

# A Journey guided by the Stars Part I

Forcing “ $\text{NS}_{\omega_1}$  is  $\omega_1$ -Dense” from Large Cardinals

Andreas Lietz

June 12th 2023

Kobe Set Theory Seminar

living.knowledge



**MM**  
Mathematics  
Münster  
Cluster of Excellence

What is true in  $V$ ?

- ZFC is too weak
- $\rightsquigarrow$  extend **naturally** to theory  $T$  approximating truth in  $V$ .
- *How?*

Guiding principles:

- (I) Truth in  $V$  should be compatible with large cardinals
- (II) Natural questions should be decided by  $T$

Canonical inner models (like Gödel's  $L$ ) are remarkably successful at (II), limited success at (I) ( $\rightsquigarrow$  Inner Model Theory Program)

Existence of large cardinals good for (I), decides all natural questions of 2nd order number theory, but not much more.

## Definition (Woodin)

$\varphi$  a  $\in$ -sentence.

$$\text{ZFC} \models_{\Omega} \varphi \Leftrightarrow \forall \text{ forcings } \mathbb{P} \forall \alpha \in \text{Ord} \left( V_{\alpha}^{\mathbb{P}} \models \text{ZFC} \rightarrow V_{\alpha}^{\mathbb{P}} \models \varphi \right)$$

$\text{ZFC} \vdash_{\Omega} \varphi$  if  $\exists$  universally Baire  $A \subseteq \mathbb{R}$  so that  $\forall$  strongly  $A$ -closed models  $N$  have  $N \models \varphi$ .

Countable transitive  $N \models \text{ZFC}$  is strongly  $A$ -closed if  $\forall$  generic extensions  $N[G]$  of  $N$ :  $A \cap N[G] \in N[G]$ .

## $\Omega$ -Conjecture (Woodin)

Suppose there are a proper class of Woodin cardinals and  $\varphi$  is a  $\Sigma_2$   $\in$ -sentence. Then

$$\text{ZFC} \models_{\Omega} \varphi \Leftrightarrow \text{ZFC} \vdash_{\Omega} \varphi.$$

Very open, but maybe we can verify consequences!

Ideals are “collections of small sets”. An ideal on  $\omega_1$  is  $I \subseteq \mathcal{P}(\omega_1)$  with

- $\emptyset \in I, \omega_1 \notin I$
- if  $X \subseteq Y \in I$  then  $X \in I$
- $X, Y \in I \Rightarrow X \cup Y \in I$

In this talk all ideals will be  $\sigma$ -closed (closed under countable unions) and uniform (contain all countable sets), even normal (Fodor's Lemma holds for  $I$ ).

- Canonical ideal is the **nonstationary ideal**  $\text{NS}_{\omega_1}$ !

Natural forcing which tries to turn  $I$  into maximal ideal: For  $A, B \subseteq \omega_1$ ,  $A \sim_I B \Leftrightarrow A \triangle B \in I$ .

- Inclusion on  $\mathcal{P}(\omega_1)$  induces partial order on  $\mathcal{P}(\omega_1) / \sim_I$ .
- Remove minimal element:  $\mathbb{P}_I := (\mathcal{P}(\omega_1) / \sim_I) \setminus \{[\emptyset]_I\}$ .

If  $G$  is  $\mathbb{P}_I$  generic then

$$U_G := \{A \in \mathcal{P}(\omega_1)^V \mid [A]_I \in G\}$$

is a  $V$ -ultrafilter, if  $N \in I$  then  $\omega_1 \setminus N \in U_G$ .

## Definition

$I$  is **precipitous** if for all generic  $G \subseteq \mathbb{P}_I$ ,  $\text{Ult}(V, U_G)$  is wellfounded. Get elementary embedding  $j_G: V \rightarrow \text{Ult}(V, U_G)$  with  $\text{crit}(j_G) = \omega_1^V$ .

## Theorem (Mitchell, 70s)

*Suppose  $\kappa$  is measurable. Then in a forcing extension there is a precipitous ideal on  $\omega_1$ .*

- Idea: Let  $U$  witness  $\kappa$  is measurable. Force with  $\text{Col}(\omega, <\kappa) \rightsquigarrow \kappa$  becomes  $\omega_1$  in  $V[G]$ .
- The ideal dual to  $U$  generates an ideal  $I$  in  $V[G]$ .
- If  $H$  is  $\mathbb{P}_I$ -generic over  $V[G]$  then  $j_H: V[G] \rightarrow \text{Ult}(V[G], U_H)$  lifts  $j: V \rightarrow \text{Ult}(V, U)$ , so  $\text{Ult}(V[G], U_H)$  is wellfounded.

## Theorem (Magidor, shortly after)

*Suppose  $\kappa$  is measurable. Then, in a forcing extension,  $\text{NS}_{\omega_1}$  is precipitous.*

Idea: turn the ideal  $I$  above into the nonstationary ideal.

## Definition

An ideal  $I$  on  $\omega_1$  is **saturated** if  $\mathbb{P}_I$  is  $\omega_2$ -cc.

Saturated ideals are precipitous (good exercise!).

## Theorem (Kunen, 70s)

*Suppose  $\kappa$  is a huge cardinal. Then in a forcing extension there is a saturated ideal on  $\omega_1 = \kappa$ .*

If  $I$  is saturated,  $j_G : V \rightarrow \text{Ult}(V, U_G) = N$  generic ultrapower by  $I$ , then in  $V[G]$ ,  $N^{< j_G(\omega_1^V)} \subseteq M$  and  $j_G(\omega_1^V) = \omega_2^V$ .

Kunen's idea: Lifting argument as before. Start with  $j : V \rightarrow M$  with  $M^{j(\kappa)} \subseteq M$  where  $\kappa = \text{crit}(j)$ . Turn  $\kappa$  into  $\omega_1$ ,  $j(\kappa)$  into  $\omega_2$ .

- What about  $\text{NS}_{\omega_1}$ ? Can it be saturated?
- Magidor's argument does not seem to preserve saturation.

## Theorem (Steel-Van Wesep, late 70s)

*Assume  $V \models \text{AD} + \text{AC}_{\mathbb{R}} + V = L(\mathcal{P}(\mathbb{R}))$ . Then in a forcing extension  $\text{ZFC} + \text{"NS}_{\omega_1} \text{ is saturated"}$  holds.*

Before AD was known to be consistent from large cardinals! Very rough idea: Under AD,  $\text{NS}_{\omega_1}$  is a maximal ideal. Force AC, preserve  $\omega_1$ ,  $\omega_2$  and hope for the best.

Can we force  $\text{NS}_{\omega_1}$  saturated over models of **ZFC**?

## Theorem (Foreman-Magidor-Shelah, 80s)

*Suppose there is a supercompact cardinal. Then there is a stationary set preserving forcing  $\mathbb{P}$  so that  $V^{\mathbb{P}} \models \text{"NS}_{\omega_1} \text{ is saturated"}$ .*

**Big surprise!** The generic embeddings do not “come from” earlier embeddings/large cardinals!  $\omega_1$  is preserved. Just verify the combinatorial property with famous “sealing forcing”.

## Definition

Suppose  $\mathcal{A}$  is a maximal antichain of stationary subsets of  $\omega_1$ , i.e.  $\{[S]_{\text{NS}_{\omega_1}} \mid S \in \mathcal{A}\}$  is a max. antichain in  $\mathbb{P}_{\text{NS}_{\omega_1}}$ . The sealing forcing  $\mathbb{S}_{\mathcal{A}}$  consists of tuples  $(f, c)$  so that for some  $\alpha < \omega_1$ ,

- $f: \alpha + 1 \rightarrow \mathcal{A}$  is a function
- $c \subseteq \alpha + 1$  is closed with  $\alpha \in c$ .

The order is given by end-extension in both arguments.

$\mathbb{S}_{\mathcal{A}}$  preserves stationary sets and in the extension,  $|\mathcal{A}| \leq \omega_1$  (arranged by the  $f$ 's) and  $\mathcal{A}$  is still a maximal antichain (arranged by the  $c$ 's). Under the forcing axiom MM, this “has already happened”.

## Theorem (Shelah)

*Suppose there is a Woodin cardinal. Then in a stationary set preserving extension,  $\text{NS}_{\omega_1}$  is saturated.*

Idea: Can iterate semiproper forcings ( $\cong$  “regular” stationary set preserving forcing), but in general not stationary set preserving forcings. Iterate sealing forcings, but only if they happen to be semiproper. The Woodin cardinal makes sure that this happens often enough.

## Definition

$I$  is  $\omega_1$ -**dense** if  $\mathbb{P}_I$  has a dense subset of size  $\omega_1$ .

Dense ideals are saturated.

Cannot reasonably strengthen this property: If  $I, J$  are  $\omega_1$ -dense ideals then  $\mathbb{P}_I \cong \mathbb{P}_J$ .

## Theorem (Woodin, late 70s)

*Assume  $V \models \text{AD}_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$ . Then in a forcing extension  $\text{ZFC} + \text{CH} + \text{“there is a dense ideal on } \omega_1 \text{”}$  holds.*

## Theorem (Woodin)

*Suppose there is an almost huge cardinal. Then in a forcing extension there is a dense ideal on  $\omega_1$  and CH hold.*

Similar strategy as Kunen, more efficient argument.

In the above models,  $\text{NS}_{\omega_1}$  is not  $\omega_1$ -dense as this implies  $\neg\text{CH}$  (Shelah, Woodin independently).

## Theorem (Woodin)

*Assume  $L(\mathbb{R}) \models \text{AD}$ . Then in a forcing extension of  $L(\mathbb{R})$ ,  $\text{ZFC} + \text{"NS}_{\omega_1} \text{ is } \omega_1\text{-dense"}$  holds. In fact, if there are a proper class of Woodin cardinals then  $\text{"NS}_{\omega_1} \text{ is } \omega_1\text{-dense"}$  is  $\Omega$ -consistent, i.e.  $\text{ZFC} \not\vdash_{\Omega} \neg(\text{"NS}_{\omega_1} \text{ is } \omega_1\text{-dense"})$ .*

- Under  $\Omega$ -conjecture and large cardinals, this implies that  $\text{"NS}_{\omega_1} \text{ is } \omega_1\text{-dense"}$  is  $\Omega$ -satisfiable, i.e. holds in a forcing extension.
- So far, Woodin's method of forcing with  $\mathbb{Q}_{\max}$  over determinacy models was the only known way to produce models of  $\text{"NS}_{\omega_1} \text{ is } \omega_1\text{-dense"}$ .

## Question (Woodin, late 90's)

Assume some large cardinal. Is there a stationary set preserving forcing  $\mathbb{P}$  so that

$$V^{\mathbb{P}} \models \text{"NS}_{\omega_1} \text{ is } \omega_1\text{-dense"}$$

The  $\Omega$ -conjecture alone does not answer this question.

## Theorem (L.)

*Suppose there is an inaccessible cardinal  $\kappa$  which is a limit of  $<\kappa$ -supercompact cardinals. Then there is a stationary set preserving forcing  $\mathbb{P}$  with*

$$V^{\mathbb{P}} \models \text{"NS}_{\omega_1} \text{ is } \omega_1\text{-dense"}.$$

## Theorem (Asperó-Schindler)

$\text{MM}^{++} \Rightarrow (*)$ .

$\text{MM}^{++}$  is a very strong forcing axiom.

$(*)$  states that  $L(\mathcal{P}(\omega_1))$  is a  $\mathbb{P}_{\max}$ -extension of a determinacy model.

## Vague Conjecture

For any wellbehaved  $\mathbb{P}_{\max}$ -variation  $\mathbb{V}_{\max}$  there is a forcing axiom which is

- (i) consistent from large cardinals and
- (ii) implies  $\mathbb{V}_{\max}-(*)$ .

Not obvious from Asperó-Schindler! What is the axiom? How to show (i)? How to show (ii)?

Interesting case:  $\mathbb{V}_{\max} = \mathbb{Q}_{\max}$ .  $\mathbb{Q}_{\max}-(*) \Rightarrow \text{"NS}_{\omega_1} \text{ is } \omega_1\text{-dense"}$ .

## Theorem (Asperó-Schindler)

$\text{MM}^{++} \Rightarrow (*)$ .

$\text{MM}^{++}$  is a very strong forcing axiom.

$(*)$  states that  $L(\mathcal{P}(\omega_1))$  is a  $\mathbb{P}_{\max}$ -extension of a determinacy model.

## Vague Conjecture

For any wellbehaved  $\mathbb{P}_{\max}$ -variation  $\mathbb{V}_{\max}$  there is a forcing axiom which is

- (i) consistent from large cardinals and
- (ii) implies  $\mathbb{V}_{\max}-(*)$ .

Not obvious from Asperó-Schindler! What is the axiom? How to show (i)? How to show (ii)?

Interesting case:  $\mathbb{V}_{\max} = \mathbb{Q}_{\max}$ .  $\mathbb{Q}_{\max}-(*) \Rightarrow \text{"NS}_{\omega_1} \text{ is } \omega_1\text{-dense"}$ .

## Theorem (Asperó-Schindler)

$\text{MM}^{++} \Rightarrow (*)$ .

$\text{MM}^{++}$  is a very strong forcing axiom.

$(*)$  states that  $L(\mathcal{P}(\omega_1))$  is a  $\mathbb{P}_{\max}$ -extension of a determinacy model.

## Vague Conjecture

For any wellbehaved  $\mathbb{P}_{\max}$ -variation  $\mathbb{V}_{\max}$  there is a forcing axiom which is

- (i) consistent from large cardinals and
- (ii) implies  $\mathbb{V}_{\max}-(*)$ .

Not obvious from Asperó-Schindler! What is the axiom? How to show (i)? How to show (ii)?

Interesting case:  $\mathbb{V}_{\max} = \mathbb{Q}_{\max}$ .  $\mathbb{Q}_{\max}-(*) \Rightarrow \text{"NS}_{\omega_1} \text{ is } \omega_1\text{-dense"}$ .

## Theorem (Asperó-Schindler)

$\text{MM}^{++} \Rightarrow (*)$ .

$\text{MM}^{++}$  is a very strong forcing axiom.

$(*)$  states that  $L(\mathcal{P}(\omega_1))$  is a  $\mathbb{P}_{\max}$ -extension of a determinacy model.

## Vague Conjecture

For any wellbehaved  $\mathbb{P}_{\max}$ -variation  $\mathbb{V}_{\max}$  there is a forcing axiom which is

- (i) consistent from large cardinals and
- (ii) implies  $\mathbb{V}_{\max}-(*)$ .

Not obvious from Asperó-Schindler! What is the axiom? How to show (i)? How to show (ii)?

Interesting case:  $\mathbb{V}_{\max} = \mathbb{Q}_{\max}$ .  $\mathbb{Q}_{\max}-(*) \Rightarrow \text{"NS}_{\omega_1} \text{ is } \omega_1\text{-dense"}$ .

- Looking for forcing axiom QM which implies  $\text{NS}_{\omega_1}$  is  $\omega_1$ -dense.
- $\text{NS}_{\omega_1}$  is  $\omega_1$ -dense  $\Leftrightarrow \exists \pi: \text{Col}(\omega, \omega_1) \rightarrow \mathbb{P}_{\text{NS}_{\omega_1}}$  dense embedding.

## Lemma (Tennenbaum (?))

*If  $\mathbb{P}$  is a forcing of size  $\omega_1$  which collapses  $\omega_1$  then there is a dense embedding  $\pi: \text{Col}(\omega, \omega_1) \rightarrow \mathbb{P}$ .*

- To force a forcing axiom, usually have “countable support style” iteration  $\mathbb{P}$  of length large  $\kappa$  of forcings of size  $< \kappa$ .  $\Rightarrow \mathbb{P}$  is  $\kappa$ -cc.
- $\Rightarrow$  version of  $\pi$  above should exist in intermediate extension along iteration to force QM. Version of  $\pi$  means

$$\pi': \text{Col}(\omega, \omega_1) \rightarrow \mathcal{P}(\omega_1) \setminus \text{NS}_{\omega_1} \text{ with } \forall p \in \text{Col}(\omega, \omega_1) [\pi'(p)]_{\text{NS}_{\omega_1}} = \pi(p).$$

- This suggests we should isolate properties of  $\pi'$ , first force  $\pi'$  to exist and then iterate forcing preserving these properties of  $\pi'$ .

## Definition (Woodin)

$\diamond(\omega_1^{<\omega})$  holds if there is an embedding  $\pi: \text{Col}(\omega, \omega_1) \rightarrow \mathcal{P}(\omega_1) \setminus \text{NS}_{\omega_1}$  so that  $\forall p \in \text{Col}(\omega, \omega_1)$  there are stationarily many countable  $X < H_{\omega_2}$  with

$p \in \{q \in \text{Col}(\omega, \omega_1) \cap X \mid \omega_1 \cap X \in \pi(q)\}$  is a filter generic over  $X$ .

## Lemma

*Suppose  $[\cdot]_{\text{NS}_{\omega_1}} \circ \pi: \text{Col}(\omega, \omega_1) \rightarrow \mathbb{P}_{\text{NS}_{\omega_1}}$  is a dense embedding. Then  $\pi$  witnesses  $\diamond(\omega_1^{<\omega})$ .*

## Proof Sketch.

Let  $p \in \text{Col}(\omega, \omega_1)$ ,  $X < H_{\omega_2}$  countable so that  $\omega_1 \cap X =: \delta^X \in \pi(p)$ . Let  $A \subseteq \text{Col}(\omega, \omega_1)$ ,  $A \in X$ , be a maximal antichain.  $\Rightarrow \mathcal{A} := [\cdot]_{\text{NS}_{\omega_1}} \circ \pi[A] \subseteq \mathbb{P}_{\text{NS}_{\omega_1}}$  is a max. antichain, thus  $\Delta \mathcal{A}$  contains a club  $C \in X$ , so  $\delta^X \in C$ . It follows that there is  $q \in X \cap A$  with  $\delta^X \in \pi(q)$ . □

If  $\pi$  will eventually witness “ $\text{NS}_{\omega_1}$  is  $\omega_1$ -dense”, we need to preserve the fact that  $\pi$  witnesses  $\diamond(\omega_1^{<\omega})$  along the iteration. This suggests the following forcing axiom:

## Definition

QM holds if  $\exists \pi$  witnessing  $\diamond(\omega_1^{<\omega})$  and  $\text{FA}_{\omega_1}(\{\mathbb{P} \mid \mathbb{P} \text{ preserves } \pi\})$  holds, i.e. whenever  $\mathbb{V}^{\mathbb{P}} \models “\pi \text{ witnesses } \diamond(\omega_1^{<\omega})”$  and  $\langle D_i \mid i < \omega_1 \rangle$  are dense subsets of  $\mathbb{P}$ , there is a  $\mathbb{P}$ -filter meeting all  $D_i$ .

If  $\pi$  witnesses QM then  $[\cdot]_{\text{NS}_{\omega_1}} \circ \pi$  is a dense embedding, so  $\text{NS}_{\omega_1}$  is  $\omega_1$ -dense. **Why?**

If  $S \subseteq \omega_1$  is stationary, no point in  $\text{ran}([\cdot]_{\text{NS}_{\omega_1}} \circ \pi)$  is below  $[S]_{\text{NS}_{\omega_1}}$ , then  $\text{CS}(\omega_1 \setminus S)$  (club shooting through  $\omega_1 \setminus S$ ) preserves  $\pi$ . But if  $\text{FA}_{\omega_1}(\{\mathbb{P}\})$  holds, then  $\mathbb{P}$  must preserve stationary sets, contradiction.

## Theorem (L.)

*Assume a supercompact limit of supercompact cardinals. Then QM holds in a stationary set preserving forcing extension.*

QM implies  $\mathbb{Q}_{\max}^{-(*)}$ .

## Two Problematic Examples

Problem: To force QM, need to iterate  $\pi$ -preserving forcing, these can kill stationary sets. Want to preserve  $\pi$ , in particular  $\omega_1$ .

- 1st example: Let  $\langle S_n \mid n < \omega \rangle$  partition of  $\omega_1$  into stationary sets. Iterate of length  $\omega$ , kill  $S_n$  in step  $n$  with  $\text{CS}(\omega_1 \setminus S_n)$ . In the limit,  $\omega_1$  must be collapsed.

**Solution:** Don't kill old stationary sets.

- 2nd example (Shelah): First force a function  $g_0: \omega_1 \rightarrow \omega_1$  above all canonical functions. Then force some  $g_1$  above all canonical functions, but below  $g_0$ . Continue like this, get

$$\text{canonical functions} < g_n < g_{n-1} < \cdots < g_1 < g_0 \pmod{\text{NS}_{\omega_1}}$$

at stage  $n$ . These forcings preserve stationary sets, but not all are semiproper. In the limit  $\omega_1$  is collapsed (as there is no infinite decreasing sequence of such functions).

**Solution:** Mostly use forcings with good “regularity properties”.

## Definition

Suppose  $\pi$  witnesses  $\diamond(\omega_1^{<\omega})$ . A  $Q$ -iteration is a nice iteration  $\langle \mathbb{P}_\alpha, \dot{Q}_\beta \mid \alpha \leq \gamma, \beta < \gamma \rangle$  of  $\pi$ -preserving forcing so that

- For  $\alpha < \gamma$  successor

$$V^{\mathbb{P}_\alpha} \models \text{“}\dot{Q}_\alpha \text{ forces SRP and that } \text{ran}(\pi) \text{ is dense for old sets”}$$

- For  $\alpha < \gamma$  limit no further requirement on  $\dot{Q}_\alpha$ .

## Work-Life-Balance Theorem (L.)

$Q$ -iterations preserve  $\pi$ .

This is a “cheapo iteration theorem”, but good enough to force QM.

To force QM, need to do three things:

1. Prove the Work-Life-Balance theorem.
2. Assuming a supercompact cardinal, find a  $\pi$ -preserving forcing which forces SRP.
3. Assuming Woodin cardinals, find a  $\pi$ -preserving forcing which makes  $\text{ran}(\pi)$  “dense for old sets”.

**This is analog of Sealing forcing!**

Plan: 1. today, 2. – 3. next week.

Iteration theorems state that some class of forcings is closed under iterations with a specific support. For example, countable support iterations of  $\sigma$ -closed forcings are  $\sigma$ -closed. Similar for proper forcings. Proofs tend to have certain form: Let  $\mathbb{P} = \langle \mathbb{P}_n, \dot{Q}_m \mid n \leq \omega, m < \omega \rangle$  iteration of length  $\omega$ ,  $X < H_\theta$  countable with  $\mathbb{P} \in X$ .

# The Killing Stationary Sets Obstacle

Suppose at some step of this argument, have

1.  $X < H_\theta$ ,  $q \in \mathbb{Q}$
2.  $S \subseteq \omega_1$  is stationary, but  $q \Vdash_{\mathbb{Q}} \check{S} \in \text{NS}_{\omega_1}$  and
3.  $\delta^X \in S$ .

Then there is no way to continue!

## Proof.

Suppose  $G \subseteq \mathbb{Q}$  is generic. Then there is a club  $C \in X[G]$  disjoint from  $S$ . Have  $\delta^{X[G]} \in C$ , but then  $\delta^{X[G]} \notin S$ . It follows that  $\delta^{X[G]} \neq \delta^X$ . □

## Definition

$I$  an ideal on  $\omega_1$ . Say that  $X$  **respects**  $I$  if  $\delta^X \notin S$  whenever  $S \in I \cap X$ .

At all costs, must maintain that  $X$  respects  $I_{\mathbb{Q}}^{\omega_1} := \{S \subseteq \omega_1 \mid q \Vdash_{\mathbb{Q}} \check{S} \in \text{NS}_{\omega_1}\}$ .

Suppose we have iteration  $\mathbb{P} = \langle \mathbb{P}_n, \dot{Q}_m \mid n \leq \omega, m < \omega \rangle$  of  $\pi$ -preserving forcing and  $p \in \mathbb{P}$ . Must start with  $X \prec H_\theta$  which respects  $\dot{I}_{p(0)}^{Q_0}$ .

After first step, hopefully have  $X \sqsubseteq X[G_1]$ . But now need that  $X[G_1]$  respects  $\dot{I}_{p(1)G_1}^{Q_1}$ . How can we arrange this?

## Definition

Suppose  $\mathbb{Q}$  is  $\omega_1$ -preserving forcing.  $\mathbb{Q}$  is **respectful** if: Whenever

- $Y \prec H_\lambda$  countable,  $\mathbb{Q} \in Y$ ,  $q \in \mathbb{Q} \cap Y$
- $\dot{I} \in Y$  is a  $\mathbb{Q}$ -name for an ideal on  $\omega_1$ .

Then one of the following:

1. There is  $r \leq q$  and  $r$  forces

$$Y \sqsubseteq Y[G] \wedge Y[G] \text{ respects } \dot{I}^G$$

2. Or:  $Y$  does **not** respect  $\dot{I}^q := \{S \subseteq \omega_1 \mid q \Vdash \check{S} \in \dot{I}\}$ .

## Lemma

*Assume SRP. Then all  $\omega_1$ -preserving forcings are respectful.*

## Proof.

Let  $\mathbb{Q}$  be  $\omega_1$ -preserving,  $Y < H_\lambda$ ,  $q \in \mathbb{Q} \cap Y$ ,  $\dot{I} \in Y$  as in definition. Have to show:

- Either there is  $r \leq q$  forcing  $Y \sqsubseteq Y[G]$  respects  $\dot{I}^G$
- or  $Y$  does not respect  $\dot{I}^q$ .

Let  $\mu = (2^{|Q|})^+ \in Y$  and  $\mathcal{S} = \{Z < H_\mu \mid \nexists r \leq q \text{ forcing } "Z \sqsubseteq Z[G] \text{ respects } \dot{I}^G"\} \in Y$ .

By SRP, can find continuous increasing  $\vec{Z} = \langle Z_\alpha \mid \alpha < \omega_1 \rangle \in Y$  s.t.:

- $\mathbb{Q}, q, \dot{I} \in Z_0$
- $Z_\alpha < H_\mu$
- Either  $Z_\alpha \in \mathcal{S}$  or there is no  $Z_\alpha \sqsubseteq Z$  with  $Z \in \mathcal{S}$ .

## Proof (Continued).

Let  $G \subseteq \mathbb{Q}$  generic,  $q \in G$ . Let  $S = \{\alpha < \omega_1 \mid Z_\alpha \in \mathcal{S}\}$ .

**Claim:**  $S \in I := \dot{j}^G$

*Proof.* Suppose otherwise,  $S \in I^+$ .  $\langle Z_\alpha[G] \mid \alpha < \omega_1 \rangle$  is continuous increasing sequence of elementary substructures of  $H_\mu^{V[G]}$ . Find club  $C \subseteq \omega_1$  with  $\alpha = \delta^{Z_\alpha} = \delta^{Z_\alpha[G]}$ . For any  $\alpha \in S \cap C$ , can find  $T_\alpha \in I \cap Z_\alpha[G]$  with  $\alpha = \delta^{Z_\alpha[G]} \in T_\alpha$ . By normality of  $I$ , there is  $S_0 \subseteq S \cap C$  in  $I^+$  and  $T$  so that  $T_\alpha = T$  for  $\alpha \in S_0$ . But then  $S_0 \subseteq T$ , contradicting  $T \in I$ .

□

Case 1:  $\delta^Y \in S$ . As  $S \in \dot{j}^q \cap Y$ ,  $Y$  does not respect  $\dot{j}^q$ .

Case 2:  $\delta^Y \notin S$ . As  $Z_{\delta^Y} \subseteq Y \cap H_\mu$ ,  $Y \cap H_\mu \notin \mathcal{S}$ . Thus there is  $r \leq q$  forcing  $Y \subseteq Y[G]$  and  $Y[G]$  respects  $\dot{j}^G$ .

□

In  $L$ ,  $\text{Add}(\omega_1, 1)$  is *not* respectful.

## Work-Life-Balance Theorem (L.)

$Q$ -iterations preserve  $\pi$ .

### Proof Sketch.

We will sketch that a  $Q$ -iteration  $\mathbb{P} = \langle \mathbb{P}_n, \dot{Q}_m \mid n \leq \omega, m < \omega \rangle$  of length  $\omega$  does not collapse  $\omega_1$ . Start with  $p \in \mathbb{P}$  and countable  $X < H_\theta$  that respects  $I_{p(0)}^{Q_0}$ .

- We are in good case of respectfulness, so find  $q \restriction 1 \leq p \restriction 1$  which forces  $X \subseteq X[G_1]$  respects  $I_{p(1)}^{Q_1}$  (actually a slightly larger ideal).
- As no old stationary sets are killed, stay in the good case for the next instance.
- Continue building  $q$  step by step.
- In the end, let  $G$  be  $\mathbb{P}$ -generic,  $q \in G$ . Have  $X \subseteq X[G_1] \subseteq X[G_2] \subseteq X[G_3] \subseteq \dots$
- If we are careful, arrange along the side that  $X[G] \cap V = \bigcup_{n < \omega} X[G_n] \cap V$ .
- This gives  $X[G_n] \subseteq X[G]$  for all  $n$ , so  $\omega_1$  is not collapsed.

Thank you for listening!

To be continued...