On splitting* game

Avoiding head-on battles is sometimes useful

Takashi Yamazoe

Kobe Set Theory Seminar

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Table

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Main Theorem

$\mathfrak{s}^*_{\mathrm{G}} < \mathfrak{s}^{**}_{\mathrm{G}}$ consistently holds.

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- $x \sqsubset^{\mathrm{s}} y$ if $x(n) = y(n) = 1$ and $x(m) = 1$ and $y(m) = 0$ for infinitely many $n, m < \omega$, i.e., $x^{-1}(\{1\})$ is split by $y^{-1}(\{1\})$.

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- $y \sqsubset^{\mathrm{r}} x$ if $\neg(x \sqsubset^{\mathrm{s}} y)$. Note that $y \sqsubset^{\mathrm{r}} x$ holds whenever $y \in \mathbb{0}$.

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- $y \sqsubset^{\mathrm{r}} x$ if $\neg(x \sqsubset^{\mathrm{s}} y)$. Note that $y \sqsubset^{\mathrm{r}} x$ holds whenever $y \in \mathbb{0}$.
- Let $j \in 2$ and $n < \omega$. $y \sqsubset^{\mathrm{r}}_{j,n} x$ if for all $m \geq n$, $x(m) = 0$ or $y(m) = 1 - j$ holds. Note that $\sqsubset^{\mathrm{r}} = \bigcup_{j \in 2} \bigcup_{n < \omega} \sqsubset^{\mathrm{r}}_{j,n}$.

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- $y \lhd_* x$ if $x \in 2^\omega \setminus \mathbb{0}$ and $y \sqsubset^{\mathrm{r}} x$.

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- $y \lhd_* x$ if $x \in 2^\omega \setminus \mathbb{0}$ and $y \sqsubset^{\mathrm{r}} x$.
- $y \lhd_{**} x$ if $y \in \mathbb{0}$ or $y \lhd_* x$.

Note that $y \lhd_* x$ [$y \lhd_{**} x$] iff I wins with the play x against the play y of II in the splitting* [splitting**] game, respectively.

notation

- Str denotes the set of all I's strategies, namely, $\mathrm{Str} := 2^{(2^{<\omega})}$. For $\sigma \in \mathrm{Str}$, $\sigma * y$ denotes the play of I according to the strategy σ and the play y of II, namely, $\sigma * y(n) := \sigma(y \restriction n)$ for $n < \omega$. $y \lhd_* \sigma$ if $y \lhd_* \sigma * y$ and $y \lhd_{**} \sigma$ if $y \lhd_{**} \sigma * y$.

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- $\mathfrak{s}_{\mathrm{G}}^* := \min\{|F| : F \subseteq 2^\omega, \neg\exists\sigma \in \mathrm{Str}\ \forall y \in F\ y \lhd_* \sigma\}$, $\mathfrak{s}_{\mathrm{G}}^{**} := \min\{|F| : F \subseteq 2^\omega, \neg\exists\sigma \in \mathrm{Str}\ \forall y \in F\ y \lhd_{**} \sigma\}$.

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Fact ([CGH23])

$\mathfrak{s} \leq \mathfrak{s}^*_{\mathrm{G}} \leq \mathrm{non}(\mathcal{M}), \mathfrak{d}, \mathrm{non}(\mathcal{N})$.
Moreover, $\{y \in 2^\omega : y \vartriangleleft_* \sigma\}$ is null for any $\sigma \in \mathrm{Str}$.

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Question

$\mathfrak{s}^{**}_{\mathrm{G}} \leq \mathfrak{d}$?

$\mathbb{P}^*$ and $\mathbb{P}^*$, which increase $\mathfrak{s}_{\mathrm{G}}^*$ and $\mathfrak{s}_{\mathrm{G}}^{**}$, respectively

We introduce posets $\mathbb{P}^*$ and $\mathbb{P}^{**}$ which generically add a winning strategy and hence increases $\mathfrak{s}_{\mathrm{G}}^*$ and $\mathfrak{s}_{\mathrm{G}}^{**}$, respectively.

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  $(\sigma', F') \leq (\sigma, F) :\Leftrightarrow \sigma' \supseteq \sigma, F' \supseteq F$ and for all $n \in [|\sigma|, |\sigma'|)$ and $y \in F$, $\sigma'(y{\restriction}n) = 0$ or $y(n) = 1$ (i.e., $\sigma'(y{\restriction}n) \leq y(n)$).

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- $\mathbb{P}^* := \{(\sigma, F) \in \mathbb{P}^{**} : F \subseteq 2^\omega \setminus \mathbb{0}\}$ and the order is defined by restriction.
- For both $\mathbb{P}^*$ and $\mathbb{P}^{**}$, σ_G denotes the generic strategy $\sigma_G := \bigcup_{(\sigma,F)\in G} \sigma$ for a generic filter G.

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Corollary

For $y \in 2^\omega$, $\Vdash_{\mathbb{P}^{**}} y \vartriangleleft_{**} \sigma_G$ and $\Vdash_{\mathbb{P}^*} y \vartriangleleft_* \sigma_G$. Hence, by iteration, $\mathbb{P}^*$ and $\mathbb{P}^{**}$ increases $\mathfrak{s}_G^*$ and $\mathfrak{s}_G^{**}$, respectively.

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We show:

1. (Key Lemma 1) $\mathbb{P}^{**}$ has Fr-limits, and
2. (Key Lemma 2) Fr-limits keep $\mathfrak{s}^{*}_{\mathrm{G}}$ small.

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Example

Singletons are ultrafilter-limit-linked and particularly Cohen forcing $\mathbb{C}$ is σ-Fr-linked.

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$Q_{\sigma,k} := \{(\sigma', F) \in \mathbb{P}^{**} : \sigma' = \sigma, |F| \leq k\}$ is ultrafilter-limit-linked for $\sigma \in \mathrm{FinStr}$ and $k < \omega$. In particular, $\mathbb{P}^{**}$ is σ-Fr-linked.

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Lemma

There are densely many determined conditions.

Proof. Induct on ξ. □

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6. For $n \in n' \setminus r'$, $\langle \xi_{n,m} : m < \delta \rangle$ is (strictly) increasing.

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For a countable uniform Δ-system, we can take the Fr-limit:

Lemma

For any (countable) uniform Δ-system $\bar{p} = \langle p_m : m < \omega \rangle \in (\mathbb{P}_\gamma)^\omega$, there is $\lim \bar{p} \in \mathbb{P}$ forcing $\exists^\infty m < \omega \ p_m \in \dot{G}$.

($\lim \bar{p}$ is obtained by basically taking limits pointwisely on the root.)

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Lemma

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Proof. $\mathfrak{s}_{\mathrm{G}}^{*,\infty} \leq \mathfrak{s}_{\mathrm{G}}^{*}$ is clear. To show $\mathfrak{s}_{\mathrm{G}}^{*} \leq \mathfrak{s}_{\mathrm{G}}^{*,\infty}$, let $F \subseteq 2^\omega$ of size $< \mathfrak{s}_{\mathrm{G}}^{*}$. Since $F' := F \cup \mathbb{0} \cup \mathbb{1}$ has size $< \mathfrak{s}_{\mathrm{G}}^{*}$, there is $\sigma \in \mathrm{Str}$ winning all $y \in F'$. This σ has to be in Str_∞. □

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Let γ be uncountable limit and $\mathbb{P}_\gamma = \langle (\mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi) : \xi < \gamma \rangle$ be a σ-Fr-iteration whose first ω_1-many iterands are Cohen forcings $\mathbb{C} = (2^{<\omega}, \supseteq)$. Then, $\mathbb{P} = \mathbb{P}_\gamma$ forces $\mathfrak{s}^*_{\mathrm{G}} = \mathfrak{s}^{*,\infty}_{\mathrm{G}} = \omega_1$, witnessed by the first ω_1-many Cohen reals $\{\dot{c}_\alpha \in 2^\omega : \alpha < \omega_1\}$.

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In particular, we have that:

$$\text{For each } \alpha \in I, p_\alpha \text{ forces } \dot{c}_\alpha \restriction n^* = s \text{ and } \dot{c}_\alpha \sqsubset^{\mathrm{r}}_{j,n^*} \dot{\sigma}. \tag{3.1}$$

Pick some countable $\{\alpha_0 < \alpha_1 < \cdots\} \in [I \setminus \nabla]^\omega$. For $m < \omega$, Define $q_m \leq p_{\alpha_m}$ by extending the α_m-th position to $q_m(\alpha_m) := s^\frown \langle j \cdots j \rangle^m \in 2^{n^*+m}$, where $\langle j \cdots j \rangle^m$ denotes the sequence of length m whose values are all j. By (3.1), for all $m < \omega$ we have:

$$q_m \Vdash \text{ for all } n < m,\ \dot{\sigma}(s^\frown \langle j \cdots j \rangle^n) = 0. \tag{3.2}$$

Since $\alpha_m \notin \nabla$, $\bar{q} \coloneqq \langle q_m : m < \omega \rangle$ also forms a uniform Δ-system (with root ∇) and hence we can take the limit $q^\infty \coloneqq \lim \bar{q}$. Let $y \coloneqq s^\frown jj \cdots \in \mathbb{0} \cup \mathbb{1}$. Since $q^\infty \Vdash \exists^\infty m < \omega\ q_m \in \dot{G}$ and by (3.2), we have:

$$q^\infty \Vdash \dot{\sigma} * y \in \mathbb{0}, \tag{3.3}$$

which contradicts $\dot{\sigma} \in \mathrm{Str}_\infty$.

Con($\mathfrak{s}^*_{\mathrm{G}} < \mathfrak{s}^{**}_{\mathrm{G}}$)

Corollary

Let $\lambda > \omega_1$ be uncountable regular with $\lambda^{\aleph_0} = \lambda$ and $\mathbb{P} = \langle (\mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi) : \xi < \lambda \rangle$ be a finite support iteration whose first ω_1-many iterands are Cohen forcings and the remaining iterands are $\mathbb{P}^{**}$. Then, $\Vdash_{\mathbb{P}} \omega_1 = \mathfrak{s}^*_{\mathrm{G}} < \mathfrak{s}^{**}_{\mathrm{G}} = 2^{\aleph_0} = \lambda$.

In fact, given uncountable regular $\kappa < \lambda$, $\kappa = \mathfrak{s}^*_{\mathrm{G}} < \mathfrak{s}^{**}_{\mathrm{G}} = \lambda$ is consistent.

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Rough Question

Can the whole argument be applied to other cardinal invariants?

References I

[BCM21] Jörg Brendle, Miguel A. Cardona, and Diego A. Mejía. Filter-linkedness and its effect on preservation of cardinal characteristics. *Annals of Pure and Applied Logic*, 172(1):Paper No. 102856, 30, 2021.

[CGH23] Jorge Antonio Cruz Chapital, Tatsuya Goto and Yusuke Hayashi. Game-theoretic variants of cardinal invariants. Preprint, arXiv:2308.12136v2, 2022.

[Mej19] Diego A. Mejía Matrix iterations with vertical support restrictions. In *InProceedings of the 14th and 15th Asian Logic Conferences*, pages 213 – 248. World Sci. Publ., Hackensack, NJ, 2019.