Cichoń's maximum with evasion number

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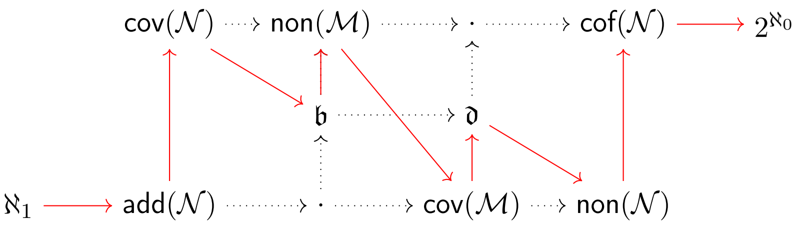
Table

History of Cichoń's maximum

In 2019, Cichoń's maximum was born in [GKS19] assuming large cardinals.
More precisely, it was shown that it is consistent modulo four strongly compact cardinals that all the ten cardinal characteristics in Cichoń's diagram are totally distinct in the following order:



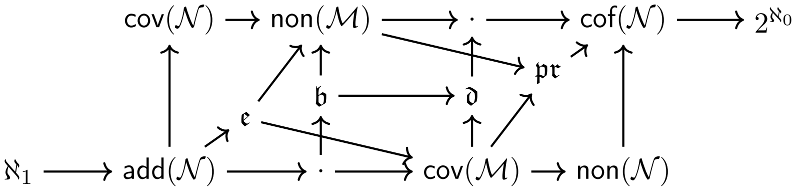
In 2022, the large cardinal assumption was eliminated in [GKMS22].
The aim of our study is to add another cardinal characteristic in Cichoń's maximum.

Evasion number $\mathfrak{e}$ and prediction number $\mathfrak{pr}$

Definition

- A pair $\pi = (D, \{\pi_n : n \in D\})$ is a predictor:$\Leftrightarrow$ $D \in [\omega]^\omega$ and each π_n is a function $\pi_n \colon \omega^n \to \omega$.
- π predicts $f \in \omega^\omega$ $:\Leftrightarrow \forall^* n \in D, f(n) = \pi_n(f \restriction n)$.
- f evades π $:\Leftrightarrow$ π does not predict f.
- $\mathfrak{pr} := \min\{|\Pi| : \Pi \subseteq \{\text{predictors}\}, \forall f, \exists \pi \in \Pi, \pi \text{ predicts } f\}$.
- $\mathfrak{e} := \min\{|F| : F \subseteq \omega^\omega, \forall \text{predictor } \pi, \exists f \in F, f \text{ evades } \pi\}$.

$\mathfrak{pr}$ and $\mathfrak{e}$ have the following relations in Cichoń's diagram:

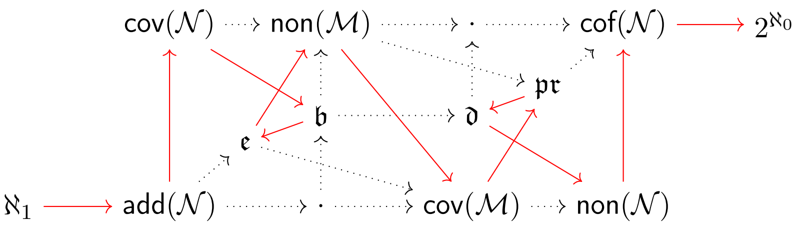


Main Theorem

The speaker showed $\mathfrak{e}$ and $\mathfrak{pr}$ can be added Cichoń's maximum.

Theorem(Y.)

It is consistent that $\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathfrak{e} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathfrak{pr} < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}$.

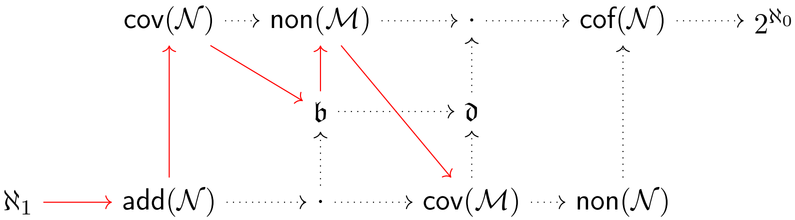


How to construct Cichoń's maximum

Construction of Cichoń's maximum consists of two steps.

First Step

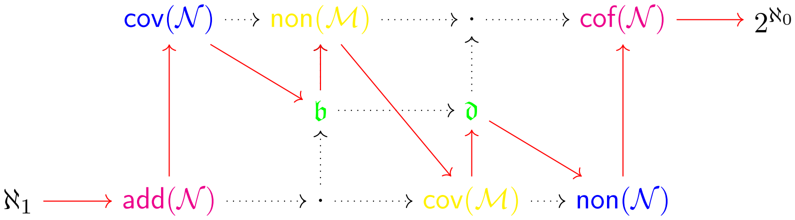
Separate the cardinals in the left side in the diagram by fsi of ccc posets.



How to construct Cichoń's maximum

Second Step

Separate the dual numbers in the right side.



- [GKS19]: using large cardinal techniques
- [GKMS22]: using submodel techniques

Both methods are so general that one can separate the right side without knowing the details of the poset used in First Step well.

For this reason, we focus on First Step in this talk.

$\mathbb{P}^5$: fsi that separates left side

Poset $\mathbb{P}^5$ introduced in [GKS19], which separates the left side is constructed as follows: For given uncountable regular cardinals (with some cardinal arithmetics) $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \lambda_5$,

$$\mathbb{P}^5 := \mathbb{C}_{\lambda_5} * \text{ fsi of length } \lambda_5 \text{ of } \begin{cases} \text{subforcing of } \mathbb{A} \text{ of size } < \lambda_1, \\ \text{subforcing of } \mathbb{B} \text{ of size } < \lambda_2, \\ \text{subforcing of } \mathbb{D} \text{ of size } < \lambda_3 \text{ or} \\ \text{subforcing of } \mathbb{E} \text{ of size } < \lambda_4 \end{cases}$$

following some bookkeeping function f_{bk}
(Here, the first λ_5 Cohen forcing is necessary for Second Step).
$\mathbb{A}$,$\mathbb{B}$,$\mathbb{D}$ and $\mathbb{E}$ are posets which increase $\mathrm{add}(\mathcal{N}), \mathrm{cov}(\mathcal{N}), \mathfrak{b}$ and $\mathrm{non}(\mathcal{M})$ respectively ($\mathrm{non}(\mathcal{M}) = \mathfrak{b}(\omega^\omega, \omega^\omega, \text{eventually different})$).

Thus, by bookkeeping argument $\mathbb{P}^5$ forces that $\mathrm{add}(\mathcal{N}) \geq \lambda_1$, $\mathrm{cov}(\mathcal{N}) \geq \lambda_2, \mathfrak{b} \geq \lambda_3, \mathrm{non}(\mathcal{M}) \geq \lambda_4, \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0} = \lambda_5$.

What $\mathbb{P}^5$ keeps small

The property of "$\mathfrak{x} \leq \theta$" (θ:regular) is preserved through ccc fsi of:

$$\begin{cases} \text{size} < \theta,\ \sigma\text{-cetered or subforcing of } \mathbb{B} & (\text{for } \mathfrak{x} = \text{add}(\mathcal{N})) \\ \text{size} < \theta \text{ or } \sigma\text{-cetered} & (\text{for } \mathfrak{x} = \text{cov}(\mathcal{N})) \\ \text{size} < \theta & (\text{for } \mathfrak{x} = \mathfrak{b}) \\ \text{size} < \theta & (\text{for } \mathfrak{x} = \text{non}(\mathcal{M})) \end{cases}$$

Since $\mathbb{D}$ and $\mathbb{E}$ are σ-centered, (assuming CH in ground model) $\mathbb{P}^5$ forces $\text{add}(\mathcal{N}) \leq \lambda_1, \text{cov}(\mathcal{N}) \leq \lambda_2, \text{non}(\mathcal{M}) \leq \lambda_4$.
$\rightarrow$ Only "$\mathfrak{b}$ is small" is remained!
Simple iteration does not seem to say more on $\mathfrak{b}$ and we may need to choose subforcings of $\mathbb{E}$ more carefully to keep $\mathfrak{b}$ small.

For this purpose, "ultrafilter method" is invented in [GKS19].

Ultrafilter limit of $\mathbb{E}$

Let us (re)define the eventually different forcing $\mathbb{E}$.

Definition

$\mathbb{E} := \{(s, k, \varphi) : s \in \omega^{<\omega}, k < \omega, \varphi \colon \omega \to [\omega]^{\leq k}\}$
$(s', k', \varphi') \leq (s, k, \varphi) :\Leftrightarrow$

- $s' \supseteq s, k' \geq k, \forall i < \omega, \varphi'(i) \supseteq \varphi(i)$
- $\forall i \in \mathrm{dom}(s' \setminus s), s'(i) \notin \varphi(i)$

For $p = (s, k, \varphi) \in \mathbb{E}$, we call $s(p) := s$ the stem of p and $k(p) := k$ the width of p.

Though we can forcing-euivalently define $\mathbb{E}$ without widths, we mention the width explicitly to define ultrafilter limit of $\mathbb{E}$ by restricting widths.

Ultrafilter limit of $\mathbb{E}$

Definition

Let D be an ultrafilter on ω, $s \in \omega^{<\omega}$ and $k < \omega$. For $\bar{p} = \langle p_m = (s, k, \varphi_m) : m < \omega \rangle \in \mathbb{E}^\omega$, define D-limit condition $\lim_D \bar{p} = (s, k, \varphi_\infty)$ by $j \in \varphi_\infty(i) :\Leftrightarrow \{m < \omega : j \in \varphi_m(i)\} \in D$.

Here is the crucial property which is used when we inductively construct names of ultrafilter through iteration afterwards.

Crucial Property of Ultrafilter Limit of $\mathbb{E}$

If $q \leq \lim_D \bar{p}$, then $\{m < \omega : p_m$ is compatible with $q\} \in D$.

Proof. Let $q := (s', k', \varphi')$ and $\lim_D \bar{p} := (s, k, \varphi_\infty)$.
Since $q \leq \lim_D \bar{p}$, $\forall i \in \mathrm{dom}(s' \setminus s), s'(i) \notin \varphi_\infty(i)$ i.e., $\{m : s'(i) \in \varphi_m(i)\} \notin D$. Since D is an ultrafilter, $A^i := \{m : s'(i) \notin \varphi_m(i)\}$ is in D for such i. If $m \in \bigcap\{A^i : i \in \mathrm{dom}(s' \setminus s)\}$, then $\forall i \in \mathrm{dom}(s' \setminus s), s'(i) \notin \varphi_m(i)$ and hence p_m is compatible with q. $\square$

Current situation

Before applying ultrafilter limit for iteration, let us clarify the current situation.

- We are constructing the fsi poset $\mathbb{P}^5 = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle_{\alpha < \lambda_5 + \lambda_5}$ which follows bookkeeping function f_{bk} such that $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha = f_{bk}(\alpha)$ (so, equivalently, we are defining f_{bk}).
- For each α, it is already determined which kind of poset the iterand $\dot{\mathbb{Q}}_\alpha$ is a subforcing of.
- Hence, we can already define S^+ as a set of ordinals of $\mathbb{E}$-position in the iteration. More precisely, $S^+ := \{\lambda_5 \leq \alpha < \lambda_5 + \lambda_5 : \Vdash_\alpha \dot{\mathbb{Q}}_\alpha \text{ is a subforcing of } \mathbb{E}\}$. Let $S^- := (\lambda_5 + \lambda_5) \setminus S^+$.
- We already know each $f_{bk}(\alpha)$ for $\alpha \in S^-$. We do not know what $f_{bk}(\alpha)$ is for $\alpha \in S^+$ yet.

In the following slides, we will define some notions mentioning $\mathbb{P}^5$, which is supposed to be not defined yet, but it makes sense since all the definitions are valid as long as f_{bk} satisfies the items above.

Guardrail

Roughly speaking, in order to define ultrafilter limits pointwisely for conditions of iteration, we consider ω-sequences of iteration with common values on S^-. For this purpose, we introduce a "guardrail".

(For some combinatorial reason) let us additionally assume that λ_3 is a successor cardinal of a regular χ with $\chi^{\aleph_0} = \chi$ and $2^\chi = \lambda_5$. Since for $\alpha \in S^-$ $\Vdash_\alpha |\dot{\mathbb{Q}}_\alpha| < \lambda_3 = \chi^+$, we can fix a name of injection $\Vdash_\alpha \dot{i}_\alpha \colon \dot{\mathbb{Q}}_\alpha \to \chi$.

Definition

- A "partial guardrail" is a function h defined on a subset of $\lambda_5 + \lambda_5$ such that $h(\alpha) \in \chi$ for $\alpha \in S^-$ and $h(\alpha) \in \omega^{<\omega} \times \omega$ for $\alpha \in S^+$ ($\omega^{<\omega} \times \omega$ represents stems and widths).
- A "countable guardrail" is a partial guardrail with countable domain. A "full guardrail" is a partial guardrail with full domain.

Guardrail

We will use the following lemma afterwards, which is a consequence of infinitary combinatorics.

Lemma

(Since $|\lambda_5| \leq 2^\chi$ and $\chi^{\aleph_0} = \chi$,) $\exists \{h_\varepsilon : \varepsilon < \chi\}$: a family of full guardrails, $\forall$countable guardrail h, $\exists \varepsilon < \chi, h \subseteq h_\varepsilon$.

Definition

A condition $p \in \mathbb{P}^5$ follows the full guardrail h, if for all $\alpha \in \mathrm{dom}(p)$, $\mathbb{P}_\alpha$ forces that:

- for $\alpha \in S^-$, $\dot{i}_\alpha(p(\alpha)) = h(\alpha)$, and
- for $\alpha \in S^+$, $(s(p(\alpha)), k(p(\alpha)) = h(\alpha)$.

Guardrail

Lemma

$D := \{p \in \mathbb{P}^5 : \exists \varepsilon < \chi, p$ follows $h_\varepsilon\}$ is dense.

Proof.
It is inductively seen that there are densely many $p \in \mathbb{P}^5$ such that

$$\exists \varepsilon < \chi, \begin{cases} \forall \alpha \in \mathrm{dom}(p) \cap S^-, p{\restriction}\alpha \Vdash \dot{i}_\alpha(p(\alpha)) = h_\varepsilon(\alpha) \text{ and} \\ \forall \alpha \in \mathrm{dom}(p) \cap S^+, p{\restriction}\alpha \Vdash (s(p(\alpha)), k(p(\alpha)) = h_\varepsilon(\alpha) \end{cases}.$$

Fix such p and h_ε. For $\alpha \in \mathrm{dom}(p)$ the followings hold:

- if $\alpha \in S^-, \Vdash_\alpha \exists x \in \dot{\mathbb{Q}}, \dot{i}_\alpha(x) = h(\alpha)$.
- if $\alpha \in S^+, \Vdash_\alpha \exists x \in \dot{\mathbb{Q}}, \begin{cases} p{\restriction}\alpha \in \dot{G}_\alpha \Rightarrow x = p(\alpha) \\ p{\restriction}\alpha \notin \dot{G}_\alpha \Rightarrow x = (h(\alpha), \emptyset) \end{cases}$.

(By maximal principle,) we can take each $\mathbb{P}_\alpha$-name τ_α for the witness. It can be (inductively) seen that if we define p' by replacing each $p(\alpha)$ with τ_α, p' is identified with p and follows h_ε. □

Ultrafilter limit for iteration

We define ultrafilter limit for ω-sequences of the iteration $\mathbb{P}^5$ which follows a common guardrail (and forms a Δ-system) by taking ultrafilter limits pointwisely.

Definition

Fix $\varepsilon < \chi$, $\beta \leq \lambda_5 + \lambda_5$ and $\dot{\bar{D}} = \{\dot{D}_\alpha : \alpha < \beta\}$ where each $\dot{D}_\alpha$ is a $\mathbb{P}_\alpha$-name of an ultrafilter on ω.

- $\bar{p} = \{p_m : m < \omega\} \in (\mathbb{P}_\beta)^\omega$ is a (countable) Δ-system with root ∇ following h_ε $:\Leftrightarrow$ $\{\mathrm{dom}(p_m) : m < \omega\}$ is a Δ-system with root ∇ and every p_m follows h_ε .
- For such $\bar{p}$, we define the $\lim_{\dot{\bar{D}}} \bar{p}$ to be the following function with domain ∇:
  - If $\alpha \in \nabla \cap S^-$, then $\Vdash_\alpha \lim_{\bar{D}} \bar{p}(\alpha) := (\dot{i}_\alpha)^{-1}(h(\alpha))$ (the common value of all $p_m(\alpha)$).
  - If $\alpha \in \nabla \cap S^+$, then $\Vdash_\alpha \lim_{\bar{D}} \bar{p}(\alpha) := \lim_{\dot{D}_\alpha} \{p_m(\alpha) : m \in \omega\}$.

Ultrafilter construction

We can inductively construct $\mathbb{P}_\alpha$-names of ultrafilter with some desirable properties.

Lemma

We can construct by induction on $\alpha \leq \lambda_5 + \lambda_5$ the χ-sequences of $\mathbb{P}_\alpha$-names of an ultrafilter $\{\dot{D}^\varepsilon_\alpha : \varepsilon < \chi\}$ (and $f_{bk}(\alpha)$ for $\alpha \in S^+$) such that for each $\varepsilon < \chi$ and countable Δ-system $\bar{p} = \{p_m : m < \omega\} \in (\mathbb{P}_\alpha)^\omega$ following h_ε, $\lim_{\{\dot{D}^\varepsilon_\beta : \beta < \alpha\}} \bar{p}$ is in $\mathbb{P}_\alpha$ and forces that $\{m \in \omega : p_m \in \dot{G}_\alpha\} \in \dot{D}^\varepsilon_\alpha$.

Note that the limit condition forces that ultrafilter many p_m's are in the generic filter, but does not decide which p_m is.

The proof is complicated and omitted in this talk, but in the proof Crucial Property of Ultrafilter Limit on $\mathbb{E}$ (and other basic trivial properties of $\mathbb{E}$) are effectively used.

Proof that $\mathfrak{b}$ is small

Theorem([GKS19])

$\Vdash_{\mathbb{P}^5} \mathfrak{b} \leq \lambda_3$. Moreover, $\mathbb{P}^5$ forces that for any regular $\lambda_3 \leq \kappa \leq \lambda_5$ and for any $\mathbb{P}^5$-name $\dot{f}$ of a member of ω^ω, there exists $i < \kappa$, for all $i \leq \alpha < \kappa$, $\mathbb{C}_\alpha$-real $\dot{c}_\alpha$ is unbounded from $\dot{f}$.

Proof. If not, $\exists \kappa, \exists p, \exists \dot{f}, p \Vdash \forall i < \kappa, i \leq \exists \alpha < \kappa, \dot{c}_\alpha \leq^* \dot{f}$.
So, $\forall i < \kappa, \exists p_i \leq p, i \leq \exists \beta_i < \kappa, \exists n_i < \omega$,
$p_i \Vdash n_i < \forall n < \omega, \dot{c}_{\beta_i}(n) \leq \dot{f}(n)$. By extending and thinning, we may assume that:

- $\forall i, \beta_i \in \mathrm{dom}(p_i)$.
- $\exists \varepsilon_0 < \chi, \forall i, p_i$ follows h_{ε_0}.
- $\{p_i : i < \kappa\}$ forms a Δ-system with root ∇.
- $\forall i, \beta_i \notin \nabla$. Hence all β_i are distinct.
- $\exists n^* < \omega, \forall i < \kappa, n_i = n^*$.
- $\exists s \in \omega^\omega$: Cohen condition of length n^*, $\forall i, p_i(\beta_i) = s$.

Proof of b small

Pick the first ω many p_i and for each $i < \omega$ define $q_i \leq p_i$ by $q_i(\beta_i) := s^\frown i$. Note that $\{q_i : i < \omega\}$ forms a Δ-system (with root ∇), following some new countable guardrail and therefore some full h_{ε_1}. Accordingly, the limit condition of $\{q_i : i < \omega\}$ forces that ultrafilter many (in particular, infinitely many) of the q_i are in the generic filter.
But each q_i forces that $\dot{c}_{\beta_i}(n^*) = i \leq \dot{f}(n^*)$, a contradiction. □

Prediction forcing $\mathbb{PR}$

Definition

The forcing poset $\mathbb{PR}$ consists of tuples (d, π, F) satisfying:

- $d \in 2^{<\omega}$.
- $\pi = \langle \pi_n : n \in d^{-1}(\{1\}) \rangle$.
- $\forall n \in d^{-1}(\{1\})$, π_n is a finite partial function of $\omega^n \to \omega$.
- $F \in [\omega^\omega]^{<\omega}$
- $\forall f, g \in F, f \restriction |d| = g \restriction |d|$ implies $f = g$.

$(d', \pi', F') \leq (d, \pi, F) :\Leftrightarrow$

- $d' \supseteq d$.
- $\forall n \in d^{-1}(\{1\}), \pi'_n \supseteq \pi_n$(as partial functions $\omega^n \to \omega$).
- $F' \supseteq F$.
- $\forall n \in (d')^{-1}(\{1\}) \setminus d^{-1}(\{1\}), \forall f \in F, \pi'_n(f \restriction n) = f(n)$.

$\mathbb{PR}$ is σ-centered and adds a predicting real (hence increase $\mathfrak{e}$).

$\mathbb{P}^6$:fsi that separates left side including $\mathfrak{e}$

Let us define fsi poset $\mathbb{P}^6$ that separates the left side including $\mathfrak{e}$.
(As explained above, once we separates the left side, the dual numbers in the right side can also be separated almost automatically by using submodel method introduced in [GKMS22].)
For given uncountable regular cardinals (with some cardinal arithmetics) $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \lambda_5 < \lambda_6$, let $\mathbb{P}^6 :=$

$$\mathbb{C}_{\lambda_6} * \text{ fsi of length } \lambda_6 \text{ of } \begin{cases} \text{subforcing of } \mathbb{A} \text{ of size } < \lambda_1, \\ \text{subforcing of } \mathbb{B} \text{ of size } < \lambda_2, \\ \text{subforcing of } \mathbb{D} \text{ of size } < \lambda_3, \\ \text{subforcing of } \mathbb{PR} \text{ of size } < \lambda_4 \text{ or} \\ \text{subforcing of } \mathbb{E} \text{ of size } < \lambda_5 \end{cases}$$

following some bookkeeping function f_{bk}. As in the case of $\mathbb{P}^5$, $\mathbb{P}^6$ forces that $\text{add}(\mathcal{N}) = \lambda_1, \text{cov}(\mathcal{N}) = \lambda_2, \mathfrak{b} \geq \lambda_3, \mathfrak{e} \geq \lambda_4, \text{non}(\mathcal{M}) = \lambda_5, \text{cov}(\mathcal{M}) = 2^{\aleph_0} = \lambda_6$.
$\rightarrow$"$\mathfrak{b}$ is small" and "$\mathfrak{e}$ is small" are remained.

Ultrafilter limit of $\mathbb{PR}$

To keep $\mathfrak{b}$ small, it is desirable if also $\mathbb{PR}$ has ultrafilter limit and hence names of ultrafilter can be constructed in the same way. The speaker showed this by modifying the proof of separating $\mathfrak{b} < \mathfrak{e}$ in [BS96].

First, we define ultrafilter limit of ω^ω.

Definition

For ultrafilter D on ω and ω-sequence $\bar{f} = \langle f^m \in \omega^\omega : m < \omega \rangle$ satisfying that:$\forall n < \omega, \exists! a_n < \omega, \{m < \omega : f^m(n) = a_n\} \in D$, define $\lim_D \bar{f} \in \omega^\omega$ by $\lim_D \bar{f}(n) = a_n$ for each $n < \omega$.

Note that $\lim_D \bar{f} \in \omega^\omega$ is not always defined.

Ultrafilter limit of $\mathbb{PR}$

We are ready to define ultrafilter limit of $\mathbb{PR}$.

Definition

Fix $k < \omega$,d, π and $\{f_l^* \in \omega^{|d|} : l < k\}$. For a countable sequence of conditions $\bar{p} = \langle p^m = (d, \pi, \{f_l^m : l < k\}) \in \mathbb{PR} : i < \omega \rangle$ with $\forall l < k, f_l^m \restriction |d| = f_l^*$, define:

- $\bar{f_l} := \langle f_l^m : m < \omega \rangle$, $A := \{l < k : \lim_D \bar{f_l}$ exists$\}, B := k \setminus A$.
- $F^\infty := \{\lim_D \bar{f_l} : l \in A\}$.
- For $l \in B$, $n_l := \min\{n < \omega : \neg \exists a < \omega, \{m < \omega : f_l^m(n) = a\} \in D\}$.
- $n^\infty := \max\{n_l + 1 : l \in B\}$ (if $B = \emptyset$, $n^\infty := |d|$).
- $d^\infty := d \cup 1^{[|d|, n^\infty)}$ (i.e., adding $(n^\infty - |d|)$ many $0's$ after d).
- $\lim_D \bar{p} := (d^\infty, \pi, F^\infty)$.

Note that $\lim_D \bar{p}$ is always defined and a condition of $\mathbb{PR}$.

$\mathbb{PR}$ has Crusial Property of ultrafilter limit

Remark

For every $l \in B$ and $a < \omega, \{m < \omega : f_l^m(n_l) > a\} \in D$.

As in the case of $\mathbb{E}$, ultrafilter limit of $\mathbb{PR}$ has the crucial property.

Crucial Property of Ultrafilter Limit of $\mathbb{PR}$

If $q \leq \lim_D \bar{p}$, then $\{m < \omega : p_m$ is compatible with $q\} \in D$.

Proof.

Let $q := (d^q, \pi^q, F^q)$. Fix $l \in A$ and $n \in (d^q)^{-1}(\{1\}) \setminus (d^\infty)^{-1}(\{1\})$. By the definition of $\lim_D \bar{f}_l$, $\exists X_0 \in D, \forall m \in X_0, f_l^m\restriction(n+1) = \lim_D \bar{f}_l\restriction(n+1)$. Along with $q \leq \lim_D \bar{p}$, $\forall m \in X_0, \pi_n^q(f_l^m\restriction n) = \lim_D \bar{f}_l(n) = f_l^m(n)$.

$\mathbb{PR}$ has Crusial Property of ultrafilter limit

Unfixing l and n, we get:

$$\exists X_1 \in D, \forall l \in A, \forall n \in (d^q)^{-1}(\{1\}) \setminus (d^\infty)^{-1}(\{1\}), \forall m \in X_1,$$
$$\pi^q_n(f^m_l \restriction n) = f^m_l(n) \tag{3.1}$$

Fix $l \in B$ and $n \in (d^q)^{-1}(\{1\}) \setminus (d^\infty)^{-1}(\{1\})$.
Let $m_n := \max\{\sigma(j) : \sigma \in \mathrm{dom}(\pi^q_n), j < n\}$.
By Remark, $\exists X_2 \in D, \forall m \in X_2, f^m_l(n_l) > m_n$. Since $n_l < n^\infty \leq n$, $\forall m \in X_2, f^m_l \restriction n \notin \mathrm{dom}(\pi^q_n)$. Unfixing l and n, we get:

$$\exists X_3 \in D, \forall l \in B, \forall n \in (d^q)^{-1}(\{1\}) \setminus (d^\infty)^{-1}(\{1\}), \forall m \in X_3,$$
$$f^m_l \restriction n \notin \mathrm{dom}(\pi^q_n) \tag{3.2}$$

We show that for all $m \in X_1 \cap X_3$, p_m and q are compatible.

$\mathbb{PR}$ has Crusial Property of ultrafilter limit

Fix such m and let $q' := (d', \pi', F')$ satisfying:

- $F' := F^q \cup \{f_l^m : l < k\}$.
- d' is an extension of d^q adding enough $0's$ after d^q to make q' be a condition.
- $\forall n \in (d^q)^{-1}(\{1\}), \pi'_n \supseteq \pi_n^q$ and $\forall l \in B, \forall n \in (d^q)^{-1}(\{1\}) \setminus (d^\infty)^{-1}(\{1\}), \pi'_n(f_l^m \restriction n) = f_l^m(n)$ (This can be done by 3.2).

$q' \leq q$ trivially holds since $(d')^{-1}(\{1\}) \setminus (d^q)^{-1}(\{1\}) = \emptyset$.
To see $q' \leq p_m$, we have to show:

$$\forall l < k, \forall n \in (d^q)^{-1}(\{1\}) \setminus (d)^{-1}(\{1\}), \pi'_n(f_l^m \restriction n) = f_l^m(n) \quad (3.3)$$

If $l \in A$, 3.1 implies 3.3, while if $l \in B$, 3.3 holds by the definition of π'. □

$\mathbb{P}^6$ forces $\mathfrak{b}$ is small

By using this Crucial Property, we can also construct $\mathbb{P}^6$ which keeps $\mathfrak{b}$ small. Define guardrails similarly by letting S^+ be a set of ordinals of $\mathbb{PR}$ or $\mathbb{E}$-position in the iteration and fix full guardrails $\{h_\varepsilon : \varepsilon < \chi\}$ such that every countable guardrail can be extended to some h_ε.

Lemma(Y.)

We can construct by induction on $\alpha \leq \lambda_6 + \lambda_6$ the χ-sequences of $\mathbb{P}_\alpha$-names of an ultrafilter $\{\dot{D}^\varepsilon_\alpha : \varepsilon < \chi\}$ such that for each $\varepsilon < \chi$ and countable Δ-system $\bar{p} = \{p_m : m < \omega\} \in \mathbb{P}^\omega_\alpha$ following h_ε, $\lim_{\{\dot{D}^\varepsilon_\beta : \beta < \alpha\}} \bar{p}$ is in $\mathbb{P}_\alpha$ and forces that $\{m \in \omega : p_m \in G_\alpha\} \in \dot{D}^\varepsilon_\alpha$.

Hence $\mathbb{P}^6$ keeps $\mathfrak{b}$ small.

Collorary(Y.)

$\mathbb{P}^6$ forces $\mathfrak{b} \leq \lambda_3$.

$\mathbb{E}$ vs $\mathbb{PR}$ on ultrafilter limits

To keep $\mathfrak{e}$ small, a simple ultrafilter limit argument as above does not work since $\mathbb{PR}$ itself has ultrafilter limit and increase $\mathfrak{e}$.
However, ultrafilter limit seems to be a strong tool.
Thus, let us see the gap between ultrafilter limits of $\mathbb{E}$ and $\mathbb{PR}$.

Gap between $\mathbb{E}$ and $\mathbb{PR}$

There is a gap between ultrafilter limits of $\mathbb{E}$ and $\mathbb{PR}$ as follows:

- $\mathbb{E}$ If $\bar{p}$ is an ω-sequence with common s and k, the limit condition also has same s and k.
- $\mathbb{PR}$ If $\bar{p}$ is an ω-sequence with common d, π, k and $\{f_l^* \in \omega^{|d|} : l < k\}$, the limit condition does not have same d in general (d^∞ might get longer by adding 0's after d).

We focus on the gap.

$\mathfrak{e}$-guardrail

Let us additionally assume that λ_4 is a successor cardinal of a regular θ with $\theta^{\aleph_0} = \theta$ and $2^\theta = \lambda_6$.

Define new guardrails by letting S^+ be a set of ordinals of $\mathbb{E}$-position in the iteration and fixing $\mathbb{P}_\alpha$-names of injection $\Vdash_\alpha \dot{j}_\alpha \to \theta$ for $\alpha \in S^-$.

We call these new guardrails $\mathfrak{e}$-guardrails and the old ones $\mathfrak{b}$-guardrails.

Fix full $\mathfrak{e}$-guardrails $\{g_\xi : \xi < \theta\}$ such that every countable $\mathfrak{e}$-guardrail can be extended to some g_ξ. For $\beta \leq \lambda_6 + \lambda_6$, $\dot{\bar{E}} = \{\dot{E}_\alpha : \alpha < \beta\}$ where each $\dot{E}_\alpha$ is a $\mathbb{P}_\alpha$-name of an ultrafilter on ω and Δ-system $\bar{p}$ with root ∇ following some g_ξ, we define the $\lim_{\dot{\bar{E}}} \bar{p}$ in the same way.

Reconstruction of $\mathbb{P}^6$

By redefining f_{bk}, we can reconstruct $\mathbb{P}^6$ to satisfy ultrafilter limit properties on $\mathfrak{e}$-guardrails, keeping those on $\mathfrak{b}$-guardrails.

Lemma

In addition to the properties of $\dot{D}$'s, we can also construct by induction on $\alpha \leq \lambda_6 + \lambda_6$ the θ-sequences of $\mathbb{P}_\alpha$-names of an ultrafilter $\{\dot{E}^\xi_\alpha : \xi < \theta\}$ (and $f_{bk}(\alpha)$ for $\alpha \in S^+$) such that for each $\xi < \chi$ and countable Δ-system $\bar{p} = \{p_m : m < \omega\} \in (\mathbb{P}_\alpha)^\omega$ following h_ξ, $\lim_{\{\dot{E}^\xi_\beta : \beta < \alpha\}} \bar{p}$ is in $\mathbb{P}_\alpha$ and forces that $\{m \in \omega : p_m \in \dot{G}_\alpha\} \in \dot{E}^\xi_\alpha$.

Strategy for proving $\mathfrak{e}$ is small

Let "$\lim_\xi$" be short for "$\lim_{\{\dot{E}^\xi_\alpha : \alpha < \lambda_6 + \lambda_6\}}$". The following Lemma holds only for $\mathfrak{e}$-guardrails, not for $\mathfrak{b}$-guardrails.

Lemma

For countable Δ-system $\bar{p}$ following g_ξ, $\lim_\xi \bar{p}$ also follows g_ξ.

Hence, we can consider "limit of limit".
The following property holds since limit condition forces that ultrafilter many conditions are in the generic filter.

Limit Preservation Property

Let $\bar{p} = \{p_m : m < \omega\}$ be a countable Δ-system following g_ξ and φ be a $\mathbb{P}^6$-forcing formula without parameter m.
If $\forall m < \omega, p_m \Vdash \varphi$, then $\lim_\xi \bar{p} \Vdash \varphi$.

Hence, the strategy is to take many limits including limits of limits, preserving desirable formulas (details below).

Proof that $\mathfrak{e}$ is small

Theorem(Y.)

$\Vdash_{\mathbb{P}^6} \mathfrak{e} \leq \lambda_4$. Moreover, $\mathbb{P}^6$ forces that for any regular $\lambda_4 \leq \kappa \leq \lambda_6$ and for any $\mathbb{P}^6$-name $\dot{\pi}$ of a predictor, there exists $i < \kappa$, for all $i \leq \alpha < \kappa$, $\mathbb{C}_\alpha$-real $\dot{c}_\alpha$ evades from $\dot{\pi}$.

Proof.

If not, $\exists \kappa, \exists p, \exists \dot{\pi} = (\dot{D}, \langle \dot{\pi}_k : k \in \dot{D} \rangle)$,
$p \Vdash \forall i < \kappa, i \leq \exists \alpha < \kappa, \dot{c}_\alpha$ is predicted by $\dot{\pi}$. So, $\forall i < \kappa, \exists p_i \leq p, i \leq \exists \beta_i < \kappa, \exists n_i < \omega, p_i \Vdash n_i < \forall k \in \dot{D}, \dot{c}_{\beta_i}(k) \leq \dot{\pi}_k(\dot{c}_{\beta_i} \restriction k)$.
By extending and thinning, we may assume:

- $\forall i, \beta_i \in \mathrm{dom}(p_i)$.
- $\exists \mathfrak{e} -$ guardrail $g_{\xi_0}, \forall i, p_i$ follows g_{ξ_0}.
- $\{p_i : i < \kappa\}$ forms a Δ-system with root ∇.
- $\forall i, \beta_i \notin \nabla$. Hence all β_i are distinct.
- $\exists n^* < \omega, \forall i < \kappa, n_i = n^*$.
- $\exists s \in \omega^\omega$: Cohen condition of length n^*, $\forall i, p_i(\beta_i) = s$.

Proof that $\mathfrak{e}$ is small

Pick the first ω many p_i and fix $n < \omega$.

Strategy

Formula φ we will preserve is:

1. "a specific point is not a predicting point." or
2. "for <u>some</u> $i < \omega$, $\dot{c}_{\beta_i}$ is predicted by $\dot{\pi}$ above n^* and the initial segment of $\dot{c}_{\beta_i}$ is a specific form."

Take many limits (including limits of limits) preserving such φ's to make the eventual limit q_n force that $[n^*, n^* + n) \cap \dot{D} = \emptyset$.
Unfix n and the limit condition of q_n's forces that for infinitely many n, $[n^*, n^* + n) \cap \dot{D} = \emptyset$, a contradiction.

Fix bijection $i \colon \omega^n \to \omega$. For each $\sigma \in \omega^n$, define $q_\sigma \leq p_{i(\sigma)}$ by $q_\sigma(\beta_{i(\sigma)}) := s^\frown \sigma$.

Proof that $\mathfrak{e}$ is small

Fix $\tau \in \omega^{n-1}$. Note that $\{q_{\tau^\frown m} : m < \omega\}$ forms a Δ-system with root ∇, following some new countable $\mathfrak{e}$-guardrail and therefore some full g_{ξ_τ}, which coincides with g_{ξ_0} on ∇. Let $q_\tau^\infty := \lim_{\xi_\tau}\{q_{\tau^\frown m} : m < \omega\}$. Since q_τ^∞ follows g_{ξ_τ} and $\mathrm{dom}(q_\tau^\infty) = \nabla$, q_τ^∞ also follows g_{ξ_0}. Each $q_{\tau^\frown m}$ forces that:

- $\dot{c}_{\beta_{i(\tau^\frown m)}} \restriction (n^* + n) = s^\frown\tau^\frown m$.
- $n^* < \forall k \in \dot{D}, \dot{c}_{\beta_{i(\tau^\frown m)}}(k) = \dot{\pi}_k(\dot{c}_{\beta_{i(\tau^\frown m)}} \restriction k)$.

Since $q_\tau^\infty \Vdash \exists \dot{E}_{\xi_\tau}$-many $m, q_{\tau^\frown m} \in \dot{G}$ and by Limit Preservation Property, q_τ^∞ forces that:

- $n^* + n - 1 \notin \dot{D}$.
- $\exists i < \omega, \begin{cases} \dot{c}_{\beta_i} \restriction (n^* + n - 1) = s^\frown\tau & \text{and} \\ n^* < \forall k \in \dot{D}, \dot{c}_{\beta_i}(k) = \dot{\pi}_k(\dot{c}_{\beta_i} \restriction k). & \end{cases}$

Proof that $\mathfrak{e}$ is small

Unfix τ and fix $\rho \in \omega^{n-2}$.
Since $\{q^\infty_{\rho^\frown m} : m < \omega\}$ forms a Δ-system with root ∇ following g_{ξ_0}, we can define $q^\infty_\rho := \lim_{\xi_0}\{q^\infty_{\rho^\frown m} : m < \omega\}$. Note that q^∞_ρ follows g_{ξ_0} and $\mathrm{dom}(q^\infty_\rho) = \nabla$. Each $q^\infty_{\rho^\frown m}$ forces that:

- $n^* + n - 1 \notin \dot{D}$.
- $\exists i < \omega, \begin{cases} \dot{c}_{\beta_i} \restriction (n^* + n - 1) = s^\frown \rho^\frown m & \text{and} \\ n^* < \forall k \in \dot{D}, \dot{c}_{\beta_i}(k) = \dot{\pi}_k(\dot{c}_{\beta_i} \restriction k). \end{cases}$

Since $q^\infty_\rho \Vdash \exists \dot{E}_{\xi_\rho}$-many $m, q^\infty_{\rho^\frown m} \in \dot{G}$ and by Limit Preservation Property, q^∞_ρ forces that:

- $n^* + n - 1 \notin \dot{D}$.
- $n^* + n - 2 \notin \dot{D}$.
- $\exists i < \omega, \begin{cases} \dot{c}_{\beta_i} \restriction (n^* + n - 2) = s^\frown \rho & \text{and} \\ n^* < \forall k \in \dot{D}, \dot{c}_{\beta_i}(k) = \dot{\pi}_k(\dot{c}_{\beta_i} \restriction k). \end{cases}$

Proof that $\mathfrak{e}$ is small

Continuing this way, we eventually get $q^n := q^\infty_\emptyset$ with the following properties:

- $\mathrm{dom}(q^n) = \nabla$ and q^n follows g_{ξ_0}.
- $q^n \Vdash [n^*, n^* + n) \cap \dot{D} = \emptyset$.

Unfix n and let $q^\infty := \lim_{\xi_0}\{q^n : n < \omega\}$. q^∞ forces that for infinitely many n, $[n^*, n^* + n) \cap \dot{D} = \emptyset$, a contradiction. □

Theorem(Y.)

It is consistent that $\aleph_1 <$ add$(\mathcal{N}) <$ cov$(\mathcal{N}) < \mathfrak{b} < \mathfrak{e} <$ non$(\mathcal{M}) <$ cov$(\mathcal{M}) < \mathfrak{pr} < \mathfrak{d} <$ non$(\mathcal{N}) <$ cof$(\mathcal{N}) < 2^{\aleph_0}$.

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