dense for old sets".

Proposition

Under $\mathsf{AD}^{L(\mathbb{R})}$, $\mathbb{Q}_{\mathsf{max}}$ *and* $\mathbb{Q}^{-}_{\mathsf{max}}$ *are forcing equivalent.*

Corollary

If NS_{ω_1} *is saturated and* π *witnesses* $\diamondsuit^{+}(\omega_1^{<\omega})+\varepsilon$, *then* $\mathbb{P}_{\diamondsuit}$ *for* $\mathbb{V}_{\mathsf{max}} = \mathbb{Q}^{-}_{\mathsf{max}}$ *preserves* π *and makes* π *"dense for old sets".*

This assumption can be forced by π-semiproper forcing assuming a Woodin cardinal. (Similar to forcing "NS_{ω_1} is saturated" from a Woodin cardinal, but turn π into a complete embedding simultaneously).

The Missing Forcing

Theorem (Woodin)

Let M be the least inner model with a proper class of Woodin cardinals and an inaccessible limit of Woodin cardinals.

1. *"NS_{ω_1} is ω_1-dense" holds in a forcing extension of M.*
2. *"NS_{ω_1} is ω_1-dense" does not hold in any ω_1-preserving forcing extension of M.*

How does a forcing witnessing $1.$ look like explicitly?

Open Questions

Question

Can the large cardinal assumptions of the main theorems be reduced?

Question

Does $(\dagger) \Rightarrow (\ddagger)$?

Question

Does "NS_{ω_1} is saturated" imply $\neg$CH?

Question

Does the existence of some large cardinal imply the existence of a precipitous ideal on ω_1?

Thank you for listening!