Chang models over derived models with supercompact measures, Part I

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Overview

This talk is all about strong form of the Axiom of Determinacy. I'll present the construction of a new determinacy model, called the Chang model over the derived model with supercompact measures (CDM$^+$).

- This week, I'm going to explain the motivation and the statement of our result. In the end, I'll show you a precise definition of our model.
- Next week, I'm going to prove the main property of our model, so the talk will be more technical. However, it only involves mild inner model theory.

This is joint work with Sandra Müller and Grigor Sargsyan.

Infinite games

For any $A \subseteq {}^{\omega}X$, the game $G_\omega(A; X)$ is as follows: two players take turns choosing elements of X.

$$\begin{array}{c||ccccccc} \mathrm{I} & x_0 & & x_2 & & \cdots & & \\ \hline \mathrm{II} & & x_1 & & x_3 & & \cdots \end{array}$$

I wins the game if and only if $\langle x_0, x_1, \dots \rangle \in A$.

- $\sigma \colon {}^{<\omega}X \to X$ is a winning strategy for I if for any $\langle x_{2n+1} \rangle_{n<\omega} \in {}^{\omega}X$, letting $x_0 = \sigma(\emptyset)$ and $x_{2n} = \sigma(\langle x_0, \dots, x_{2n-1} \rangle)$ for any $n > 1$, then
$$\langle x_0, x_1, \dots \rangle \in A.$$
- A winning strategy for II is defined in an analogous way.

We say A is **determined** if I or II has a winning strategy in $G_\omega(A; X)$.

Axiom of Determinacy

ZFC+ Large Cardinal Axiom implies determinacy of subsets of $\mathbb{R} := {}^\omega\omega$.

- (Martin) ZFC $\Rightarrow$ All Borel subsets of $\mathbb{R}$ are determined.
- (Martin) ZFC+ a measurable $\Rightarrow$ All analytic subsets of $\mathbb{R}$, i.e. the projections of Borel sets, are determined.
- (Martin–Steel) ZFC+ ω many Woodins + a measurable above them $\Rightarrow$ All subsets of $\mathbb{R}$ in $L(\mathbb{R})$, i.e. the subsets that are constructible from $\mathbb{R}$, are determined.

Definition

The Axiom of Determinacy (AD) *is the statement that any $A \subseteq {}^\omega\omega$ is determined.*

Remark: ZFC $\Rightarrow \neg$AD.

Stronger forms of determinacy

During this talk, we consider the following stronger axioms of determinacy.

1. AD^+ is Woodin's technical improvement of AD. It is conjectured that AD^+ is equivalent to AD.
2. $\mathsf{AD}_{\mathbb{R}}$ is the statement that any $A \subseteq {}^{\omega}\mathbb{R}$ is determined.
3. Let $\Theta = \sup\{\alpha \in \mathrm{Ord} \mid \exists f : \mathbb{R} \to \alpha \, (f \text{ is surjective})\}$.
   The theory "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular" is much stronger than $\mathsf{AD}_{\mathbb{R}}$.

Sargsyan showed that the consistency strength of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular" is strictly below a Woodin limit of Woodin cardinals.

However, no determinacy axiom that has consistency strength exactly at the level of a Woodin limit of Woodin cardinals has been found yet. We consider one of such candidates — $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular + ω_1 is supercompact.

Supercompactness

For a filter μ on $\wp_\kappa(X) = \{\sigma \subseteq X \mid |X| < \kappa\}$, we say

1. μ is countably complete if it is closed under countable intersections.
2. μ is fine if for any $x \in X$, $\{\sigma \in \wp_\kappa(X) \mid x \in \sigma\} \in \mu$.
3. μ is normal if it is closed under diagonal intersections, i.e. whenever $\langle A_x \mid x \in X\rangle$ is a sequence such that $A_x \in \mu$ for all $x \in X$, then $\triangle_{x \in X} A_x := \{\sigma \in \wp_\kappa(X) \mid \sigma \in \bigcap_{x \in \sigma} A_x\} \in \mu$.

The club filter on $\wp_\kappa(X)$ has all these properties.

Definition

*We say κ is X-**supercompact** if there is a countably complete fine normal ultrafilter on $\wp_\kappa(X)$. Also, we say κ is supercompact if κ is X-supercompact for any set X.*

This definition is meaningful even in the absence of the Axiom of Choice.

Determinacy and supercompactness of ω_1

Many theorems are known so far regarding determinacy and supercompact measures on ω_1 in models of determinacy. The following are such examples.

1. (Solovay) $\mathrm{AD}_{\mathbb{R}}$ implies that ω_1 is $\mathbb{R}$-supercompact.
2. (Harington, Kechris, Neeman & Woodin?) AD^+ implies that ω_1 is $< \Theta$-supercompact.
3. (Woodin) The theory "$\mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact" is equiconsistent with the existence of ω^2 Woodin cardinals.
4. (Trang–Wilson) If $\mathrm{DC} + \omega_1$ is $\wp(\mathbb{R})$-supercompact, then there is a sharp for a model of $\mathrm{AD}_{\mathbb{R}} + \mathrm{DC}$.
5. (Trang) The theory "$\mathrm{AD}_{\mathbb{R}} + \Theta$ is regular $+ \omega_1$ is $\wp(\mathbb{R})$-supercompact" is equiconsistent with the theory "$\mathrm{AD}_{\mathbb{R}} +\ \Theta$ is measurable."
6. (Ikegami–Trang) If ω_1 is supercompact and HPC (Hod Pair Capturing) holds, then any Suslin-co-Suslin subsets of $\mathbb{R}$ is determined.

Woodin's result on full supercompactness of ω_1

Theorem (Woodin)

Assuming the existence of a proper class of Woodin limits of Woodin cardinals, there is an inner model containing $\mathrm{Ord} \cup \mathbb{R}$ *of* $\mathrm{AD}^+ + \omega_1$ *is supercompact.*

Regarding stronger forms of determinacy in this model, the following are known:

- (Ikegami–Trang) If ω_1 is supercompact, then AD^+ is equivalent to $\mathrm{AD}_{\mathbb{R}}$.
- (Woodin) Assuming the determinacy of some definable game of length ω_1, which is still unknown to be consistent, the model of the theorem also satisfies that Θ is regular.

Conjecture on supercompactness of ω_1

A strong determinacy axiom that has consistency strength exactly at the level of a Woodin limit of Woodin cardinals has not been found. Based on the previous results, we conjecture the following.

Conjecture

The following theories are equiconsistent:

1. $\mathsf{ZFC} +$ *there is a Woodin limit of Woodin cardinals.*
2. $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular* $+ \omega_1$ *is supercompact.*

Toward this conjecture, we took a different path from Woodin. Starting from an inner model theoretic assumption (the existence of a hod mouse), which is known to be consistent relative to a Woodin limit of Woodin cardinals, we construct a model of $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular that has possibly high degree of supercompactness of ω_1.

Chang model

The **Chang model** is the minimal inner model of ZF closed under countable sequences. More precisely,

$$\mathsf{CM} = \bigcup_{\alpha \in \mathsf{Ord}} L({}^{\omega}\alpha)$$

Theorem (Woodin)

If there is a proper class of Woodin limits of Woodin cardinlas, then the theory of CM *cannot be changed by a set-sized forcing.*

The theory of CM is still mysterious and almost nothing is known yet. The following theorem is one of few exceptions.

Theorem

1. *(Woodin) If there is a proper class of Woodin limits of Woodin cardinlas, then* $\mathsf{CM} \models \mathsf{AD}^+$ *in any generic extension.*
2. *(G.–Sargsyan) If there is a hod mouse with a Woodin limit of Woodin cardinal, then* $\mathsf{CM} \models \mathsf{AD}^+$ *in any generic extension.*

Generalized Chang model

Woodin also introduced a variant of the Chang model:

$$\mathsf{CM}^+ = \bigcup_{\alpha \in \mathsf{Ord}} L({}^{\omega}\alpha)[\mu_\alpha],$$

where μ_α is the club filter on $\wp_{\omega_1}({}^{\omega}\alpha)$.

Theorem (Woodin)

If there is a proper class of Woodin limits of Woodin cardinals, then

$$\mathsf{CM}^+ \models \mathsf{AD}^+ + \omega_1 \textit{ is supercompact.}$$

Our basic idea

The basic idea of our construction is as follows.

1. We start with a derived model, which is a canonical determinacy model of the form $V = L(\wp(\mathbb{R}))$.
2. First, we add ω-sequences of ordinals to a derived model. This was done by Sargsyan and we call the resulting model CDM (the Chang model over the derived model).
3. To get (partial) supercompactness of ω_1, we add club filters to CDM. The resulting model is called CDM^+.
4. The construction of CDM and CDM^+ is done in a collapse extension of a hod mouse.

The advantage of this approach is that we can often generalize properties for derived models to CDM and CDM^+ without difficulty. In particular, the proof of $\mathrm{CDM}^+ \models \mathrm{AD}_{\mathbb{R}} + \Theta$ is regular is the same as one for derived models.

Universally Baire sets

A set $T \subseteq {}^{<\omega}(\omega \times \mathrm{Ord})$ is a tree if whenever $t \in T$ and $n < \mathrm{lh}(t)$, $t \restriction n \in T$. Let $[T] \subseteq {}^{\omega}(\omega \times \mathrm{Ord}) \approx {}^{\omega}\omega \times {}^{\omega}\mathrm{Ord}$ be the set of infinite branches of T. Also, let $p[T] = \{x \in {}^{\omega}\omega \mid \exists y \, \langle x, y \rangle \in [T]\}$.

Definition

Let λ be an infinite cardinal. We say that $A \subseteq {}^{\omega}\omega$ is $<\lambda$-**universally Baire** *($<\lambda$-uB) if there are trees $T, U \subseteq {}^{<\omega}(\omega \times \mathrm{Ord})$ such that*

- *$A = p[T] = {}^{\omega}\omega \setminus p[U]$ and*
- *for any poset $\mathbb{P}$ of size $< \lambda$ and any V-generic $g \subseteq \mathbb{P}$,*

$$V[g] \models p[T] = {}^{\omega}\omega \setminus p[U].$$

We also say that A is universally Baire if A is $<\lambda$-uB for all λ.

Let $A \subseteq {}^{\omega}\omega$ be a uB set and take (T, U) as above. For any V-generic $g \subseteq \mathbb{P}$,

$$A^g := (p[T])^{V[g]} \subseteq ({}^{\omega}\omega)^{V[g]}.$$

is independent of the choice of (T, U).

Derived models

There is a canonical way to get determinacy model of the form $V = L(\wp(\mathbb{R}))$ from a large cardinal. The resulting model is called a **derived model**.

Suppose that λ is a limit of Woodin cardinals and let $g \subseteq \mathrm{Col}(\omega, < \lambda)$ be V-generic. Then a derived model of V at λ is defined as

$$L(\Gamma_g^*, \mathbb{R}_g^*),$$

where

$$\mathbb{R}_g^* = \bigcup_{\alpha<\lambda} ({}^\omega\omega)^{V[g\restriction\alpha]},$$

$$\Gamma_g^* = \{A_g^* \subseteq \mathbb{R}_g^* \mid \exists\alpha < \lambda\ (V[g \restriction \alpha] \models A \text{ is } < \lambda\text{-uB})\}.$$

Here, we write $A_g^* = \bigcup_{\beta\in(\alpha,\lambda)} A^{g\restriction\beta}$. Woodin showed that $L(\Gamma_g^*, \mathbb{R}_g^*) \models \mathsf{AD}^+$.

- If λ is also a limit of $< \lambda$-strong cardinals, then $L(\Gamma_g^*, \mathbb{R}_g^*) \models \mathsf{AD}_{\mathbb{R}}$.
- If $\exists\kappa < \lambda$ (κ is λ-supercompact), then $L(\Gamma_g^*, \mathbb{R}_g^*) \models \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular.

Hod mice

As we mentioned, our target models of determinacy are constructed from a hod mouse.

A **hod mouse** is a canonical inner model theoretic object, which is designed for representing HOD under the Axiom of Determinacy.

It is defined as an iterable structure (explained later) of the form

$$L_\alpha[\vec{E}, \Sigma],$$

where

- $\vec{E}$ is a sequence of extenders ($\rightsquigarrow$ large cardinals) and
- Σ is a fragment of its own iteration strategy acting on iteration trees in it ($\rightsquigarrow$ self-iterability).

Theorem (Steel)

Assuming $\mathrm{AD}_{\mathbb{R}} + \mathrm{HPC}$ *(Hod Pair Capturing),* V_Θ^{HOD} *is a hod mouse.*

Chang models over derived models

Sargsyan introduced a new kind of determinacy model called **the Chang model over the derived model**. Assume the following:

- Let $\mathcal{V}$ be a countable model of ZFC that is a **hod mouse**.
- δ is a regular limit of Woodin cardinals in $\mathcal{V}$. (The regularity of δ is assumed just for slight simplicity.)
- Let $g \subseteq \mathrm{Col}(\omega, < \delta)$ is $\mathcal{V}$-generic.

Let $L(\Gamma^*_g, \mathbb{R}^*_g)$ be the derived model of $\mathcal{V}$ at δ.

Theorem (Sargsyan)

In $\mathcal{V}[g]$, there is a transitive model $\mathcal{M}_\infty$ of ZFC − Power *such that*

- $\mathcal{M}_\infty \cap \mathrm{Ord} = \omega_2^{\mathcal{V}[g]} = (\delta^+)^{\mathcal{V}}$,
- $\mathcal{M}_\infty$ *has a largest cardinal* $\delta_\infty > \delta$, *and*

$$L(\mathcal{M}_\infty, {}^\omega\delta_\infty, \Gamma^*_g, \mathbb{R}^*_g) \models \mathrm{AD}^+.$$

We denote the above model by CDM. This model plays a crucial role in G.–Sargsyan's proof of $\mathrm{CM} \models \mathrm{AD}^+$.

Chang models over derived models with supercompact measures

Assume the following:

- Let $\mathcal{V}$ be a countable model of ZFC that is a hod mouse.
- δ is a regular limit of Woodin cardinals in $\mathcal{V}$.
- Let $g \subseteq \mathrm{Col}(\omega, <\delta)$ is $\mathcal{V}$-generic.

Let $L(\Gamma_g^*, \mathbb{R}_g^*)$ be the derived model of $\mathcal{V}$ at δ.

Theorem (G.–Müller–Sargsyan)

In $\mathcal{V}[g]$, there is a transitive model $\mathcal{M}'_\infty$ of ZFC − Power *such that*

- $\mathcal{M}'_\infty \cap \mathrm{Ord} = \omega_2^{\mathcal{V}[g]} = (\delta^+)^{\mathcal{V}}$,
- $\mathcal{M}'_\infty$ *has a largest cardinal* $\delta'_\infty > \delta$, *and*

$$L(\mathcal{M}'_\infty, {}^\omega\delta'_\infty, \Gamma_g^*, \mathbb{R}_g^*)[\langle \mu_\alpha \mid \alpha < \delta'_\infty \rangle] \models \mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}} + \Theta \text{ is regular} \\ + \omega_1 \text{ is } < \delta'_\infty\text{-supercompact}$$

where μ_α is the club filter on $\wp_{\omega_1}(\alpha)$. Furthermore, if δ is a limit of $<\delta$-strong cardinals of $\mathcal{V}$, then $\delta'_\infty > \Theta$.

Iteration strategies

A model of set theory is called iterable if one can avoid to reach an ill-founded model during a construction of its iterated ultrapowers. In general, we need to consider non-linear iterations called **iteration trees**.

$$
\begin{tikzcd}
 & & & \mathcal{M}_7 & \\
\mathcal{M} = \mathcal{M}_0 \arrow[r, "E_1"] \arrow[dr, "E_0"] & \mathcal{M}_2 \arrow[r, "E_3"] & \mathcal{M}_4 \arrow[r, "E_5"] \arrow[ur, "E_6"] \arrow[dr, "E_4"] & \mathcal{M}_6 \arrow[r, "E_7"] & \cdots \\
 & \mathcal{M}_1 \arrow[r, "E_2"] & \mathcal{M}_3 & \mathcal{M}_5 &
\end{tikzcd}
$$

An **iteration strategy** Σ is a function such that for any iteration tree $\mathcal{T}$, $\Sigma(\mathcal{T})$ is a cofinal branch whose direct limit model is well-founded. So $\mathcal{M}$ is iterable if there is an iteration strategy Σ defined on iteration trees on $\mathcal{M}$.

Basic objects: $\mathcal{V}, \delta, g, \mathcal{P}$ and Σ

Throughout the rest of this talk, we fix the following objects.

- Let $\mathcal{V}$ be a countable model of ZFC that is a hod mouse.
- δ is a regular limit of Woodin cardinals in $\mathcal{V}$.
- Let $g \subseteq \mathrm{Col}(\omega, < \delta)$ is $\mathcal{V}$-generic.
- $\mathcal{P} = \mathcal{V}|(\delta^+)^{\mathcal{V}}$.
- Σ is the iteration strategy for $\mathcal{P}$ acting on iteration trees in $\mathcal{V}$ based on $\mathcal{P}|\delta$ determined by the internal strategy of $\mathcal{V}$. Actually, Σ can be uniquely extended to an iteration strategy in $\mathcal{V}[g]$, so we also denote it by Σ.

We work in $\mathcal{V}[g]$ throughout this talk.

The internal direct limit system

Remember that $g \subseteq \mathrm{Col}(\omega, < \delta)$ is the fixed $\mathcal{V}$-generic and we work in $\mathcal{V}[g]$.

- Let $I_g(\mathcal{P}, \Sigma)$ be the set of a Σ-iterate $\mathcal{Q}$ of $\mathcal{P}$ via $\mathcal{T}$ such that $\pi^{\mathcal{T}}(\delta) = \delta$.
- For $\mathcal{Q} \in I^g(\mathcal{P}, \Sigma)$ and $\eta < \delta$, define

$$\mathcal{F}_g(\mathcal{Q}, \eta) = \{\mathcal{R} \mid \mathcal{R} \text{ is a non-dropping } \Sigma_{\mathcal{Q}}\text{-iterates of } \mathcal{Q} \text{ via } \mathcal{T} \text{ of length} < \delta \text{ such that } \mathrm{crit}(\pi^{\mathcal{T}}) > \eta.\}$$

  Here, "non-dropping" means that an iteration map from $\mathcal{Q}$ to $\mathcal{R}$ exists. Also, $\Sigma_{\mathcal{Q}}$ is the iteration strategy for $\mathcal{Q}$ defined by $\Sigma_{\mathcal{Q}}(\mathcal{U}) = \Sigma(\mathcal{T}^\frown\mathcal{U})$.
- For any $\mathcal{Q}, \mathcal{R} \in \mathcal{F}_g(\mathcal{P})$, define

$$\mathcal{Q} \preceq \mathcal{R} \iff \mathcal{R} \text{ is a } \Sigma_{\mathcal{Q}}\text{-iterate of } \mathcal{Q}.$$

  We assume that a system $(\mathcal{F}_g(\mathcal{Q}, \eta), \preceq)$ is a directed system under iteration maps.

Comparison

By **Comparison Theorem** due to Steel, under AD$^+$, for any two non-dropping $\Sigma_{\mathcal{Q}}$-iterates $\mathcal{R}_0$ and $\mathcal{R}_1$, there is an $\mathcal{S}$ such that it is a common non-dropping iterate of $\mathcal{R}_0$ and $\mathcal{R}_1$ and the following diagram commutes.

$$\begin{tikzcd} & \mathcal{R}_0 \arrow[dr] & \\ \mathcal{Q} \arrow[ur] \arrow[dr] & & \mathcal{S} \\ & \mathcal{R}_1 \arrow[ur] & \end{tikzcd}$$

This is one of the reasons why we need to assume that $\mathcal{V}$ is a hod mouse.

Because we do not assume AD$^+$ as the background theory, we implicitly include such a comparison result, together with other properties that follows from AD$^+$, as an assumption on Σ. Then we set

$$\mathcal{M}_\infty(\mathcal{Q},\eta) = \text{the direct limit of } (\mathcal{F}_g(\mathcal{Q},\eta), \preceq).$$

and let $\delta_\infty^{\mathcal{Q},\eta}$ be the direct limit image of δ in $\mathcal{M}_\infty(\mathcal{Q},\eta)$.

Precise definition of CDM$^+$

For $\mathcal{Q} \in I_g(\mathcal{P}, \Sigma)$ and $\eta < \delta$, define

$$\mathrm{CDM}^+(\mathcal{Q}, \eta) = L(\mathcal{M}_\infty(\mathcal{Q}, \eta), {}^\omega(\delta_\infty^{\mathcal{Q},\eta}), \Gamma_g^*, \mathbb{R}_g^*)[\langle \mu_\alpha \mid \alpha < \delta_\infty^{\mathcal{Q},\eta} \rangle],$$

where $L(\Gamma_g^*, \mathbb{R}_g^*)$ be the derived model of $\mathcal{V}$ at δ and μ_α is the club filter on $\wp_{\omega_1}(\alpha)$. Now we restate our theorem.

Theorem (G.–Müller–Sargsyan)

In $\mathcal{V}[g]$, there are a Σ-iterate $\mathcal{Q}$ of $\mathcal{P}$ and $\eta < \delta$ such that

$$\mathrm{CDM}^+(\mathcal{Q}, \eta) \models \mathrm{AD}^+ + \mathrm{AD}_\mathbb{R} + \Theta \textit{ is regular} \\ + \omega_1 \textit{ is } < \delta_\infty^{\mathcal{Q},\eta}\textit{-supercompact.}$$

Furthermore, if δ is a limit of $< \delta$-strong cardinals of $\mathcal{V}$, then $\delta_\infty^{\mathcal{Q},\eta} > \Theta$.

CDM is defined as $L(\mathcal{M}_\infty(\mathcal{P}, 0), {}^\omega(\delta_\infty^{\mathcal{P},0}), \Gamma_g^*, \mathbb{R}_g^*)$. We do not know if $\mathrm{CDM}^+(\mathcal{P}, 0)$ satisfies the above properties. Also, we are still in progress to show that $\delta_\infty^{\mathcal{Q},\eta} > \Theta^+$ is possible under stronger assumption on δ.

Main references

- Takehiko Gappo, Sandra Müller & Grigor Sargsyan, *Chang models over derived models with supercompact measures*, in preparation.
- Grigor Sargsyan, *Covering with Chang models over derived models*, Adv. Math. **384** (2021), Paper No. 107717, 21.
- John R. Steel, *A comparison process for mouse pairs*, Lecture Notes in Logic, vol. 51, Association for Symbolic Logic, Ithaca, NY; Cambridge University Press, Cambridge, ©2023.