Global evasion and prediction associated with ideals

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$\mathrm{non}(M_{\mathscr{I}})$ and $\mathrm{non}(K_{\mathscr{I}})$

Let $\mathscr{I}$ be an ideal on ω. We have been studying the cardinal invariants $\mathrm{non}(M_{\mathscr{I}})$ and $\mathrm{non}(K_{\mathscr{I}})$ defined as follows:

Definition

- For a function $\phi \colon \omega^{<\omega} \to \mathscr{I}$, let:

$$M_\phi := \{x \in \omega^\omega : \forall^\infty n < \omega\ x(n) \in \phi(x \restriction n)\}.$$

$$M_{\mathscr{I}} := \{A \subseteq \omega^\omega : \exists \phi \in \mathscr{I}^\omega, A \subseteq M_\phi\}.$$

- For a function $\phi \colon \omega \to \mathscr{I}$, let:

$$K_\phi := \{x \in \omega^\omega : \forall^\infty n < \omega\ x(n) \in \phi(n)\}.$$

$$K_{\mathscr{I}} := \{A \subseteq \omega^\omega : \exists \phi \in \mathscr{I}^\omega, A \subseteq K_\phi\}.$$

$M_{\mathscr{I}}$ and $K_{\mathscr{I}}$ form σ-ideals. $\mathrm{non}(M_{\mathscr{I}})$ and $\mathrm{non}(K_{\mathscr{I}})$ denotes their uniformities.

Blass' article in Handbook of set theory

These cardinal invariants are formalized in Blass' framework [Bla10] of evasion and prediction.

6. Combinatorial Cardinal Characteristics of the Continuum

Andreas Blass

Contents

Global Prediction

Blass studied many variants of evasion number $\mathfrak{e}$. In this talk, we focus on *global prediction* in his framework.

Definition

- A predictor is a function $\pi \colon \omega^{<\omega} \to \mathcal{P}(\omega)$.
- A predictor π globally predicts $x \in \omega^\omega$ if:

$$\forall^\infty n < \omega \; x(n) \in \pi(x \restriction n).$$

- A non-adaptive predictor is a function $\pi \colon \omega \to \mathcal{P}(\omega)$.
- A non-adaptive predictor π globally predicts $x \in \omega^\omega$ if:

$$\forall^\infty n < \omega \; x(n) \in \pi(n).$$

"globally" corresponds to "$\forall^\infty n < \omega$".

Global evasion numbers

Let $\mathscr{F} \subseteq \mathcal{P}(\omega)$ be a family.

Definition

A predictor π is an $\mathscr{F}$-type if $\mathrm{ran}(\pi) \subseteq \mathscr{F}$.
$\mathfrak{e}(\text{global, adaptive}, \mathscr{F})$ is the least size of $X \subseteq \omega^\omega$ such that no single $\mathscr{F}$-type predictor can globally predict all $x \in X$.
Define $\mathfrak{e}(\text{global, non-adaptive}, \mathscr{F})$ analogously.

Thus,

- $\mathrm{non}(M_{\mathscr{I}}) = \mathfrak{e}(\text{global, adaptive}, \mathscr{I})$.
- $\mathrm{non}(K_{\mathscr{I}}) = \mathfrak{e}(\text{global, non-adaptive}, \mathscr{I})$.

If $\mathscr{F} \subseteq \mathscr{F}'$, then:

$$\begin{array}{ccc} \mathfrak{e}(\text{global, non-adaptive}, \mathscr{F}) & \longrightarrow & \mathfrak{e}(\text{global, adaptive}, \mathscr{F}) \\ \downarrow & & \downarrow \\ \mathfrak{e}(\text{global, non-adaptive}, \mathscr{F}') & \longrightarrow & \mathfrak{e}(\text{global, adaptive}, \mathscr{F}') \end{array}$$

Blass' table

Blass summarized evasion numbers for global prediction associated with the six families $\mathscr{F}$ which collects $G \subseteq \omega$ which satisfy each of the following properties (G represents "guess"):

Table 1: Evasion numbers for global prediction

	Non-adaptive	Adaptive
$\lvert G \rvert = 1$	2	$\aleph_1$
$\lvert G \rvert = k$	$k+1$	$\mathfrak{m}(\sigma\text{-}k\text{-linked}) \leq ? \leq \mathbf{add}(\mathcal{L})$
$\lvert G \rvert = f(n)$	$\mathbf{add}(\mathcal{L})$	$\mathbf{add}(\mathcal{L})$
G finite	$\mathfrak{b}$	$\mathfrak{b}$
$\omega - G$ infinite	$\mathbf{non}(\mathcal{B})$	$\mathbf{non}(\mathcal{B})$
$G \subsetneq \omega$	$\mathbf{non}(\mathcal{B})$	$\mathbf{non}(\mathcal{B})$

Fig: Blass' table. $f \in \omega^\omega$ tends to infinity.

We deal with ideals $\mathscr{I}$ with $\mathscr{I} \supseteq \mathrm{Fin} = [\omega]^{<\omega}$, so we are in the red zone.

adaptive and non-adaptive numbers can be different

One might expect that the adaptive and non-adaptive numbers are the same in the red zone. However,

Theorem (Cieślak, Gappo, Martínez-Celis and Y.)

It is consistent that $\mathrm{non}(K_{\mathrm{Fin}\otimes\mathrm{Fin}}) < \mathrm{non}(M_{\mathrm{Fin}\otimes\mathrm{Fin}})$.

$\mathrm{Fin} \otimes \mathrm{Fin}$ is the ideal on $\omega \times \omega \cong \omega$ defined as the Fubini product of two Fin's:

$$\mathrm{Fin} \otimes \mathrm{Fin} \coloneqq \{A \subseteq \omega \times \omega : \forall^\infty n < \omega \, |(A)_n| < \omega\},$$

where $(A)_n \coloneqq \{m < \omega : (n, m) \in A\}$ denotes the n-th vertical section of $A \subseteq \omega \times \omega$. Thus for $A \subseteq \omega \times \omega$,

$$A \in \mathrm{Fin} \otimes \mathrm{Fin} \Leftrightarrow \exists k < \omega, \exists h \in \omega^\omega \text{ such that}$$

$$A \subseteq \{(a, b) \in \omega \times \omega : a \leq k \text{ or } b \leq h(a)\}.$$

$\mathrm{non}(K_{\mathrm{Fin}\otimes\mathrm{Fin}}) = \mathfrak{b}$

Lemma

$\mathrm{non}(K_{\mathrm{Fin}\otimes\mathrm{Fin}}) = \mathfrak{b}$.

Proof. It suffices to show $\mathrm{non}(K_{\mathrm{Fin}\otimes\mathrm{Fin}}) \leq \mathfrak{b}$. Let $B \subseteq \omega^\omega$ be an unbounded family consisting of strictly increasing functions. We shall show $F := \{f \times g : f, g \in B\} \subseteq (\omega \times \omega)^\omega$ is not in $K_{\mathrm{Fin}\otimes\mathrm{Fin}}$. Let $\phi\colon \omega \to \mathrm{Fin} \otimes \mathrm{Fin}$ be arbitrary. For each $n < \omega$, there are $k_n < \omega$ and $h_n \in \omega^\omega$ such that $\phi(n) \subseteq \{(a, b) \in \omega \times \omega : a \leq k_n$ or $b \leq h_n(a)\}$. Since B is unbounded, there are $f \in B$ and $D \in [\omega]^\omega$ such that $k_n < f(n)$ for $n \in D$. For $n < \omega$, put $h(n) := h_n(f(n))$ and $h'(n) := h(d_n)$, where d_n denotes the n-th element of D. Since B is unbounded, there are $g \in B$ and $E \in [\omega]^\omega$ such that $h'(n) < g(n)$ for $n \in E$. Then, for any $n \in E$, $g(d_n) \geq g(n) > h'(n) = h(d_n) = h_{d_n}(f(d_n))$. Thus, for all $m \in \{d_n : n \in E\}$, we have $(f(m), g(m)) \notin \phi(m)$, so $F \notin K_{\mathrm{Fin}\otimes\mathrm{Fin}}$. □

On $\mathrm{non}(M_{\mathrm{Fin}\otimes\mathrm{Fin}})$

Let us move on to $\mathrm{non}(M_{\mathrm{Fin}\otimes\mathrm{Fin}})$.

It turns out that $\mathrm{non}(M_{\mathrm{Fin}\otimes\mathrm{Fin}})$ has a connection with the notion of constant prediction, introduced by Kamo ([Kam00], [Kam01]).

Definition

Let $2 \leq k < \omega$. A predictor $\pi\colon \omega^{<\omega} \to \mathcal{P}(\omega)$ k-constantly predicts $f \in \omega^\omega$ if:

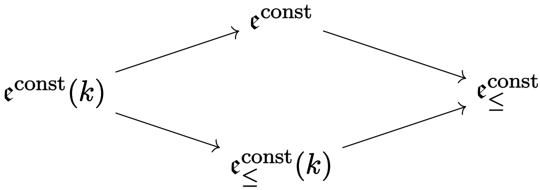
$$\forall^\infty i < \omega \, \exists j \in [i, i+k) \, (f(j) \in \pi(f \restriction j)).$$

π constantly predicts $f \in \omega^\omega$ if π k-constantly predicts f for some k. Define the evasion number $\mathfrak{e}((k\text{-})\text{constant, adaptive}, \mathscr{F})$ of the $\mathscr{F}$-type (k-)constant prediction analogously.

Constant evasion numbers

Define the following notation for typical constant evasion numbers:

- $\mathfrak{e}^{\mathrm{const}}(k) := \mathfrak{e}(k\text{-constant, adaptive}, \mathrm{Singletons})$.
- $\mathfrak{e}^{\mathrm{const}} := \mathfrak{e}(\text{constant, adaptive}, \mathrm{Singletons})$.
- $\mathfrak{e}^{\mathrm{const}}_{\leq}(k) := \mathfrak{e}(k\text{-constant, adaptive}, \mathrm{Fin})$.
- $\mathfrak{e}^{\mathrm{const}}_{\leq} := \mathfrak{e}(\text{constant, adaptive}, \mathrm{Fin})$.



In the cases of $\mathrm{Singletons}$- and Fin- type predictions, we often use predictors π with $\mathrm{ran}(\pi) \in \omega$ and instead of "$\in$" in the definition of "$f(j) \in \pi(f \restriction j)$", we use "$=$" and "$\leq$", respectively.

$\mathrm{non}(M_{\mathrm{Fin}\otimes\mathrm{Fin}}) = \mathfrak{e}^{\mathrm{const}}_{\leq}(2)$

We show $\mathrm{non}(M_{\mathrm{Fin}\otimes\mathrm{Fin}}) = \mathfrak{e}^{\mathrm{const}}_{\leq}(2)$. To see this, we introduce the weak version of constant prediction as follows:

Definition

Let $2 \leq k < \omega$. A predictor $\pi\colon \omega^{<\omega} \to \mathcal{P}(\omega)$ weakly k-constantly predicts $f \in \omega^\omega$ if:

$$\forall^\infty m < \omega\, \exists j \in [mk, mk+k)\, (f(j) \in \pi(f \restriction j)).$$

Define the evasion number $\mathfrak{e}(\text{weakly } k\text{-constant, adaptive}, \mathscr{F})$ of the $\mathscr{F}$-type weakly k-constant prediction analogously.

Put $\mathfrak{e}^{\mathrm{wconst}}_{\leq}(k) := \mathfrak{e}(\text{weakly } k\text{-constant, adaptive}, \mathrm{Fin})$. Clearly $\mathfrak{e}^{\mathrm{const}}_{\leq}(k) \leq \mathfrak{e}^{\mathrm{wconst}}_{\leq}(k)$.

$\mathfrak{e}_{\leq}^{\mathrm{wconst}}(k) = \mathfrak{e}_{\leq}^{\mathrm{const}}(k)$

Lemma

$\mathfrak{e}_{\leq}^{\mathrm{wconst}}(k) = \mathfrak{e}_{\leq}^{\mathrm{const}}(k)$.

Proof. It suffices to prove $\mathfrak{e}_{\leq}^{\mathrm{wconst}}(k) \leq \mathfrak{e}_{\leq}^{\mathrm{const}}(k)$. Let $F \subseteq \omega^\omega$ of size $< \mathfrak{e}_{\leq}^{\mathrm{wconst}}(k)$. For $f \in \omega^\omega$ and $l < k$, define the l-shift $f^l \in \omega^\omega$ of f by:

$$f^l(n) = \begin{cases} 0 & \text{if } n < l \\ f(n-l) & \text{if } n \geq l \end{cases}$$

For $t \in \omega^{<\omega}$, define t^l analogously (so $\mathrm{dom}(t^l) = l + |t|$). Let $F' := \{f^l : f \in F, l < k\}$. Since F' has size $< \mathfrak{e}_{\leq}^{\mathrm{wconst}}(k)$, there is $\pi' : \omega^{<\omega} \to \mathrm{Fin}$ which weakly k-constant predicts f^l for all $f \in F$ and $l < k$. Define $\pi : \omega^{<\omega} \to \mathrm{Fin}$ by $\pi(t) = \bigcup\{\pi'(t^l) : l < k\}$. Then it is routine to see that π k-constantly predicts all $f \in F$. □

Lemma

$\mathfrak{e}_{\leq}^{\text{wconst}}(2) \leq \operatorname{non}(M_{\mathrm{Fin}\otimes\mathrm{Fin}})$.

Proof. Let $F \subseteq (\omega \times \omega)^\omega$ be of size $< \mathfrak{e}_{\leq}^{\text{wconst}}(2)$. For $(f, g) \in F$, $f * g \in \omega^\omega$ is given by $f * g(2n) = f(n)$ and $f * g(2n+1) = g(n)$. Put $F^* = \{f * g : (f, g) \in F\} \subseteq \omega^\omega$. Since F^* has size $< \mathfrak{e}_{\leq}^{\text{wconst}}(2)$, there is a predictor $\pi\colon \omega^{<\omega} \to \omega$ which weakly 2-constantly predicts all $f * g$ (in the sense of $\leq$). Thus, we have

$$\forall (f, g) \in F\ \forall^\infty n < \omega \text{ either } \begin{cases} f(n) \leq \pi((f * g) \restriction 2n), & \text{or} \\ g(n) \leq \pi((f * g) \restriction 2n+1). \end{cases}$$

For $(u, v) \in (\omega \times \omega)^{<\omega}$, let $\sigma(u, v) := \pi(u * v)$ and $s(u, v)(a) := \pi((u * v)^\frown a)$ for $a \in \omega$, where $u * v \in \omega^{<\omega}$ is defined analogously. Again for $(u, v) \in (\omega \times \omega)^{<\omega}$, let $\phi(u, v) = \{(a, b) \in \omega \times \omega : a \leq \sigma(u, v) \text{ or } b \leq s(u, v)(a)\}$. Note $\phi(u, v) \in \mathrm{Fin} \otimes \mathrm{Fin}$. Then it is routine to check $F \subseteq M_\phi$. □

Lemma

$\mathrm{non}(M_{\mathrm{Fin}\otimes\mathrm{Fin}}) \leq \mathfrak{e}^{\mathrm{wconst}}_{\leq}(2)$.

Proof. Basically do the opposite argument. Let $F \subseteq \omega^\omega$ be of size $< \mathrm{non}(M_{\mathrm{Fin}\otimes\mathrm{Fin}})$. For each $f \in F$, define $f_{\mathrm{even}}, f_{\mathrm{odd}} \in \omega^\omega$ by $f_{\mathrm{even}}(n) = f(2n)$ and $f_{\mathrm{odd}}(n) = f(2n+1)$. Put $F_* = \{\langle f_{\mathrm{even}}, f_{\mathrm{odd}}\rangle : f \in F\}$. Since F_* has size $< \mathrm{non}(M_{\mathrm{Fin}\otimes\mathrm{Fin}})$, there is $\phi\colon \omega^{<\omega} \to \mathrm{Fin} \otimes \mathrm{Fin}$ such that $F_* \subseteq M_\phi$. Take $\sigma\colon (\omega \times \omega)^{<\omega} \to \omega$ and $s\colon (\omega \times \omega)^{<\omega} \to \omega^\omega$ such that $\phi(u, v) \subseteq \{(a, b) \in \omega \times \omega : a \leq \sigma(u, v) \text{ or } b \leq s(u, v)(a)\}$. Now define $\pi\colon \omega^{<\omega} \to \omega$ by

$$\pi(t) = \sigma(t_{\mathrm{even}}, t_{\mathrm{odd}})$$
$$\pi(t^\frown\langle n\rangle) = s(t_{\mathrm{even}}, t_{\mathrm{odd}})(n)$$

for all $t \in \omega^\omega$ of even length and all $n < \omega$, where $t_{\mathrm{even}}, t_{\mathrm{odd}} \in \omega^{<\omega}$ are defined analogously. Then it is easy to see that π weakly 2-constantly predicts all $f \in F$ (in the sense of $\leq$). □

$\mathrm{non}(K_{\mathrm{Fin}\otimes\mathrm{Fin}}) < \mathrm{non}(M_{\mathrm{Fin}\otimes\mathrm{Fin}})$ is consistent

Thus we obtain $\mathrm{non}(M_{\mathrm{Fin}\otimes\mathrm{Fin}}) = \mathfrak{e}^{\mathrm{wconst}}_{\leq}(2) = \mathfrak{e}^{\mathrm{const}}_{\leq}(2)$.

Brendle [Bre03] showed the consistency of $\mathfrak{e}^{\mathrm{const}} > \mathfrak{b}$, but essentially proved more:

Theorem ([Bre03])

Given $\kappa < \lambda = \lambda^{<\kappa}$ regular uncountable, there is a poset forcing $\mathfrak{b} = \kappa$ and $\mathfrak{e}^{\mathrm{const}}(2) = \lambda = \mathfrak{c}$.

By $\mathfrak{e}^{\mathrm{const}}(2) \leq \mathfrak{e}^{\mathrm{const}}_{\leq}(2)$ and $\mathrm{non}(K_{\mathrm{Fin}\otimes\mathrm{Fin}}) = \mathfrak{b}$, we have:

Theorem (Cieślak, Gappo, Martínez-Celis and Y.)

Given $\kappa < \lambda$ as above, there is a poset forcing $\mathrm{non}(K_{\mathrm{Fin}\otimes\mathrm{Fin}}) = \kappa < \mathrm{non}(M_{\mathrm{Fin}\otimes\mathrm{Fin}}) = \lambda = \mathfrak{c}$.

1. Blass' global prediction
2. On the ideal $\mathrm{Fin} \otimes \mathrm{Fin}$
3. On the ideal $\mathcal{ED}$
4. Question

Eventually different ideal $\mathcal{ED}$

How much can we extend this result of the adaptive/non-adaptive gap in global prediction? Take a look at the eventually different ideal $\mathcal{ED}$:

Definition

$\mathcal{ED}$ is the ideal on $\omega \times \omega \cong \omega$ given by:

$$\mathcal{ED} := \{A \subseteq \omega \times \omega : \exists l < \omega \, \forall^\infty n < \omega \, |(A)_n| \leq l\}.$$

Thus for $A \subseteq \omega \times \omega$,

$$A \in \mathcal{ED} \Leftrightarrow \exists k, l < \omega, \exists h \colon \omega \to [\omega]^{\leq l} \text{ such that}$$

$$A \subseteq \{(a, b) \in \omega \times \omega : a \leq k \text{ or } b \in h(a)\}.$$

$\mathcal{ED} \subseteq \mathrm{Fin} \otimes \mathrm{Fin}$, so $\mathfrak{b} \leq \mathrm{non}(K_{\mathcal{ED}}) \leq \mathrm{non}(K_{\mathrm{Fin}\otimes\mathrm{Fin}}) = \mathfrak{b}$, i.e., $\mathrm{non}(K_{\mathcal{ED}}) = \mathfrak{b}$.

Lemma

$\mathfrak{e}^{\mathrm{const}}(2) \leq \mathrm{non}(M_{\mathcal{ED}})$.

Proof. The proof is basically the same as $\mathfrak{e}^{\mathrm{wconst}}_{\leq}(2) \leq \mathrm{non}(M_{\mathrm{Fin}\otimes\mathrm{Fin}})$. Let $F \subseteq (\omega \times \omega)^\omega$ of size $< \mathfrak{e}^{\mathrm{const}}(2)$. Put $F^* \coloneqq \{f * g : (f, g) \in F\} \subseteq \omega^\omega$. Since F^* has size $< \mathfrak{e}^{\mathrm{const}}(2)$, there is a predictor $\pi \colon \omega^{<\omega} \to \omega$ which 2-constantly predicts all $f * g$ (in the sense of $=$). In particular,

$$\forall (f, g) \in F\ \forall^\infty n < \omega \text{ either } \begin{cases} f(n) = \pi((f * g) \restriction 2n), & \text{or} \\ g(n) = \pi((f * g) \restriction 2n + 1). \end{cases}$$

Define $\sigma \colon (\omega \times \omega)^{<\omega} \to \omega$ by $\sigma(u, v) \coloneqq \pi(u * v)$. For $(u, v) \in (\omega \times \omega)^{<\omega}$, let $\sigma(u, v) \coloneqq \pi(u * v)$ and $s(u, v)(a) \coloneqq \pi((u * v)^\frown a)$ for $a \in \omega$. Again for $(u, v) \in (\omega \times \omega)^{<\omega}$, let $\phi(u, v) \coloneqq \{(a, b) \in \omega \times \omega : a \leq \sigma(u, v) \text{ or } b = s(u, v)(a)\}$. Note $\phi(u, v) \in \mathcal{ED}$. Then it is routine to check $(f, g) \in M_\phi$ for all $(f, g) \in F$. □

$\max\{\mathfrak{b}, \mathfrak{e}^{\mathrm{const}}(2)\} < \mathrm{non}(M_{\mathcal{ED}})$

We know $\mathfrak{b}, \mathfrak{e}^{\mathrm{const}}(2) \leq \mathrm{non}(M_{\mathcal{ED}})$. However,

Theorem (Cieślak, Gappo, Martínez-Celis and Y.)

It is consistent that $\max\{\mathfrak{b}, \mathfrak{e}^{\mathrm{const}}(2)\} < \mathrm{non}(M_{\mathcal{ED}})$.

Roughly speaking, the result suggests that the $\mathcal{ED}$-type prediction is not just a mixture of domination and constant prediction.

Poset $\mathbb{P}^{\mathcal{ED}}$

We introduce the poset $\mathbb{P}^{\mathcal{ED}}$ defined as follows:
Conditions are $p = (\sigma^p, s^p, n^p, F^p)$ such that:

- $\sigma^p \colon \{(u, v) : u, v \in \omega^{<n^p}\} \to \omega$ finite partial function.
- $s^p \colon \{(u^\frown a, v) : u, v \in \omega^{<n^p}, a \in \omega\} \to \omega$ finite partial function.
- $n^p \in \omega$.
- $F^p \subseteq \omega^\omega$ finite.

The order $q \leq p$ is given by:

- $\sigma^q \supseteq \sigma^p$, $s^q \supseteq s^p$, $n^q \geq n^p$ and $F^q \supseteq F^p$.
- $\forall i \in [n^p, n^q) \, \forall x, y \in F^p$, either:
  - $(x \restriction i, y \restriction i) \in \mathrm{dom}(\sigma^q)$ and $x(i) \leq \sigma^q(x \restriction i, y \restriction i)$, or
  - $(x \restriction (i+1), y \restriction i) \in \mathrm{dom}(s^q)$ and $y(i) = s^q(x \restriction (i+1), y \restriction i)$.

Let G be a generic filter. In $V[G]$, $\phi_G \colon (\omega \times \omega)^{<\omega} \to \mathcal{ED}$ is given by:
$\phi_G(u, v) \coloneqq \bigcup_{p \in G} \{(a, b) \in \omega \times \omega : a \leq \sigma^p(u, v) \text{ or } b = s^p(u^\frown a, b)\}$.

We see basic some properties of $\mathbb{P}^{\mathcal{ED}}$. Recall the definition:
Conditions $p = (\sigma^p, s^p, n^p, F^p)$ are such that:

- $\sigma^p : \{(u, v) : u, v \in \omega^{<n^p}\} \to \omega$ finite partial function.
- $s^p : \{(u^\frown a, v) : u, v \in \omega^{<n^p}, a \in \omega\} \to \omega$ finite partial function.
- $n^p \in \omega$.
- $F^p \subseteq \omega^\omega$ finite.

$q \leq p$ if:

- $\sigma^q \supseteq \sigma^p$, $s^q \supseteq s^p$, $n^q \geq n^p$ and $F^q \supseteq F^p$.
- $\forall i \in [n^p, n^q) \, \forall x, y \in F^p$, either:
  - $(x \restriction i, y \restriction i) \in \mathrm{dom}(\sigma^q)$ and $x(i) \leq \sigma^q(x \restriction i, y \restriction i)$, or
  - $(x \restriction (i+1), y \restriction i) \in \mathrm{dom}(s^q)$ and $y(i) = s^q(x \restriction (i+1), y \restriction i)$.

Lemma

Fix σ, s, n as above. Then, $Q_{\sigma,s,n} := \{p : \sigma^p = \sigma, s^p = s, n^p = n\}$ is centered. In particular, $\mathbb{P}^{\mathcal{ED}}$ is σ-centered.

Conditions $p = (\sigma^p, s^p, n^p, F^p)$ are such that:

- $\sigma^p : \{(u,v) : u,v \in \omega^{<n^p}\} \to \omega$ finite partial function.
- $s^p : \{(u^\frown a, v) : u,v \in \omega^{<n^p}, a \in \omega\} \to \omega$ finite partial function.
- $n^p \in \omega$.
- $F^p \subseteq \omega^\omega$ finite.

$q \leq p$ if:

- $\sigma^q \supseteq \sigma^p$, $s^q \supseteq s^p$, $n^q \geq n^p$ and $F^q \supseteq F^p$.
- $\forall i \in [n^p, n^q) \, \forall x, y \in F^p$, either:
  - $(x{\restriction}i, y{\restriction}i) \in \mathrm{dom}(\sigma^q)$ and $x(i) \leq \sigma^q(x{\restriction}i, y{\restriction}i)$, or
  - $(x{\restriction}(i+1), y{\restriction}i) \in \mathrm{dom}(s^q)$ and $y(i) = s^q(x{\restriction}(i+1), y{\restriction}i)$.

$\phi_G(u,v) := \bigcup_{p \in G} \{(a,b) \in \omega \times \omega : a \leq \sigma^p(u,v) \text{ or } b = s^p(u^\frown a, b)\}$.

Lemma

- For $n < \omega$, $\{p : n_p \geq n\}$ is dense. Thus ϕ_G has a valid definition.
- For $f, g \in \omega^\omega$, $\{p : f, g \in F^p\}$ is dense. Thus $\mathbb{P}^{\mathcal{ED}} \Vdash \forall (f,g) \in (\omega \times \omega)^\omega \cap V, \ (f,g) \in M_{\phi_G}$.

Thus the iteration of $\mathbb{P}^{\mathcal{ED}}$ increases $\mathrm{non}(M_{\mathcal{ED}})$.

Fr-linkedness to keep $\mathfrak{b}$ small

To see that $\mathbb{P}^{\mathcal{ED}}$ keep $\mathfrak{b}$ small, we use Fr-linkedness:

Definition

Let $\mathbb{P}$ be a poset. $Q \subseteq \mathbb{P}$ is Fr-linked if for any countable sequence $\bar{q} = \langle q_m : m < \omega \rangle \in Q^\omega$, there is $q^\infty \in \mathbb{P}$ such that:

$$q^\infty \Vdash \exists^\infty m < \omega \; q_m \in \dot{G}.$$

$\mathbb{P}$ is σ-Fr-linked if $\mathbb{P}$ is a union countably many Fr-linked components.

Fact ([Mej19])

Any σ-Fr-linked forcing $\mathbb{P}$ is "ω^ω-good". I.e., given a $\mathbb{P}$-name $\dot{h}$ of a member of ω^ω, there is a countable set $H \subseteq \omega^\omega$ such that if $f \in \omega^\omega$ is unbounded from any $h \in H$, then $\Vdash_{\mathbb{P}} f$ is unbounded from $\dot{h}$.

It is known that ω^ω-goodness is preserved through finite support iterations and ω^ω-good posets keep $\mathfrak{b}$ small.

Recall the definition of $\mathbb{P}^{\mathcal{ED}}$:

Conditions $p = (\sigma^p, s^p, n^p, F^p)$ are such that:

- $\sigma^p : \{(u, v) : u, v \in \omega^{<n^p}\} \to \omega$ finite partial function.
- $s^p : \{(u^\frown a, v) : u, v \in \omega^{<n^p}, a \in \omega\} \to \omega$ finite partial function.
- $n^p \in \omega$.
- $F^p \subseteq \omega^\omega$ finite.

$q \leq p$ if:

- $\sigma^q \supseteq \sigma^p$, $s^q \supseteq s^p$, $n^q \geq n^p$ and $F^q \supseteq F^p$.
- $\forall i \in [n^p, n^q)\, \forall x, y \in F^p$, either:
  - $(x \restriction i, y \restriction i) \in \mathrm{dom}(\sigma^q)$ and $x(i) \leq \sigma^q(x \restriction i, y \restriction i)$, or
  - $(x \restriction (i+1), y \restriction i) \in \mathrm{dom}(s^q)$ and $y(i) = s^q(x \restriction (i+1), y \restriction i)$.

Lemma

Fix σ, s, n as above. Let $L < \omega$ and $\bar{x}^* = \{x_l^* : l < L\} \subseteq \omega^n$ such that all x_l^* are pairwise different. Then, $Q_{\sigma,s,n,\bar{x}^*} := \{p : \sigma^p = \sigma, s^p = s, n^p = n, \{x \restriction n : x \in F^p\} = \bar{x}^*\}$ is Fr-linked. In particular, $\mathbb{P}^{\mathcal{ED}}$ is σ-Fr-linked.

Proof. Omitted. □

Constant prediction on 2^ω

To see that $\mathbb{P}^{\mathcal{ED}}$ keep $\mathfrak{e}^{\mathrm{const}}(2)$ small, we consider constant predictions on 2^ω instead of ω^ω.

Let $\mathfrak{e}_2^{\mathrm{const}}$ (and $\mathfrak{e}_2^{\mathrm{const}}(k)$) denote the ($k$-)constant evasion number (in the sense of "=") with respect to 2^ω.
Note $\mathfrak{e}_2^{\mathrm{const}} \geq \mathfrak{e}^{\mathrm{const}}$ and $\mathfrak{e}_2^{\mathrm{const}}(k) \geq \mathfrak{e}^{\mathrm{const}}(k)$.

Brendle and Shelah [BS03] showed that σ-2^k-linked posets does not increase $\mathfrak{e}_2^{\mathrm{const}}(k)$ (in the sense of "goodness"):

Lemma ([BS03])

Let $2 \leq k < \omega$ and $\mathbb{P}$ be a σ-2^k-linked poset. Then given a $\mathbb{P}$-name $\dot{\pi}$ of a predictor (in 2^ω), there is a countable set Π of predictors such that if $x \in 2^\omega$ is not k-constantly predicted by any $\pi \in \Pi$, $\Vdash_{\mathbb{P}} x$ is not k-constantly predicted by $\dot{\pi}$.

$\max\{\mathfrak{b}, \mathfrak{e}^{\mathrm{const}}(2)\} < \mathrm{non}(M_{\mathcal{ED}})$

Now we show the consistency of $\max\{\mathfrak{b}, \mathfrak{e}^{\mathrm{const}}(2)\} < \mathrm{non}(M_{\mathcal{ED}})$.

Theorem

Assume CH and let $\mathbb{P}$ be a finite support iteration of $\mathbb{P}^{\mathcal{ED}}$ of length ω_2. Then,
$\Vdash_{\mathbb{P}} \mathfrak{b} = \mathfrak{e}^{\mathrm{const}}(2) = \mathfrak{e}_2^{\mathrm{const}}(2) = \omega_1 < \mathrm{non}(M_{\mathcal{ED}}) = \mathfrak{c} = \omega_2$.

Sketch of Proof.

- Why $\mathrm{non}(M_{\mathcal{ED}})$ is large: Because $\mathbb{P}^{\mathcal{ED}}$ increases $\mathrm{non}(M_{\mathcal{ED}})$.
- Why $\mathfrak{b}$ is small: Because $\mathbb{P}^{\mathcal{ED}}$ is σ-Fr-linked and σ-Fr-linked posets keep $\mathfrak{b}$ small.
- Why $\mathfrak{e}_2^{\mathrm{const}}(2)$ is small: Because $\mathbb{P}^{\mathcal{ED}}$ is σ-centered and σ-4-linked posets keep $\mathfrak{e}_2^{\mathrm{const}}(2)$ small.

□

This result can be generalized as follows:

Theorem (Cieślak, Gappo, Martínez-Celis and Y.)

Given $\kappa < \lambda = \lambda^{<\kappa}$ regular uncountable, there is a poset forcing $\mathfrak{b} = \mathfrak{e}^{\mathrm{const}}(2) = \mathfrak{e}_2^{\mathrm{const}} = \kappa$ and $\mathrm{non}(M_{\mathcal{ED}}) = \lambda = \mathfrak{c}$.

Question

We have seen that $\mathrm{non}(K_{\mathscr{I}}) < \mathrm{non}(M_{\mathscr{I}})$ is consistent when $\mathscr{I} = \mathrm{Fin} \otimes \mathrm{Fin}, \mathcal{ED}$.

Question

What other ideal $\mathscr{I}$ has the consistency of $\mathrm{non}(K_{\mathscr{I}}) < \mathrm{non}(M_{\mathscr{I}})$?

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