Chang models over derived models with supercompact measures, Part II

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Our context

We are interested in the relation between strong form of determinacy and supercompactness of ω_1.

Conjecture

The following theories are equiconsistent:

1. $\mathsf{ZFC}+$ *there is a Woodin limit of Woodin cardinals.*
2. $\mathsf{ZF}+\mathsf{AD}_{\mathbb{R}}+\Theta$ *is regular* $+\,\omega_1$ *is supercompact.*

Toward this conjecture, we construct a model called CDM^+, the Chang model over the derived model with supercompact measures.

- Our assumpetion is the existence of a hod mouse with some large cardinals, which is known to be consistent relative to a Woodin limit of Woodin cardinals.
- In a collapse extension of the hod mouse, we define a model of $\mathsf{AD}_{\mathbb{R}}+\Theta$ is regular that has possibly high degree of supercompactness of ω_1.

Iteration strategies

A model of set theory is called iterable if one can avoid to reach an ill-founded model during a construction of its iterated ultrapowers. In general, we need to consider non-linear iterations called **iteration trees**.

$$\begin{tikzcd}
& & & \mathcal{M}_7 & \\
\mathcal{M} = \mathcal{M}_0 \arrow[r, "E_1"] \arrow[dr, "E_0"] & \mathcal{M}_2 \arrow[r, "E_3"] & \mathcal{M}_4 \arrow[r, "E_5"] \arrow[ur, "E_6"] \arrow[dr, "E_4"] & \mathcal{M}_6 \arrow[r, "E_7"] & \cdots \\
& \mathcal{M}_1 \arrow[r, "E_2"] & \mathcal{M}_3 & \mathcal{M}_5 &
\end{tikzcd}$$

An **iteration strategy** Σ is a function such that for any iteration tree $\mathcal{T}$, $\Sigma(\mathcal{T})$ is a cofinal branch whose direct limit model is well-founded. So $\mathcal{M}$ is iterable if there is an iteration strategy Σ defined on iteration trees on $\mathcal{M}$.

Basic objects: $\mathcal{V}, \delta, g, \mathcal{P}$ and Σ

Throughout the rest of this talk, we fix the following objects.

- Let $\mathcal{V}$ be a countable model of ZFC that is a hod mouse.
- δ is a regular limit of Woodin cardinals in $\mathcal{V}$.
- Let $g \subseteq \mathrm{Col}(\omega, <\delta)$ is $\mathcal{V}$-generic.
- $\mathcal{P} = \mathcal{V}|(\delta^+)^{\mathcal{V}}$.
- Σ is the iteration strategy for $\mathcal{P}$ acting on iteration trees in $\mathcal{V}$ based on $\mathcal{P}|\delta$ determined by the internal strategy of $\mathcal{V}$. Actually, Σ can be uniquely extended to an iteration strategy in $\mathcal{V}[g]$, so we also denote it by Σ.

We work in $\mathcal{V}[g]$ throughout this talk.

The internal direct limit system

Remember that $g \subseteq \mathrm{Col}(\omega, < \delta)$ is the fixed $\mathcal{V}$-generic and we work in $\mathcal{V}[g]$.

- Let $I_g(\mathcal{P}, \Sigma)$ be the set of a Σ-iterate $\mathcal{Q}$ of $\mathcal{P}$ via $\mathcal{T}$ such that $\pi^{\mathcal{T}}(\delta) = \delta$.
- For $\mathcal{Q} \in I^g(\mathcal{P}, \Sigma)$ and $\eta < \delta$, define

$$\mathcal{F}_g(\mathcal{Q}, \eta) = \{\mathcal{R} \mid \mathcal{R} \text{ is a non-dropping } \Sigma_{\mathcal{Q}}\text{-iterates of } \mathcal{Q} \text{ via } \mathcal{T} \text{ of length} < \delta \text{ such that } \mathrm{crit}(\pi^{\mathcal{T}}) > \eta.\}$$

  Here, "non-dropping" means that an iteration map from $\mathcal{Q}$ to $\mathcal{R}$ exists. Also, $\Sigma_{\mathcal{Q}}$ is the iteration strategy for $\mathcal{Q}$ defined by $\Sigma_{\mathcal{Q}}(\mathcal{U}) = \Sigma(\mathcal{T}^\frown\mathcal{U})$.
- For any $\mathcal{Q}, \mathcal{R} \in \mathcal{F}_g(\mathcal{P})$, define

$$\mathcal{Q} \preceq \mathcal{R} \iff \mathcal{R} \text{ is a } \Sigma_{\mathcal{Q}}\text{-iterate of } \mathcal{Q}.$$

  We assume that a system $(\mathcal{F}_g(\mathcal{Q}, \eta), \preceq)$ is a directed system under iteration maps.

Precise definition of CDM$^+$

For $\mathcal{Q} \in I_g(\mathcal{P}, \Sigma)$ and $\eta < \delta$, define

$$\mathrm{CDM}^+(\mathcal{Q}, \eta) = L(\mathcal{M}_\infty(\mathcal{Q}, \eta), {}^\omega(\delta_\infty^{\mathcal{Q},\eta}), \Gamma_g^*, \mathbb{R}_g^*)[\langle \mu_\alpha \mid \alpha < \delta_\infty^{\mathcal{Q},\eta} \rangle],$$

where $L(\Gamma_g^*, \mathbb{R}_g^*)$ be the derived model of $\mathcal{V}$ at δ and μ_α is the club filter on $\wp_{\omega_1}(\alpha)$.

Main Theorem (G.–Müller–Sargsyan)

In $\mathcal{V}[g]$, there are a Σ-iterate $\mathcal{Q}$ of $\mathcal{P}$ and $\eta < \delta$ such that

$$\mathrm{CDM}^+(\mathcal{Q}, \eta) \models \mathsf{AD}^+ + \mathsf{AD}_\mathbb{R} + \Theta \text{ is regular} + \omega_1 \text{ is } < \delta_\infty^{\mathcal{Q},\eta}\text{-supercompact}.$$

Furthermore, if δ is a limit of $< \delta$-strong cardinals of $\mathcal{V}$, then $\delta_\infty^{\mathcal{Q},\eta} > \Theta$.

Determinacy in CDM^+

Regarding determinacy in the derived model, the following are known:

- (Woodin) The derived model of V satisfies AD^+.
- (Steel) The derived model of a hod mouse satisfies $\mathsf{AD}_{\mathbb{R}}$.
- (G.–Sargsyan) The derived model of a "self-iterable" structure at a regular limit of Woodin cardinals satisfies $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular.

We can show that

$$\mathsf{CDM}^+(\mathcal{Q}, \eta) \cap \wp(\mathbb{R}) = \Gamma_g^* = L(\Gamma_g^*, \mathbb{R}_g^*)$$

by Sargsyan's proof of the same fact for CDM. So $\mathsf{CDM}^+(\mathcal{Q}, \eta) \models \mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}}$. Also, G.–Sargsyan's proof shows that $\mathsf{CDM}^+(\mathcal{Q}, \eta) \models \mathsf{DC} + \Theta$ is regular.

To show the "furthermore" part, let $\kappa^{\mathcal{Q},\eta} < \delta$ be the least $< \delta$-strong cardinal above η in $\mathcal{Q}$. Then in $\mathsf{CDM}^+(\mathcal{Q}, \eta)$,

$$\Theta = \kappa_\infty^{\mathcal{Q},\eta} < \delta_\infty^{\mathcal{Q},\eta}.$$

This is not really new. Steel showed it in his book on hod mice.

Outline of our proof

Theorem

Let $\mathcal{Q} \in I_g^*(\mathcal{P}, \Sigma)$ and $\eta < \delta$ be such that $\mathcal{Q}$ is a genericity iterate of $\mathcal{P}$ and $(\mathcal{Q}, \eta)$ stabilizes δ_∞. Then for each $\alpha < \delta_\infty^{\mathcal{Q},\eta}$, $\mu_\alpha \cap \mathrm{CDM}^+(\mathcal{Q}, \eta)$ is a supercompact measure on $\wp_{\omega_1}(\alpha)$ in $\mathrm{CDM}^+(\mathcal{Q}, \eta)$.

1. Introduce some terminologies and choose $\mathcal{Q}$ and η.
2. Reduce Theorem to two main lemmas by taking a "better" iterate $\mathcal{R}$ of $\mathcal{Q}$ and considering $\mathcal{F}_g(\mathcal{R}, \eta')$ for some $\eta' > \eta$.
3. Show Main Lemma 1. This is easier one and our argument doesn't depend on the choice of $\mathcal{R}$ and η'.
4. Show Main Lemma 2, which is the core of our proof. We start with describing how to choose $\mathcal{R}$ and η'.

Genericity iterates

The following theorem makes an iterable structure with a Woodin cardinal of special interest.

Theorem (Neeman; there is another version due to Woodin)

Let M be a sufficiently iterable structure and let δ be a Woodin cardinal of M that is countable in V. Then for any $x \subseteq \omega$, there is an iterate N of M such that x is generic over N via $\mathrm{Col}(\omega, j(\delta))$, where $j \colon M \to N$ be the iteration map.

Using this, one can show that for any $\mathcal{P}^* \in \mathcal{F}_g(\mathcal{P}, \eta)$ and $\eta' \in (\eta, \delta)$, there is $\mathcal{Q} \in I^g(\mathcal{P}, \Sigma)$ such that $\mathcal{Q}$ is an iterate of $\mathcal{P}^*$,

- $\mathbb{R}^{\mathcal{P}[g]} = \mathbb{R}^{\mathcal{Q}[h]}$ for some $h \subseteq \mathrm{Col}(\omega, < \delta)$ in $\mathcal{V}[g]$, and
- $\mathrm{crit}(\pi_{\mathcal{P}^*, \mathcal{Q}}) > \eta'$, where $\pi_{\mathcal{P}^*, \mathcal{Q}} \colon \mathcal{P}^* \to \mathcal{Q}$ is the iteration map.

Such a $\mathcal{Q}$ is called a **genericity iterate** of $\mathcal{P}$ above η.

$\mathcal{Q}$ is obtained by making initial segments of $\mathcal{P}$ generic using Woodin cardinals above η'. The length of the iteration tree from $\mathcal{P}$ to $\mathcal{Q}$ is δ, but the iteration map does not move δ.

Lemma

Let $\mathcal{Q}$ be a genericity iterate of $\mathcal{P}$ above η and let $h \subseteq \mathrm{Col}(\omega, <\delta)$ such that $\mathbb{R}^{\mathcal{P}[g]} = \mathbb{R}^{\mathcal{Q}[h]}$. Then

$$\mathcal{M}_\infty(\mathcal{P}, \eta) = (\mathcal{M}_\infty(\mathcal{Q}, \eta))^{\mathcal{V}_\mathcal{Q}[h]}$$

and $\pi^{\mathcal{P},\eta}_{\mathcal{P},\infty} = \pi^{\mathcal{Q},\eta}_{\mathcal{Q},\infty} \circ \pi_{\mathcal{P},\mathcal{Q}}$. Furthermore, $\mathsf{CDM}^+(\mathcal{P}, \eta) = (\mathsf{CDM}^+(\mathcal{Q}, \eta))^{\mathcal{V}_\mathcal{Q}[h]}$.

When applying the $\mathcal{P}$-to-$\mathcal{Q}$ iteration tree to $\mathcal{V}$, its last model is denoted by $\mathcal{V}_\mathcal{Q}$. Then $\pi_{\mathcal{V},\mathcal{V}_\mathcal{Q}}$ extends $\pi_{\mathcal{P},\mathcal{Q}}$.

$$\begin{tikzcd}
 & & \mathcal{M}_\infty(\mathcal{P}, \eta) \\
\mathcal{P} \arrow[urr, bend left, "\pi^{\mathcal{P},\eta}_{\mathcal{P},\infty}"] \arrow[r, "\pi_{\mathcal{P},\mathcal{P}^*}"'] \arrow[d, phantom, "\trianglelefteq"] & \mathcal{P}^* \arrow[ur, "\pi^{\mathcal{P},\eta}_{\mathcal{P}^*,\infty}"] \arrow[r, "\pi_{\mathcal{P}^*,\mathcal{Q}}"'] & \mathcal{Q} \arrow[u, "\pi^{\mathcal{Q},\eta}_{\mathcal{Q},\infty}"] \arrow[d, phantom, "\trianglelefteq"] \\
\mathcal{V} \arrow[rr, "\pi_{\mathcal{V},\mathcal{V}_\mathcal{Q}}"] & & \mathcal{V}_\mathcal{Q}
\end{tikzcd}$$

Stabilizing δ_∞: how to choose $\mathcal{Q}$ and η

If $\eta \leq \eta' < \delta$ then $\delta_\infty^{\mathcal{Q},\eta} \geq \delta_\infty^{\mathcal{Q},\eta'}$ because $\mathcal{F}_g(\mathcal{Q},\eta')$ is a subsystem of $\mathcal{F}_g(\mathcal{Q},\eta)$.

Lemma

There is a genericity iterate $\mathcal{Q}$ of $\mathcal{P}$ and an ordinal $\eta < \delta$ such that for any genericity iterate $\mathcal{R}$ of $\mathcal{Q}$ above η and any ordinal $\eta' \in [\eta, \delta)$, $\delta_\infty^{\mathcal{Q},\eta} = \delta_\infty^{\mathcal{R},\eta'}$.

This lemma immediately follows from another fact on genericity iterates: a genericity iterate of a genericity iterate of $\mathcal{Q}$ is a genericity iterate of $\mathcal{Q}$.

Proof. Suppose not. Then one can inductively find $\langle \mathcal{Q}_n, \eta_n \mid n < \omega \rangle$ such that for any $n < \omega$, $\mathcal{Q}_{n+1}$ is a genericity iterate of $\mathcal{Q}_n$, $\eta_n < \eta_{n+1}$, and $\delta_\infty^{\mathcal{Q}_n,\eta_n} > \delta_\infty^{\mathcal{Q}_{n+1},\eta_{n+1}}$. This is a contradiction as we have found a decreasing infinite sequence of ordinals.

We say $(\mathcal{Q},\eta)$ stabilizes δ_∞ if it satisfies the conclusion of the lemma.

Reducing Theorem to two Main Lemmas

Theorem

Let $\mathcal{Q} \in I_g^*(\mathcal{P}, \Sigma)$ and $\eta < \delta$ be such that $\mathcal{Q}$ is a genericity iterate of $\mathcal{P}$ and $(\mathcal{Q}, \eta)$ stabilizes δ_∞. Then for each $\alpha < \delta_\infty^{\mathcal{Q},\eta}$, $\mu_\alpha \cap \mathrm{CDM}^+(\mathcal{Q}, \eta)$ is a supercompact measure on $\wp_{\omega_1}(\alpha)$ in $\mathrm{CDM}^+(\mathcal{Q}, \eta)$.

Proof. Let $A \subseteq \wp_{\omega_1}(\alpha)$ be such that $A \in \mathrm{CDM}^+(\mathcal{Q}, \eta)$. Then we will choose some genericity iterate $\mathcal{R}$ of $\mathcal{Q}$ and $\eta' \in (\eta, \delta)$ such that $(\mathcal{R}, \eta)$ "stabilizing parameters in the definition of A."

For any $\mathcal{R}^* \in \mathcal{F}_k(\mathcal{R}, \eta')$, let

$$\sigma_{\mathcal{R}^*} = \mathrm{ran}(\pi^{\mathcal{R},\eta'}_{\mathcal{R}^*,\infty}) \cap \alpha \in \wp_{\omega_1}(\alpha).$$

Here, note that $\alpha < \delta_\infty^{\mathcal{R},\eta'} = \delta_\infty^{\mathcal{Q},\eta}$. Also, note that for any $\xi < \delta$, $\mathcal{R}|\xi$ is countable in $\mathcal{V}[g]$, where $g \subseteq \mathrm{Col}(\omega, < \delta)$.

Remember that $A \subseteq \wp_{\omega_1}(\alpha)$ be such that $A \in \mathrm{CDM}^+(\mathcal{Q}, \eta)$ and we took some $\mathcal{R}$ and η'. For any $\mathcal{R}^* \in \mathcal{F}_k(\mathcal{R}, \eta')$, let $\sigma_{\mathcal{R}^*} = \mathrm{ran}(\pi^{\mathcal{R},\eta'}_{\mathcal{R}^*,\infty}) \cap \alpha \in \wp_{\omega_1}(\alpha)$.

Main Lemma 1

If $\alpha \in [\delta, \delta^{\mathcal{Q},\eta}_\infty)$, then the set

$$\{\sigma_{\mathcal{R}^*} \mid \mathcal{R}^* \in \mathcal{F}_k(\mathcal{R}, \eta') \wedge \alpha \in \mathrm{ran}(\pi^{\mathcal{R},\eta'}_{\mathcal{R}^*,\infty})\}$$

contains a club subset of $\wp_{\omega_1}(\alpha)$.

Main Lemma 2

1. *If $\sigma_{\mathcal{R}} \in A$, then $\{\sigma_{\mathcal{R}^*} \mid \mathcal{R}^* \in \mathcal{F}_k(\mathcal{R}, \eta') \wedge \alpha \in \mathrm{ran}(\pi^{\mathcal{R},\eta'}_{\mathcal{R}^*,\infty})\} \subseteq A$*
2. *If $\sigma_{\mathcal{R}} \notin A$, then $\{\sigma_{\mathcal{R}^*} \mid \mathcal{R}^* \in \mathcal{F}_k(\mathcal{R}, \eta') \wedge \alpha \in \mathrm{ran}(\pi^{\mathcal{R},\eta'}_{\mathcal{R}^*,\infty})\} \subseteq \wp_{\omega_1}(\alpha) \setminus A$.*

These lemmas imply that $\mu_\alpha \cap \mathrm{CDM}^+(\mathcal{R}, \eta)$ is an ultrafilter in $\mathrm{CDM}^+(\mathcal{Q}, \eta)$. Its countably completeness, fineness and normality easily follows as the club filter μ_α has these properties.

Catching elements in $\mathcal{M}_\infty$

We frequently use the following simple argument: let $\alpha \in \mathcal{M}_\infty(\mathcal{Q}, \eta)$. Then there is a $\mathcal{Q}^* \in \mathcal{F}_h(\mathcal{Q}, \eta)$ catching α, i.e., $\alpha \in \operatorname{ran}(\pi^{\mathcal{Q},\eta}_{\mathcal{Q}^*,\infty})$.

- If $\mathcal{Q}^*$ catches α, then its iterates also catch α. In particular, one can find an iterate $\mathcal{R}$ of $\mathcal{Q}^*$ such that $\mathcal{R}$ is a genericity iterate of $\mathcal{Q}$ catching α.

$$
\begin{tikzcd}
 & & & \mathcal{M}_\infty(\mathcal{Q},\eta) \arrow[r, phantom, "\ni"] & \alpha \\
\mathcal{Q} \arrow[rrru, bend left, "\pi^{\mathcal{Q},\eta}_{\mathcal{Q},\infty}"] \arrow[r, "\pi_{\mathcal{Q},\mathcal{Q}^*}"'] & \mathcal{Q}^* \arrow[rru, "\pi^{\mathcal{Q},\eta}_{\mathcal{Q}^*,\infty}"] \arrow[rr, "\pi_{\mathcal{Q}^*,\mathcal{R}}"'] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & \mathcal{R} \arrow[u, "\pi^{\mathcal{R},\eta}_{\mathcal{R},\infty}"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \\
 & \alpha_{\mathcal{Q}^*} \arrow[rr, mapsto] & & \alpha_{\mathcal{R}} \arrow[ruu, mapsto, bend right] &
\end{tikzcd}
$$

- Since $\mathcal{F}_h(\mathcal{Q}, \eta)$ is countably closed, for any countable $\sigma \subseteq \mathcal{M}_\infty(\mathcal{Q}, \eta)$, there is a $\mathcal{Q}^* \in \mathcal{F}_h(\mathcal{Q}, \eta)$ such that $\sigma \subseteq \operatorname{ran}(\pi^{\mathcal{Q},\eta}_{\mathcal{Q}^*,\infty})$.

Main Lemma 1: Finding a club set

Main Lemma 1

If $\alpha \in [\delta, \delta_\infty^{\mathcal{Q},\eta})$, then the set

$$\{\sigma_{\mathcal{R}^*} \mid \mathcal{R}^* \in \mathcal{F}_k(\mathcal{R}, \eta') \wedge \alpha \in \mathrm{ran}(\pi^{\mathcal{R},\eta'}_{\mathcal{R}^*,\infty})\}$$

contains a club subset of $\wp_{\omega_1}(\alpha)$.

Proof. Fix a bijection $f : \delta \to \wp_{\omega_1}(\alpha)$ (in $\mathcal{V}[g]$). We inductively define $\mathcal{R}_\xi \in \mathcal{F}_k(\mathcal{R}, \eta')$ for $\xi < \delta$ as follows:

1. Let $\mathcal{R}_0 \in \mathcal{F}_k(\mathcal{R}, \eta')$ be any $\mathcal{R}^*$ with $\alpha \in \mathrm{ran}(\pi^{\mathcal{R},\eta'}_{\mathcal{R}^*,\infty})$.
2. For each $\xi < \delta$, let $\mathcal{R}_{\xi+1} \in \mathcal{F}_k(\mathcal{R}, \eta')$ be an iterate of $\mathcal{R}_\xi$ such that $f(\xi) \subseteq \mathrm{ran}(\pi^{\mathcal{R},\eta'}_{\mathcal{R}^*,\infty})$.
3. For each limit ordinal $\lambda < \delta$, let $\mathcal{R}_\lambda$ be the direct limit of $\mathcal{R}_\beta$'s.

By the construction, $\alpha \in \mathrm{ran}(\pi^{\mathcal{R},\eta'}_{\mathcal{R}_\xi,\infty})$ for any $\xi < \delta$ and $\{\sigma_{\mathcal{R}_\xi} \mid \xi < \delta\}$ is a closed unbounded subset of $\wp_{\omega_1}(\alpha)$.

Stabilizing parameters: how to choose $\mathcal{R}$

For any $\mathcal{R}^* \in \mathcal{F}_k(\mathcal{R}, \eta')$, let $\sigma_{\mathcal{R}^*} = \mathrm{ran}(\pi^{\mathcal{R},\eta'}_{\mathcal{R}^*,\infty}) \cap \alpha \in \wp_{\omega_1}(\alpha)$.

Main Lemma 2

1. *If $\sigma_{\mathcal{R}} \in A$, then $\{\sigma_{\mathcal{R}^*} \mid \mathcal{R}^* \in \mathcal{F}_k(\mathcal{R}, \eta') \wedge \alpha \in \mathrm{ran}(\pi^{\mathcal{R},\eta'}_{\mathcal{R}^*,\infty})\} \subseteq A$*
2. *If $\sigma_{\mathcal{R}} \notin A$, then $\{\sigma_{\mathcal{R}^*} \mid \mathcal{R}^* \in \mathcal{F}_k(\mathcal{R}, \eta') \wedge \alpha \in \mathrm{ran}(\pi^{\mathcal{R},\eta'}_{\mathcal{R}^*,\infty})\} \subseteq \wp_{\omega_1}(\alpha) \setminus A$.*

We only prove (1). Let $\alpha < \delta^{\mathcal{Q},\eta}_\infty$ and let $A \subseteq \wp_{\omega_1}(\alpha)$ in $\mathsf{CDM}^+(\mathcal{Q}, \eta)$. Then for some formula ϕ in the language for $\mathsf{CDM}^+(\mathcal{Q}, \eta)$ and for some ordinal γ,

$$A = \{\sigma \in \wp_{\omega_1}(\alpha) \mid \mathsf{CDM}^+(\mathcal{Q}, \eta)|\gamma \models \phi(\sigma, Y, Z, x, \vec{\beta})\},$$

where $Y = \langle Y(i) \mid i < \omega\rangle \in {}^\omega\xi$ for some $\xi < \delta^{\mathcal{Q},\eta}_\infty$, $Z \in \Gamma^*_g$, $x \in \mathbb{R}^*_g$, and $\vec{\beta} \in {}^{<\omega}\gamma$.

Then we can take a genericity iterate $\mathcal{R}$ of $\mathcal{Q}$ above η such that

$$\{\alpha, \vec{\beta}, \gamma\} \cup \mathrm{ran}(Y) \subseteq \mathrm{ran}(\pi^{\mathcal{R},\eta}_{\mathcal{V}_{\mathcal{R}},\infty}).$$

Claim

Whenever $\mathcal{S}$ is a genericity iterate of $\mathcal{R}$ above η, $\pi_{\mathcal{V}_\mathcal{R},\mathcal{V}_\mathcal{S}}((\alpha,\vec{\beta},\gamma)) = (\alpha,\vec{\beta},\gamma)$ and $\pi_{\mathcal{V}_\mathcal{R},\mathcal{V}_\mathcal{S}}(Y(i)) = Y(i)$ for any $i < \omega$.

Proof. Let $\alpha_\mathcal{R} < \delta$ be such that $\alpha = \pi^{\mathcal{R},\eta}_{\mathcal{V}_\mathcal{R},\infty}(\alpha_\mathcal{R})$. Then we have

$$\begin{aligned}\pi_{\mathcal{V}_\mathcal{R},\mathcal{V}_\mathcal{S}}(\alpha) &= \pi_{\mathcal{V}_\mathcal{R},\mathcal{V}_\mathcal{S}}(\pi^{\mathcal{R},\eta}_{\mathcal{V}_\mathcal{R},\infty}(\alpha_\mathcal{R}))\\ &= \pi^{\mathcal{S},\eta}_{\mathcal{V}_\mathcal{S},\infty}(\pi_{\mathcal{V}_\mathcal{R},\mathcal{V}_\mathcal{S}}(\alpha_\mathcal{R}))\\ &= \pi^{\mathcal{R},\eta}_{\mathcal{V}_\mathcal{R},\infty}(\alpha_\mathcal{R}) = \alpha.\end{aligned}$$

The second equation follows from the elementarity of $\pi_{\mathcal{V}_\mathcal{R},\mathcal{V}_\mathcal{S}}$ and the third equation holds since $\pi^{\mathcal{R},\eta}_{\mathcal{V}_\mathcal{R},\infty} = \pi^{\mathcal{S},\eta}_{\mathcal{V}_\mathcal{S},\infty} \circ \pi_{\mathcal{V}_\mathcal{R},\mathcal{V}_\mathcal{S}}$.

Replacing parameters with reals: how to choose η'

Let $k \subseteq \mathrm{Col}(\omega, <\delta)$ be $\mathcal{R}$-generic such that $\mathbb{R}^{\mathcal{P}[g]} = \mathbb{R}^{\mathcal{R}[k]}$. We code the parameters Y and Z by reals y and z respectively as follows.

- Let $\xi_Y < \delta$ be such that $Y \subseteq \pi^{\mathcal{R},\eta}_{\mathcal{R},\infty}[\xi_Y]$. Let $y \in \mathbb{R}^*_k$ code a function $f_y \colon \omega \to \xi_Y$ such that for any $i \in \omega$,

$$Y(i) = \pi^{\mathcal{R},\eta}_{\mathcal{R},\infty}(f_y(i)).$$

- Note that $\{\mathrm{Code}(\Sigma^g_{\mathcal{P}|\xi}) \mid \xi < \delta\}$ is Wadge cofinal in Γ^*_g (Standard fact). So we may assume that $Z = \mathrm{Code}(\Sigma^g_{\mathcal{P}|\xi_Z})$ for some $\xi_Z < \delta$. Let $z \in \mathbb{R}^*_k$ be a real coding $\pi_{\mathcal{P},\mathcal{R}} \restriction \mathcal{P}|\xi_Z \colon \mathcal{P}|\xi_Z \to \mathcal{R}|\pi_{\mathcal{P},\mathcal{R}}(\xi_Z)$. Then Z can be defined from z as the code of $\pi_{\mathcal{P},\mathcal{R}}$-pullback of the strategy for $\mathcal{R}|\pi_{\mathcal{P},\mathcal{R}}(\xi_Z)$ determined by the strategy predicate of $\mathcal{R}$.

Now choose any $\eta' \in [\max\{\eta, \xi_Y, \pi_{\mathcal{Q},\mathcal{R}}(\xi_Z)\}, \delta)$ be such that $x, y, z \in \mathcal{R}[h \restriction \eta']$.

Recall that

$$A = \{\sigma \in \wp_{\omega_1}(\alpha) \mid \mathsf{CDM}^+(\mathcal{Q},\eta)|\gamma \models \phi(\sigma, Y, Z, x, \vec{\beta})\},$$

and for any $\mathcal{R}^* \in \mathcal{F}_k(\mathcal{R},\eta')$, let

$$\sigma_{\mathcal{R}^*} = \mathrm{ran}(\pi^{\mathcal{R},\eta'}_{\mathcal{R}^*,\infty}) \cap \alpha \in \wp_{\omega_1}(\alpha).$$

Suppose that $\sigma_{\mathcal{R}} \in A$. Then

$$\mathcal{V}_{\mathcal{R}}[x,y,z] \models \phi^*(\mathrm{ran}(\pi^{\mathcal{R},\eta'}_{\mathcal{R}^*,\infty}) \cap \alpha, x, y, z, \eta, \delta, \vec{\beta}, \gamma),$$

where ϕ^* is defined as follows:

- y codes a function $f\colon \omega \to \xi$ for some $\xi < \delta$, and
- z codes an elementary embedding $\pi\colon \mathcal{M} \to \mathcal{N}$, where $\mathcal{N}$ is an initial segment of $\mathcal{R}$, and
- letting $Y = \langle \pi^{\mathcal{R},\eta}_{\mathcal{R},\infty}(f(i)) \mid i \in \omega\rangle$ and Z be the code of the π-pullback of the strategy for $\mathcal{N}$ determined by the strategy predicate of $\mathcal{R}$, the maximal element of $\mathrm{Col}(\omega, {<}\delta)$ forces $\mathsf{CDM}^+(\mathcal{R},\eta)|\gamma \models \phi(u, Y, Z, x, \vec{\beta})$.

Let $\mathcal{R}^* \in \mathcal{F}_k(\mathcal{R}, \eta')$ such that $\alpha \in \mathrm{ran}(\pi^{\mathcal{R},\eta'}_{\mathcal{R}^*,\infty})$. We want to show $\sigma_{\mathcal{R}^*} \in A$.

Since $\alpha < \delta^{\mathcal{Q},\eta}_\infty = \delta^{\mathcal{R},\eta'}_\infty$, we have $\alpha_{\mathcal{R}^*} := (\pi^{\mathcal{R},\eta'}_{\mathcal{R}^*,\infty})^{-1}(\alpha) < \delta$. Then there is an iterate $\mathcal{S}$ of $\mathcal{R}^*$ such that it is a genericity iterate of $\mathcal{R}$ with $\mathrm{crit}(\pi_{\mathcal{R}^*,\mathcal{S}}) > \alpha_{\mathcal{R}^*}$.

$$\begin{tikzcd}
 & & \mathcal{M}_\infty(\mathcal{R},\eta') \\
\mathcal{R} \arrow[r] \arrow[urr, bend left] & \mathcal{R}^* \arrow[r] \arrow[ur] & \mathcal{S} \arrow[u]
\end{tikzcd}$$

The elementarity of $\pi^+_{\mathcal{V}_\mathcal{R},\mathcal{V}_\mathcal{S}} : \mathcal{V}_\mathcal{R}[x,y,z] \to \mathcal{V}_\mathcal{S}[x,y,z]$, which is the canonical liftup of $\pi_{\mathcal{V}_\mathcal{R},\mathcal{V}_\mathcal{S}}$, implies that

$$\mathcal{V}_\mathcal{S}[x,y,z] \models \phi^*(\mathrm{ran}(\pi^{\mathcal{S},\eta'}_{\mathcal{S},\infty}) \cap \alpha, x, y, z, \eta, \delta, \vec{\beta}, \gamma).$$

Unraveling the definition of ϕ^*, we get $\sigma_\mathcal{S} \in A$.

$$\begin{tikzcd}
 & & \mathcal{M}_\infty(\mathcal{R},\eta') & \ni & \alpha \\
\mathcal{R} \arrow[r, "\pi_{\mathcal{R},\mathcal{R}^*}"'] \arrow[urr, bend left, "\pi^{\mathcal{R},\eta'}_{\mathcal{R},\infty}"] & \mathcal{R}^* \arrow[r, "\pi_{\mathcal{R}^*,\mathcal{S}}"'] \arrow[ur, "\pi^{\mathcal{R},\eta'}_{\mathcal{R}^*,\infty}"] & \mathcal{S} \arrow[u, "\pi^{\mathcal{S},\eta'}_{\mathcal{S},\infty}"] & & \\
 & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & & \\
 & \alpha_{\mathcal{R}^*} \arrow[r, equal] & \alpha_{\mathcal{S}} \arrow[uuurr, mapsto, bend right] & &
\end{tikzcd}$$

Because $\mathrm{crit}(\pi_{\mathcal{R}^*,\mathcal{S}}) > \alpha_{\mathcal{R}^*}$ and $\pi^{\mathcal{R},\eta'}_{\mathcal{R}^*,\infty} = \pi^{\mathcal{S},\eta'}_{\mathcal{S},\infty} \circ \pi_{\mathcal{R}^*,\mathcal{S}}$, we have

$$\sigma_{\mathcal{R}^*} = \pi^{\mathcal{R},\eta'}_{\mathcal{R}^*,\infty}[\alpha_{\mathcal{R}^*}] = \pi^{\mathcal{S},\eta'}_{\mathcal{S},\infty}[\alpha_{\mathcal{S}}] = \sigma_{\mathcal{S}}.$$

It follows that $\sigma_{\mathcal{R}^*} \in A$. This completes the proof of Main Lemma 2.

Another related result

Steel independently constructed CDM with supercompactness measures assuming that δ is a measurable Woodin (which is not known to be consistent). Adapting G.–Sargsyan's proof of CM $\models$ AD$^+$, he showed that

$$\mathsf{CM}^+ \models \mathsf{AD}^+ + \omega_1 \text{ is supercompact.}$$

Questions and future works

It seems that there is plenty of room for research on Chang-type models of determinacy.

- How does stronger large cardinal property (e.g., a Woodin limit of Woodin cardinals) in a hod mouse affect the property of CDM or its variants?
- Find other definable objects that can be added to CDM.
- Analyze the internal theory of Chang-type models in more detail.
  $\rightsquigarrow$ new consistency results via the $\mathbb{P}_{\max}$ forcing.

I believe that Chang-type models, or more generally, determinacy models satisfying $V \neq L(\wp(\mathbb{R}))$ will play critical roles in inner model theory.

Main references

- Grigor Sargsyan, *Covering with Chang models over derived models*, Adv. Math. **384** (2021), Paper No. 107717, 21.
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- Takehiko Gappo, Sandra Müller & Grigor Sargsyan, *Chang models over derived models with supercompact measures*, in preparation.