

On the cardinal characteristics associated with the σ -ideal generated by closed measure zero sets of reals

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Additivity of \mathcal{I} : $\text{add}(\mathcal{I}) = \min\{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I}, \bigcup \mathcal{J} \notin \mathcal{I}\}.$

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Denote by

- ① \mathcal{N} : the σ -ideal of Lebesgue measure zero (null) subsets of the Cantor Space 2^ω .
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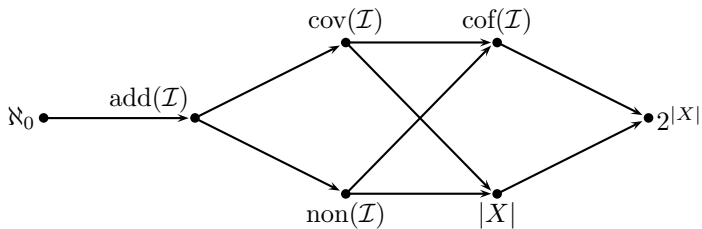
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It is well-known that $\mathcal{E} \subseteq \mathcal{N} \cap \mathcal{M}$. Even more, $\mathcal{E} \subsetneq \mathcal{N} \cap \mathcal{M}$

Provable inequalities



Cardinal characteristics of the continuum II

For $f, g \in \omega^\omega$ we write

$$f \leq^* g \text{ iff } \exists m < \omega \forall n \geq m (f(n) \leq g(n))$$

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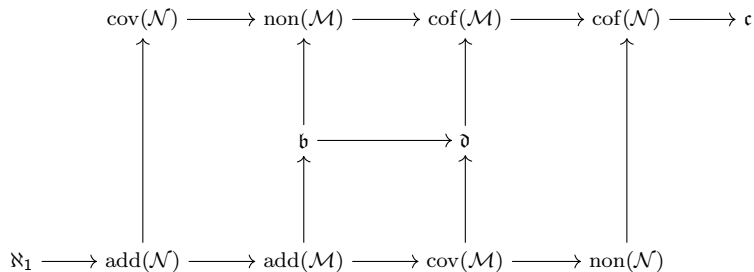
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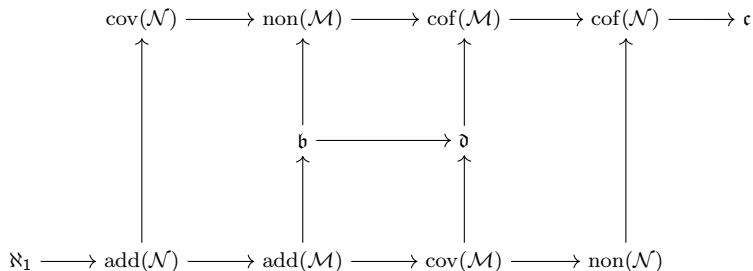
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- ③ $\mathfrak{c} := 2^\omega.$

Cichoń's diagram

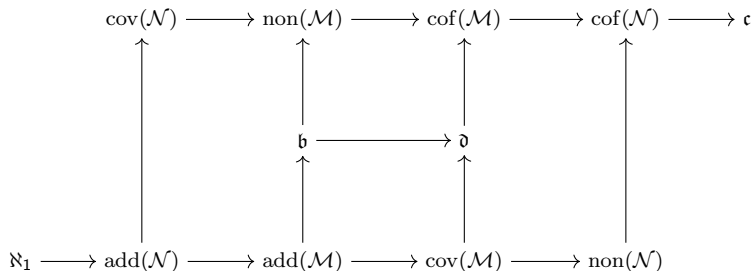


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Completeness: Bartoszyński, Judah, Miller, Shelah.

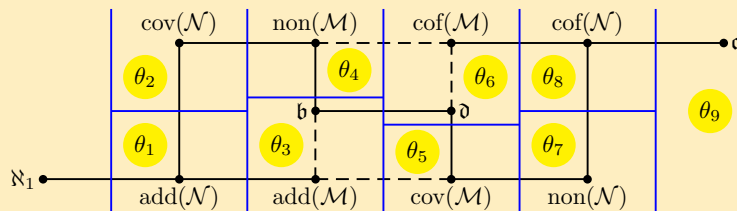
In the context of this diagram, a natural question arises:

Is it consistent that all the cardinals in Cichoń's diagram (with the exception of the dependent values $\text{add}(\mathcal{M})$ and $\text{cof}(\mathcal{M})$) are pairwise different?

Theorem (Goldstern, Kellner and Shelah [GKS19])

Assume GCH and that

$\theta_1 < \theta_9 < \theta_1 < \theta_8 < \theta_2 < \theta_7 < \theta_3 < \theta_6 < \theta_4 \leq \theta_5 \leq \theta_6 \leq \theta_7 \leq \theta_8 \leq \theta_9$ are regular, θ_i strongly compact for $i = 6, 7, 8, 9$. Then there is a ccc poset forcing



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Theorem (Goldstern, Kellner, Mejía, and Shelah [GKMS21])

No large cardinals are needed for Cichoń's Maximum.

Question 1

Is it consistent that all the cardinals in Cichoń's diagram (with the exception of the dependent values $\text{add}(\mathcal{M})$ and $\text{cof}(\mathcal{M})$) are pairwise different where $\text{cov}(\mathcal{M}) < \text{non}(\mathcal{M})$?

Theorem (Bartoszyński and Shelah [BS92])

$\text{add}(\mathcal{E}) = \text{add}(\mathcal{M})$ and $\text{cof}(\mathcal{E}) = \text{cof}(\mathcal{M})$.

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Theorem ([BS92])

- ① $\max\{\text{cov}(\mathcal{M}), \text{cov}(\mathcal{N})\} \leq \text{cov}(\mathcal{E}) \leq \max\{\mathfrak{d}, \text{cov}(\mathcal{N})\}.$
- ② $\min\{\mathfrak{b}, \text{non}(\mathcal{N})\} \leq \text{non}(\mathcal{E}) \leq \min\{\text{non}(\mathcal{M}), \text{non}(\mathcal{N})\}.$

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In particular,

Corollary ([BS92])

- ① If $\mathfrak{d} = \text{cov}(\mathcal{M})$, then $\text{cov}(\mathcal{E}) = \max\{\text{cov}(\mathcal{M}), \text{cov}(\mathcal{N})\}.$
- ② If $\mathfrak{b} = \text{non}(\mathcal{M})$, then $\text{non}(\mathcal{E}) = \min\{\text{non}(\mathcal{M}), \text{non}(\mathcal{N})\}.$

For ideals $\mathcal{I} \subseteq \mathcal{J}$ define

$$\text{cof}(\mathcal{I}, \mathcal{J}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{J} \text{ and } \forall A \in \mathcal{I} \exists B \in \mathcal{F} (A \subseteq B)\}.$$

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Theorem (Brendle [Bre99])

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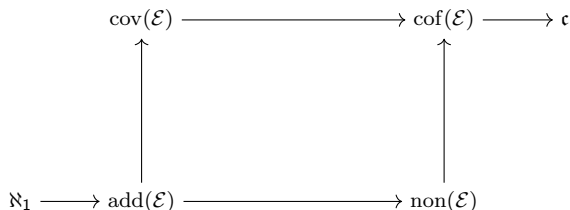
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Lemma ([Bre99])

- ① $\text{cof}(\mathcal{E}) = \text{cof}(\mathcal{E}_0, \mathcal{E}) = \text{cof}(\mathcal{E}_0).$
- ② $\text{cof}(\mathcal{E}, \mathcal{M}) = \text{cof}(\mathcal{E}_0, \mathcal{M}).$

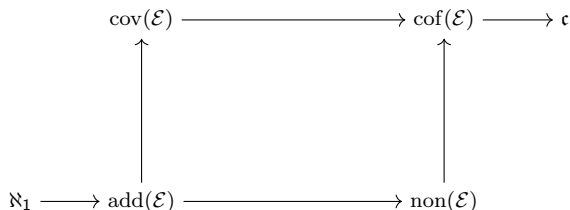
Here, \mathcal{E}_0 , denotes the ideal of the set with closure \bar{A} of measure zero.



Main problem

Is it consistent that all the four cardinal characteristics associated with \mathcal{E} in the diagram above are pairwise different?

Motivation

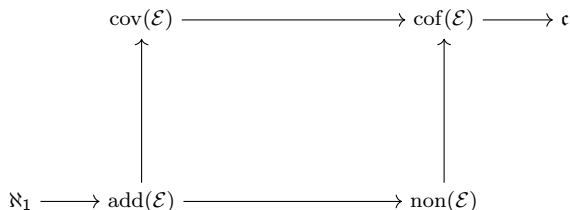


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Note that there can be at most two instances of the Main problem, namely

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(A2) $_{\mathcal{E}}$ $\text{add}(\mathcal{E}) < \text{non}(\mathcal{E}) < \text{cov}(\mathcal{E}) < \text{cof}(\mathcal{E})$.

- ④ (Mejía 2013) $(A1)_{\mathcal{N}}$ is consistent with ZFC.

- Let $\langle \sigma(n) : n < \omega \rangle$, $\sigma(n) \in 2^{<\omega}$. Denote

$$[\sigma]_{\infty} := \bigcap_{n < \omega} \bigcup_{m \geq n} [\sigma(m)]$$

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Definition (Yorioka 2002)

Let $f : \omega \rightarrow \omega$ increasing. The *Yorioka ideal* \mathcal{I}_f is defined by

$$\mathcal{I}_f := \{X \subseteq 2^{\omega} : \exists \sigma \in (2^{<\omega})^{\omega} (X \subseteq [\sigma]_{\infty} \text{ and } \text{ht}_{\sigma} \gg f)\}.$$

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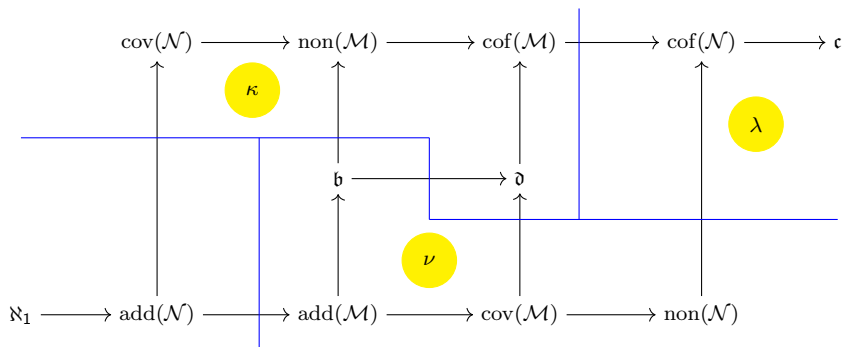
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- 6 (Brendle 2021 [Bre19]) $(A2)_{\aleph_1}$ is consistent with ZFC.



The constellation of Cichoń's diagram forced in [Br21] where $\aleph_1 < \nu < \kappa < \lambda$ with κ and ν regular.

Strong measure sets

Given a sequence $\langle \sigma(n) : n < \omega \rangle$, $\sigma(n) \in 2^{<\omega}$ define $\text{ht}_\sigma : \omega \rightarrow \omega$, $\text{ht}_\sigma(n) := |\sigma(n)|$ for each $n < \omega$.

Definition

Let $X \subseteq 2^\omega$. Say that X *has strong measure zero* iff for every $f \in \omega^\omega$ there is some $\sigma \in (2^{<\omega})^\omega$ such that

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Let $\mathcal{SN} := \{X \subseteq 2^\omega : X \text{ has strong measure zero}\}$.

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The result

Theorem (C. [Car22a])

Let $\theta \leq \mu \leq \nu$ be uncountable regular cardinals and let λ be a cardinal such that $\nu \leq \lambda = \lambda^{<\theta}$. Then there is a ccc poset forcing

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Sketch Proof.

- Use UF-extendable matrix iteration framework from [BCM21] for forcing $\text{cov}(\mathcal{N}) = \text{non}(\mathcal{M}) = \mu$, $\text{cov}(\mathcal{M}) = \text{non}(\mathcal{N}) = \nu$, $\mathfrak{b} = \theta$ and $\mathfrak{d} = \lambda$.

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Since $\mathcal{E} \subseteq \mathcal{M} \cap \mathcal{N}$,

$$\text{non}(\mathcal{E}) \leq \min\{\text{non}(\mathcal{M}), \text{non}(\mathcal{N})\}$$

and

$$\max\{\text{cov}(\mathcal{M}), \text{cov}(\mathcal{N})\} \leq \text{cov}(\mathcal{E}).$$

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- Use UF-extendable matrix iteration framework from [BCM21] for forcing $\text{cov}(\mathcal{N}) = \text{non}(\mathcal{M}) = \mu$, $\text{cov}(\mathcal{M}) = \text{non}(\mathcal{N}) = \nu$, $\mathfrak{b} = \theta$ and $\mathfrak{d} = \lambda$.
- Note that $\text{add}(\mathcal{E}) = \theta$ and $\text{cof}(\mathcal{E}) = \lambda$ (because $\text{add}(\mathcal{E}) = \text{add}(\mathcal{M})$ and $\text{cof}(\mathcal{E}) = \text{cof}(\mathcal{M})$).

Since $\mathcal{E} \subseteq \mathcal{M} \cap \mathcal{N}$,

$$\text{non}(\mathcal{E}) \leq \min\{\text{non}(\mathcal{M}), \text{non}(\mathcal{N})\}$$

and

$$\max\{\text{cov}(\mathcal{M}), \text{cov}(\mathcal{N})\} \leq \text{cov}(\mathcal{E}).$$

Then

- $\text{non}(\mathcal{E}) \leq \mu$ and $\text{cov}(\mathcal{E}) \geq \nu$.

The result

Theorem (C. [C22])

Let $\theta \leq \mu \leq \nu$ be uncountable regular cardinals and let λ be a cardinal such that $\nu \leq \lambda = \lambda^{<\theta}$. Then there is a ccc poset forcing

$$\text{add}(\mathcal{E}) = \theta \leq \text{non}(\mathcal{E}) = \mu \leq \text{cov}(\mathcal{E}) = \nu \leq \text{cof}(\mathcal{E}) = \lambda$$

Sketch Proof.

- Use UF-extendable matrix iteration framework from [BCM21] for forcing $\text{cov}(\mathcal{N}) = \text{non}(\mathcal{M}) = \mu$, $\text{cov}(\mathcal{M}) = \text{non}(\mathcal{N}) = \nu$, $\mathfrak{b} = \theta$ and $\mathfrak{d} = \lambda$.
- $\text{add}(\mathcal{E}) = \theta$ and $\text{cof}(\mathcal{E}) = \lambda$ (because $\text{add}(\mathcal{E}) = \text{add}(\mathcal{M})$ and $\text{cof}(\mathcal{E}) = \text{cof}(\mathcal{M})$).
- $\text{non}(\mathcal{E}) \leq \mu$ and $\text{cov}(\mathcal{E}) \geq \nu$.

What about the converse?

$$\mu \leq \text{non}(\mathcal{E}) \text{ and } \text{cov}(\mathcal{E}) \leq \nu.$$

How about $\mu \leq \text{non}(\mathcal{E})$ and $\text{cov}(\mathcal{E}) \leq \nu$?

To solve this, we find a lower bound to $\text{non}(\mathcal{E})$ and an upper bound to $\text{cov}(\mathcal{E})$.

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To solve this, we find a lower bound to $\text{non}(\mathcal{E})$ and an upper bound to $\text{cov}(\mathcal{E})$.

Given a sequence of non-empty sets $b = \langle b(n) : n \in \omega \rangle$ and $h: \omega \rightarrow \omega$, define

① $\prod b := \prod_{n \in \omega} b(n).$

② $\mathcal{S}(b, h) := \prod_{n \in \omega} [b(n)]^{\leq h(n)}.$

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For two functions x and φ with domain ω , write

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Definition

Let $b = \langle b(n) : n < \omega \rangle$ be a sequence of non-empty sets and let $h \in \omega^\omega$. Define the cardinal numbers $\mathfrak{b}_{b,h}^{\text{Lc}}$, $\mathfrak{d}_{b,h}^{\text{Lc}}$, called *localization cardinals*, as follows:

$$\mathfrak{b}_{b,h}^{\text{Lc}} := \min \left\{ |F| : F \subseteq \prod b, \neg \exists \varphi \in \mathcal{S}(b, h) \forall x \in F (x \in^* \varphi) \right\},$$

$$\mathfrak{d}_{b,h}^{\text{Lc}} := \min \left\{ |D| : D \subseteq \mathcal{S}(b, h), \forall x \in \prod b \exists \varphi \in D (x \in^* \varphi) \right\}$$

A variation of $\mathfrak{b}_{b,h}^{\text{Lc}}$ and $\mathfrak{d}_{b,h}^{\text{Lc}}$

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Lemma

With the notation from the previous definition. If $\limsup_{n \rightarrow \infty} \frac{h(n)}{|b(n)|} < 1$, then $\text{cov}(\mathcal{E}) \leq \mathfrak{d}_{b,h}^{\text{Lc}*} \leq \mathfrak{d}_{b,h}^{\text{Lc}}$ and $\mathfrak{b}_{b,h}^{\text{Lc}} \leq \mathfrak{b}_{b,h}^{\text{Lc}*} \leq \text{non}(\mathcal{E})$.

How to increase $\mathfrak{b}_{b,h}^{\text{Lc}}$?

For $b, h \in \omega^\omega$ such that $\forall i < \omega (b(i) > 0)$ and h going to infinity, the *localization forcing* is defined by

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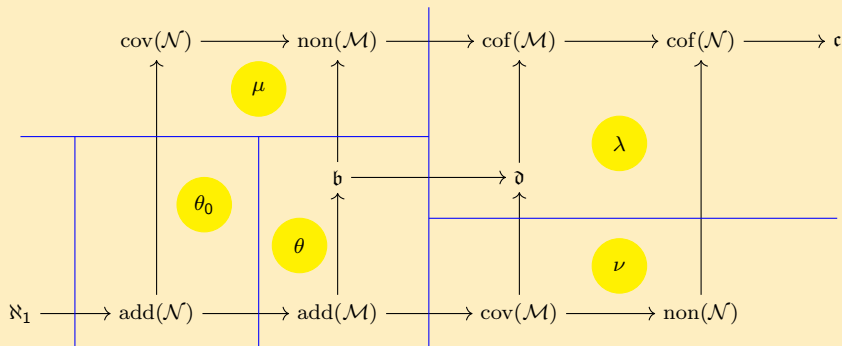
The key point is to iterate, in addition: $\text{LOC}_{b,h}$ to increase $\mathfrak{b}_{b,h}^{\text{Lc}}$. Hence,

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The result

Theorem (C. [Car22a])

Let $\theta_0 \leq \theta \leq \mu \leq \nu$ be uncountable regular cardinals and let λ be a cardinal such that $\nu \leq \lambda = \lambda^{<\theta}$. Then there is a ccc poset forcing



and $\text{add}(\mathcal{E}) = \theta$, $\text{non}(\mathcal{E}) = \mu$, $\text{cov}(\mathcal{E}) = \nu$, and $\text{cof}(\mathcal{E}) = \lambda$.

Question 4

Are each one the following statements consistent with ZFC?

$$\aleph_1 < \text{add}(\mathcal{N}) < \mathfrak{b} < \text{cov}(\mathcal{N}) < \text{non}(\mathcal{E}) < \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) \\ < \text{cov}(\mathcal{E}) = \text{non}(\mathcal{N}) = \mathfrak{d} = \mathfrak{c}. \quad (1)$$

$$\aleph_1 < \text{add}(\mathcal{N}) < \mathfrak{b} < \text{non}(\mathcal{E}) < \text{cov}(\mathcal{N}) < \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) \\ < \text{cov}(\mathcal{E}) = \text{non}(\mathcal{N}) = \mathfrak{d} = \mathfrak{c}. \quad (2)$$

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In [KST19] (Kellner, Shelah, and Tănăsiei), it was constructed FAMS (finitely additive measures) along a FS (finite support) iteration to force

$$\aleph_1 < \text{add}(\mathcal{N}) < \mathfrak{b} < \text{cov}(\mathcal{N}) < \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) = \mathfrak{c}.$$

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It is consistent with ZFC

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Also we may ask:

Question 5

Does eventually different real forcing preserve $\text{non}(\mathcal{E})$ small?

One positive answer to Question 4 along with the method of submodels of [GKMS21] would help solving:

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Question 6

Is it consistent with ZFC

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Question 7

Is it consistent with ZFC

$$\aleph_1 < \text{add}(\mathcal{N}) < \text{cov}(\mathcal{N}) < \mathfrak{b} < \text{non}(\mathcal{E}) < \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) \\ < \text{cov}(\mathcal{E}) < \text{non}(\mathcal{N}) < \mathfrak{d} < \text{cof}(\mathcal{N}) < \mathfrak{c}?$$

Question 8

- ❶ $(A1)_{\mathcal{E}}$.
- ❷ $(A1)_{\mathcal{M}}$.
- ❸ $(A2)_{\mathcal{SN}}$.
- ❹ $(A2)_{\mathcal{I}_f}$ for any $f \in \omega^\omega$.

Question 8

- ❶ $(A1)_{\mathcal{E}}$.
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- ❸ $(A2)_{\mathcal{SN}}$.
- ❹ $(A2)_{\mathcal{I}_f}$ for any $f \in \omega^\omega$.

FS iterations of ccc forcings will not work to solve Question 8 because any such iteration forces $\text{non}(\mathcal{M}) \leq \text{cov}(\mathcal{M})$.

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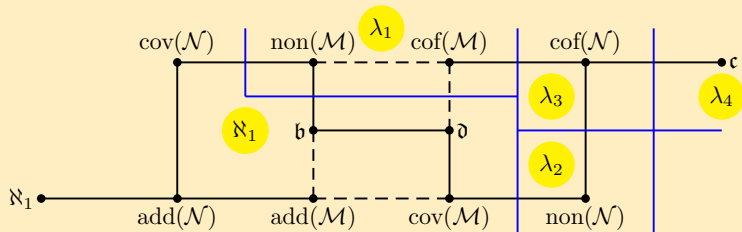
Roughly speaking, there are two approaches it could be used to solve these problems.

- Creature forcing method based on the notion of decisiveness (Kellner and Shelah [KS09, KS12]).
- Shattered iteration ([Bre19]).

Example

Theorem (Fischer, Goldstern, Kellner, and Shelah [FGKS17])

Under CH, if $\lambda_1 \leq \lambda_3 \leq \lambda_4$ and $\lambda_2 \leq \lambda_3$ are infinite cardinals such that $\lambda_i^{\aleph_0} = \lambda_i$ for $i \in \{1, 2, 3, 4\}$, then there is some proper ω^ω -bounding poset with \aleph_2 -cc forcing

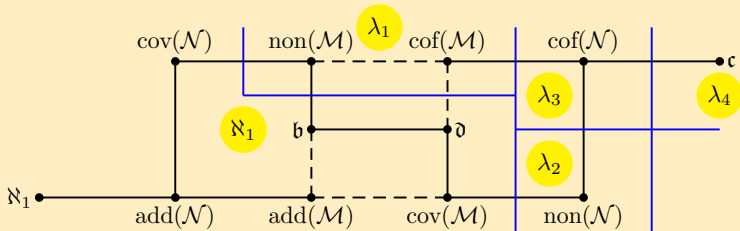


The constellation of Cichoń's diagram forced in [FGKS17], [GK21] (Goldstern and Klausner 2021).

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The main problem with this approach is that it is restricted to ω^ω -bounding forcings.

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