Revisiting the cofinality of the ideal of strong measure zero sets

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Joint work (in progress) with
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1. Relational systems and Tukey order

Relational systems

Definition 1.1

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Definition 1.2

Let $\mathcal{I}$ be an ideal on a set X (containing $[X]^{<\aleph_0}$).

1. $\mathcal{I} = \langle \mathcal{I}, \mathcal{I}, \subseteq \rangle$ as a relational system.
2. $\mathbf{C}_{\mathcal{I}} := \langle X, \mathcal{I}, \in \rangle$, which is a relational system.

Cardinal characteristics associated with relational systems

Definition 1.3

Let $\mathbf{R} = \langle X, Y, R \rangle$ be a relational system.

1. $B \subseteq X$ is $\mathbf{R}$-bounded if $\exists\, y \in Y\ \forall\, x \in B \colon xRy$.
2. $D \subseteq Y$ is $\mathbf{R}$-dominating if $\forall\, x \in X\ \exists\, y \in D \colon xRy$.

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3. $\mathfrak{b}(\mathbf{R}) := \min\{|F| : F \subseteq X \text{ is } \mathbf{R}\text{-unbounded}\}$.
4. $\mathfrak{d}(\mathbf{R}) := \min\{|D| : D \subseteq Y \text{ is } \mathbf{R}\text{-dominating}\}$.

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Fact 1.4

If $\mathcal{I}$ is an ideal on a set X then

(a) $\mathfrak{b}(\mathcal{I}) = \mathrm{add}(\mathcal{I}) := \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I},\ \bigcup_{A \in \mathcal{F}} A \notin \mathcal{I}\}$.

(b) $\mathfrak{d}(\mathcal{I}) = \mathrm{cof}(\mathcal{I}) := \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I},\ \forall\, A \in \mathcal{I}\ \exists\, B \in \mathcal{F}\colon A \subseteq B\}$.

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(c) $\mathfrak{b}(\mathbf{C}_{\mathcal{I}}) = \mathrm{non}(\mathcal{I}) := \min\{|Z| : Z \subseteq X,\ Z \notin \mathcal{I}\}$.

(d) $\mathfrak{d}(\mathbf{C}_{\mathcal{I}}) = \mathrm{cov}(\mathcal{I}) := \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I},\ \bigcup_{A \in \mathcal{F}} A = X\}$.

Examples

Example 1.5

1. $\mathcal{N}$: the ideal of measure zero (null) subsets of the Cantor space 2^ω.
2. $\mathcal{M}$: the ideal of first category (meager) subsets of 2^ω.
3. When $\kappa \leq |X|$ is an infinite cardinal, $[X]^{<\kappa} := \{A \subseteq X : |A| < \kappa\}$.

Tukey connections

Definition 1.6

Let $\mathbf{R} = \langle X, Y, R \rangle$ and $\mathbf{R}' = \langle X', Y', R' \rangle$ be relational systems.
A pair $(F, G) : \mathbf{R} \to \mathbf{R}'$ is a Tukey connection from $\mathbf{R}$ to $\mathbf{R}'$ if

$$F : X \to X', \quad G : Y' \to Y, \quad \forall\, x \in X \;\forall\, y' \in Y' \colon F(x) R' y' \Rightarrow x R G(y').$$

$$\begin{array}{ccccc}
 & \mathbf{R} & & \mathbf{R}' & \\
G(y') & Y & \xleftarrow[G]{\qquad} & Y' & y' \\
R & & & & R' \\
x & X & \xrightarrow[F]{\qquad} & X' & F(x) \\
 & x & & &
\end{array}$$

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$\mathbf{R} \preceq_{\mathrm{T}} \mathbf{R}'$ ($\mathbf{R}$ is Tukey-below $\mathbf{R}'$) if $\exists\,(F, G) : \mathbf{R} \to \mathbf{R}'$ Tukey connection.
$\mathbf{R} \cong_{\mathrm{T}} \mathbf{R}'$ ($\mathbf{R}$ and $\mathbf{R}'$ are Tukey equivalent) if $\mathbf{R} \preceq_{\mathrm{T}} \mathbf{R}'$ and $\mathbf{R}' \preceq_{\mathrm{T}} \mathbf{R}$.

$$\begin{tikzcd}
\mathbf{R} & \mathbf{R}' \\
Y & Y' \arrow[l, "G"'] \\
X \arrow[r, "F"'] & X'
\end{tikzcd}$$

Tukey connections and cardinal characteristics

Lemma 1.7

1. $\mathbf{R} \preceq_{\mathrm{T}} \mathbf{R}'$ *implies* $\mathfrak{b}(\mathbf{R}') \leq \mathfrak{b}(\mathbf{R})$ *and* $\mathfrak{d}(\mathbf{R}) \leq \mathfrak{d}(\mathbf{R}')$.
2. $\mathbf{R} \cong_{\mathrm{T}} \mathbf{R}'$ *implies* $\mathfrak{b}(\mathbf{R}) = \mathfrak{b}(\mathbf{R}')$ *and* $\mathfrak{d}(\mathbf{R}) = \mathfrak{d}(\mathbf{R}')$.

Products of relational systems

Definition 1.8

For each $i \in I$ let $\mathbf{R}_i := \langle X_i, Y_i, R_i \rangle$ be a relational system.
Define $\prod_{i \in I} \mathbf{R}_i := \langle \prod_{i \in I} X_i, \prod_{i \in I} Y_i, R^{\Pi} \rangle$ where

$$x \, R^{\Pi} \, y \text{ iff } \forall \, i \in I \colon x_i \, R_i \, y_i.$$

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Fact 1.9

(a) $\mathbf{R}_i \preceq_{\mathrm{T}} \prod_{i\in I} \mathbf{R}_i$.

(b) $\mathfrak{b}\left(\prod_{i\in I} \mathbf{R}_i\right) \overset{\le}{=} \min_{i\in I} \mathfrak{b}(\mathbf{R}_i)$ *and*
$\sup_{i\in I} \mathfrak{d}(\mathbf{R}_i) \le \mathfrak{d}\left(\prod_{i\in I} \mathbf{R}_i\right) \le \prod_{i\in I} \mathfrak{d}(R_i)$.

$X_i \to \prod_{i\in I} X_i$ $\qquad$ $\prod_{i\in I} Y_i \to Y_i$

$a_i \mapsto \langle \cdots, a_i, \cdots \rangle$ $\qquad$ $\bar{y} \mapsto y_i$

i

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$\sup_{i \in I} \mathfrak{d}(\mathbf{R}_i) \leq \mathfrak{d}\left(\prod_{i \in I} \mathbf{R}_i\right) \leq \prod_{i \in I} \mathfrak{d}(R_i)$.

$\mathfrak{b}\left(\prod_{i \in I} \mathbf{R}\right) \geq \min_{i \in I} \mathfrak{b}(\mathbf{R}_i)$:

If $F \subseteq \prod_{i \in I} X_i$ and $|F| < \min_{i \in I} \mathfrak{b}(\mathbf{R}_i)$
then $\{x_i : x \in F\}$ is bounded by some $y_i \in Y_i$,
so F is bounded by $y := \langle y_i : i \in I \rangle$.

Powers of relational systems

When $\mathbf{R}_i = \mathbf{R}$ for all $i \in I$, we denote $\mathbf{R}^I := \prod_{i\in I} \mathbf{R}$.

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Example 1.10

Let $\mathcal{I}$ be an ideal on X.

1. $\mathcal{I}^J \cong_{\mathrm{T}} \mathcal{I}^{(J)}$ where $\mathcal{I}^{(J)}$ is the ideal on X^J generated by $\prod_{i\in J} A_i$ for $\langle A_i : i \in J\rangle \in \mathcal{I}^J$.
2. $\mathbf{C}_{\mathcal{I}}^J \cong_{\mathrm{T}} \mathbf{C}_{\mathcal{I}^{(J)}}$.

So we denote $\mathrm{add}(\mathcal{I}^J) := \mathfrak{b}(\mathcal{I}^J) = \mathrm{add}(\mathcal{I}^{(J)})$, etc.

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$$\mathrm{add}(\mathcal{I}^J) = \mathrm{add}(\mathcal{I}) \text{ and } \mathrm{non}(\mathcal{I}^J) = \mathrm{non}(\mathcal{I}).$$

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$$\mathrm{add}(\mathcal{I}^J) = \mathrm{add}(\mathcal{I}) \text{ and } \mathrm{non}(\mathcal{I}^J) = \mathrm{non}(\mathcal{I}).$$

Example 1.11

When S is a directed preorder, denote

$$\mathfrak{b}_S^I := \mathfrak{b}(S^I), \qquad \mathfrak{d}_S^I := \mathfrak{d}(S^I).$$

2. The ideal of strong measure zero sets and Yorioka ideals

Strong measure zero sets

Definition 2.1

1. For $\sigma = \langle \sigma_i : i < \omega \rangle \in (2^{<\omega})^\omega$, define $\mathrm{ht}_\sigma \colon \omega \to \omega$ s.t. $\mathrm{ht}_\sigma(i) := |\sigma_i|$.
2. A set $Z \subseteq 2^\omega$ has strong measure zero (in 2^ω) if
$$\forall\, f \in \omega^\omega\ \exists\, \sigma \in (2^{<\omega})^\omega \colon f \leq^* \mathrm{ht}_\sigma \text{ and } Z \subseteq \bigcup_{i<\omega} [\sigma_i]$$
where $[s] := \{x \in 2^\omega : s \subseteq x\}$ for $s \in 2^{<\omega}$.
3. $\mathcal{SN}$: the collection of strong measure zero subsets of 2^ω.

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$$\forall f \in \omega^\omega \ \exists \sigma \in (2^{<\omega})^\omega : f \leq^* \mathrm{ht}_\sigma \text{ and } Z \subseteq [\sigma]_\infty$$

where $[\sigma]_\infty := \{x \in 2^\omega : x \text{ extends infinitely many } \sigma_i\}$.

Yorioka ideals

Definition 2.3

1. For $x, y \in \omega^\omega$,

$$x \ll y \text{ iff } \forall\, k < \omega\ \exists\, m_k < \omega\ \forall\, i \geq m_k \colon x(i^k) \leq y(i).$$

2. $\omega^{\uparrow\omega} := \{f \in \omega^\omega : f \text{ is increasing}\}$.
3. For $f \in \omega^{\uparrow\omega}$ define the Yorioka ideal

$$\mathcal{I}_f := \{A \subseteq 2^\omega : \exists\, \sigma \in (2^{<\omega})^\omega \colon f \ll \mathrm{ht}_\sigma \text{ and } A \subseteq [\sigma]_\infty\}.$$

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Theorem 2.4 (Yorioka 2002)

Each $\mathcal{I}_f$ is a σ-ideal and $\mathcal{SN} = \bigcap_{f \in \omega^{\uparrow\omega}} \mathcal{I}_f$.

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Theorem 2.5 (Kamo & Osuga 2008)

We do not get an ideal when replacing $f \ll \mathrm{ht}_\sigma$ by $f \leq^* \mathrm{ht}_\sigma$.

Fact 2.6

1. *If* $f \leq^* g$ *then* $\mathcal{I}_g \subseteq \mathcal{I}_f$.
2. *If* $D \subseteq \omega^{\uparrow\omega}$ *is a dominating family, then* $\mathcal{SN} = \bigcap_{f \in D} \mathcal{I}_f$.

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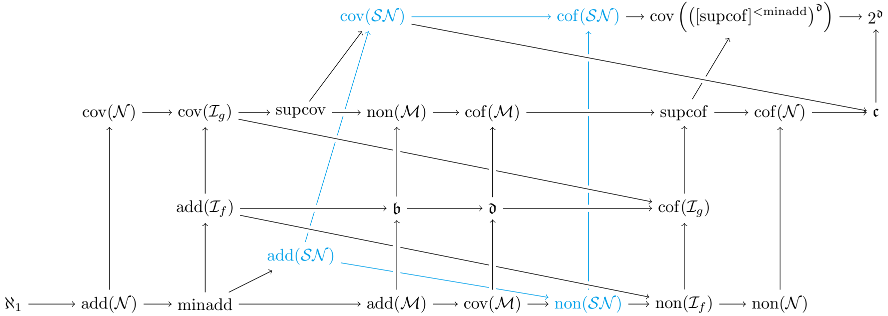
Definition 2.7

$$\begin{aligned}
\text{minadd} &:= \min_{f \in \omega^{\uparrow\omega}} \text{add}(\mathcal{I}_f), \\
\text{supcov} &:= \sup_{f \in \omega^{\uparrow\omega}} \text{cov}(\mathcal{I}_f), \\
\text{minnon} &:= \min_{f \in \omega^{\uparrow\omega}} \text{non}(\mathcal{I}_f) \\
\text{supcof} &:= \sup_{f \in \omega^{\uparrow\omega}} \text{cof}(\mathcal{I}_f).
\end{aligned}$$

supadd = add($\mathcal{I}_{id}$)

New arrows: Miller, Yorioka, Kamo, Osuga, Cardona & M., Brendle



About the cardinal characteristics

Fact 2.8 (Miller 1981, Osuga 2008)

1. $\mathrm{non}(\mathcal{SN}) = \mathrm{minnon}$.
2. $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{non}(\mathcal{SN})\}$.
3. $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{supcov}\}$.

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Theorem 2.9 (Pawlikowski 1990)

Any FS (finite support) iteration of precaliber $\aleph_1$ posets with length of uncountable cofinality forces $\mathrm{cov}(\mathcal{SN}) \leq \aleph_1$.

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Theorem 2.10 (Cardona & M. & Rivera-Madrid 2021)

In Sacks' model, $\mathrm{cov}(\mathcal{SN}) = \mathfrak{c}$.

Theorem 2.11 (Carlson 1993)

$\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathcal{SN})$.

Theorem 2.12 (Goldstern, Judah & Shelah 1993)

It is consistent with ZFC *that* $\mathrm{cof}(\mathcal{M}) < \mathrm{add}(\mathcal{SN})$.

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Theorem 2.13 (Kamo – proof by Cardona & M. 2019)

$\mathrm{add}(\mathcal{N}) \leq \mathrm{minadd} \leq \mathrm{add}(\mathcal{SN})$.

Yorioka's Characterization Theorem

Theorem 2.14 (Yorioka 2022)

If $\mathrm{minadd} = \mathrm{supcof} = \lambda$ *then* $\mathcal{SN} \cong_{\mathrm{T}} \lambda^{\lambda}$.
In particular $\mathrm{add}(\mathcal{SN}) = \lambda$ *and* $\mathrm{cof}(\mathcal{SN}) = \mathfrak{d}_{\lambda}$.

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In particular $\mathrm{add}(\mathcal{SN}) = \lambda$ *and* $\mathrm{cof}(\mathcal{SN}) = \mathfrak{d}_\lambda$.

Theorem 2.15 (Yorioka 2002)

ZFC *does not prove any relation between* $\mathrm{cof}(\mathcal{SN})$ *and* $\mathfrak{c}$.

$CH \to c < \mathrm{cof}(SN)$

$\times\ \mathrm{Con}(\mathrm{cof}(SN) < c)$

3. Developments about $\mathrm{cof}(\mathcal{SN})$

Improvements

Definition 3.1 (Cardona 2022)

Let S be a directed preorder with minimal element i_0.

For $f \in \omega^{\uparrow\omega}$, $\bar{A} = \langle A_i : i \in S \rangle$ is an $\mathcal{I}_f$-directed system on S if

(I) $\forall\, i \in S\colon A_i \in \mathcal{I}_f$,

(II) A_{i_0} is dense G_δ,

(III) if $i \leq_S j$ then $A_i \subseteq A_j$, and

(IV) $\{A_i : i \in S\}$ is cofinal in $\mathcal{I}_f$.

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(IV) $\{A_i : i \in S\}$ is cofinal in $\mathcal{I}_f$.

If $\{f_\alpha : \alpha < \lambda\}$ is a dominating family on ω^ω then $\langle \bar{A}^{f_\alpha} : \alpha < \lambda \rangle$ is a λ-dominating directed system on S if each $\bar{A}^{f_\alpha}$ is an $\mathcal{I}_{f_\alpha}$-directed system on S and

(V) $\forall\, \alpha < \lambda\colon \bigcap_{\xi<\alpha} A_{i_0}^{f_\xi} \notin \mathcal{I}_{f_\alpha}$.

Lemma 3.2 (Yorioka 2002)

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Lemma 3.3 (Cardona 2022)

If $\mathrm{cov}(\mathcal{M}) = \mathfrak{d} = \lambda$, $D \subseteq \omega^{\uparrow\omega}$ *is dominating and, for each* $f \in D$*, there is some* $\mathcal{I}_f$*-directed system on* S*, then there is a* λ*-dominating directed system on* S*.*

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Lemma 3.4 (Cardona 2022)

If there is a λ*-dominating directed system on* S *then* $\mathcal{SN} \preceq_{\mathrm{T}} S^{\lambda}$*. In particular*

$$\mathfrak{b}(S) \leq \mathrm{add}(\mathcal{SN}) \text{ and } \mathrm{cof}(\mathcal{SN}) \leq \mathfrak{d}^{\lambda}_{S}.$$

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If $\mathrm{minadd} = \mathrm{supcof} = \lambda$ *then there is a* λ*-dominating system on* λ.

Lemma 3.3 (Cardona 2022)

If $\mathrm{cov}(\mathcal{M}) = \mathfrak{d} = \lambda$, $D \subseteq \omega^{\uparrow\omega}$ *is dominating and, for each* $f \in D$, *there is some* $\mathcal{I}_f$*-directed system on* S, *then there is a* λ*-dominating directed system on* S.

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If there is a λ*-dominating directed system on* S *then* $\mathcal{SN} \preceq_{\mathrm{T}} S^\lambda$. *In particular*

$$\mathfrak{b}(S) \leq \mathrm{add}(\mathcal{SN}) \text{ and } \mathrm{cof}(\mathcal{SN}) \leq \mathfrak{d}_S^\lambda.$$

Point

If there is some $\mathcal{I}_f$-directed system on S then $\mathcal{I}_f \preceq_{\mathrm{T}} S$.

Lemma 3.5

$\mathcal{SN} \preceq_{\mathrm{T}} \prod_{f\in D} \mathcal{I}_f$ for any dominating family $D \subseteq \omega^{\uparrow\omega}$.

In particular, $\mathrm{minadd} \leq \mathrm{add}(\mathcal{SN})$ *and*

$$\mathrm{cof}(\mathcal{SN}) \leq \mathfrak{d}\left(\prod_{f\in D} \mathcal{I}_f\right) \leq \prod_{f\in D} \mathrm{cof}(\mathcal{I}_f) = 2^{\mathfrak{d}},$$

the last equality when $|D| = \mathfrak{d}$.

$$\mathcal{SN} \longrightarrow \prod_{f\in D} \mathcal{I}_f \qquad \prod_{f\in D} \mathcal{I}_f \longrightarrow \mathcal{SN}$$

$$z \longmapsto \langle \ldots, z, \ldots \rangle \qquad \langle \ldots, A_f, \ldots \rangle \mapsto \bigcap_{f\in D} A_f \in \bigcap_{f\in D} \mathcal{I}_f = \mathcal{SN}$$

$$\langle \ldots, z, \ldots \rangle \subseteq^{\Pi} \langle \ldots, A_f, \ldots \rangle \Rightarrow z \subseteq \bigcap_{f\in D} A_f$$

Lemma 3.5

$\mathcal{SN} \preceq_{\mathrm{T}} \prod_{f\in D}\mathcal{I}_f$ for any dominating family $D \subseteq \omega^{\uparrow\omega}$.

In particular, $\mathrm{minadd} \leq \mathrm{add}(\mathcal{SN})$ and

$$\mathrm{cof}(\mathcal{SN}) \leq \mathfrak{d}\left(\prod_{f\in D}\mathcal{I}_f\right) \leq \prod_{f\in D}\mathrm{cof}(\mathcal{I}_f) = 2^{\mathfrak{d}},$$

the last equality when $|D| = \mathfrak{d}$.

The existence of an $\mathcal{I}_f$-directed system on S for all $f \in D$ implies

$$\mathcal{SN} \preceq_{\mathrm{T}} \prod_{f\in D}\mathcal{I}_f \preceq_{\mathrm{T}} \prod_{f\in D} S = S^D.$$

A more direct Tukey-connection

Fact 3.6

1. *If* $X \subseteq Y$ *and* $\theta \leq \kappa$ *then* $\mathbf{C}_{[X]^{<\kappa}} \preceq_{\mathrm{T}} \mathbf{C}_{[Y]^{<\theta}}$.
2. *If* S *is a directed preorder then* $S \preceq_{\mathrm{T}} \mathbf{C}_{[\mathfrak{d}(S)]^{<\mathfrak{b}(S)}}$.

$\{Y_\alpha : \alpha < \mathfrak{d}(S)\}$ dom. on S.

$S \longrightarrow \mathfrak{d}(S)$

$x \longmapsto \alpha_x$ s.t. $x \leq Y_{\alpha_x}$

$[\mathfrak{d}(S)]^{<\mathfrak{b}(S)} \longrightarrow S$

$A \longmapsto z_A$: upper bound of $\{Y_\alpha : \alpha \in A\}$

size $< \mathfrak{b}(S)$

$\alpha_x \in A \rightarrow x \leq Y_{\alpha_x} \leq z_A$

A more direct Tukey-connection

Fact 3.6

1. *If* $X \subseteq Y$ *and* $\theta \leq \kappa$ *then* $\mathbf{C}_{[X]^{<\kappa}} \preceq_{\mathrm{T}} \mathbf{C}_{[Y]^{<\theta}}$.
2. *If* S *is a directed preorder then* $S \preceq_{\mathrm{T}} \mathbf{C}_{[\mathfrak{d}(S)]^{<\mathfrak{b}(S)}}$.

Theorem 3.7 (BCM)

$\mathcal{SN} \preceq_{\mathrm{T}} \mathbf{C}^{\mathfrak{d}}_{[\mathrm{supcof}]^{<\mathrm{minadd}}}$, *in particular*

$$\mathrm{cof}(\mathcal{SN}) \leq \mathrm{cov}\left(\left([\mathrm{supcof}]^{<\mathrm{minadd}}\right)^{\mathfrak{d}}\right).$$

$$\mathcal{SN} \preceq_{T} \prod_{f\in D} \mathcal{I}_f \preceq_{T} \prod_{f\in D} C_{[\mathrm{cof}(\mathcal{I}_f)]^{<\mathrm{add}(\mathcal{I}_f)}} \preceq_{T} \prod_{f\in D} C_{[\mathrm{supcof}]^{<\mathrm{minadd}}} = C^{\mathfrak{d}}_{[\mathrm{supcof}]^{<\mathrm{min}}}$$

$|D| = \mathfrak{d}$

Lower bounds of $\mathrm{cof}(\mathcal{SN})$

Lemma 3.8 (Cardona 2022)

Asumme that κ and λ are cardinals such that $0 < \kappa \leq \lambda \leq \mathrm{non}(\mathcal{SN})$ and that there is some λ-dominating directed system on $\kappa \times \lambda$.
Then $\lambda^\lambda \preceq_{\mathrm{T}} \mathcal{SN}$, in particular $\mathrm{add}(\mathcal{SN}) \leq \lambda$ and $\mathfrak{d}_\lambda \leq \mathrm{cof}(\mathcal{SN})$.

Lower bounds of $\mathrm{cof}(\mathcal{SN})$

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Then $\lambda^\lambda \preceq_{\mathrm{T}} \mathcal{SN}$, in particular $\mathrm{add}(\mathcal{SN}) \leq \lambda$ and $\mathfrak{d}_\lambda \leq \mathrm{cof}(\mathcal{SN})$.*

Using a matrix iteration:

Theorem 3.9 (Cardona 2022)

It can be forced that $\mathrm{cov}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$ and the continuum $\mathfrak{c}$ in any position with respect to $\mathrm{cof}(\mathcal{SN})$.

Lower bounds of $\mathrm{cof}(\mathcal{SN})$

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What is the role of κ?

Non-directed systems

Definition 3.10

Let I be a set and $i_0 \in I$. Given $f \in \omega^{\uparrow\omega}$, a family $A^f = \langle A_i^f : i \in I \rangle$ is an $\mathcal{I}_f$-system on (I, i_0) if it satisfies:

(I) $\forall\, i \in I\colon A_i^f \in \mathcal{I}_f$,

(II) $A_{i_0}^f \in \mathcal{I}_f$ is dense G_δ,

(III) $\forall\, i \in I\colon A_{i_0}^f \subseteq A_i^f$, and

(IV) $\langle A_i^f : i \in I \rangle$ is cofinal in $\mathcal{I}_f$.

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(III) $\forall\, i \in I \colon A_{i_0}^f \subseteq A_i^f$, and

(IV) $\langle A_i^f : i \in I \rangle$ is cofinal in $\mathcal{I}_f$.

If $\{f_\alpha : \alpha < \lambda\}$ is a dominating family on $\omega^{\uparrow\omega}$ then $\langle \bar{A}^{f_\alpha} : \alpha < \lambda \rangle$ is a λ-dominating system on (I, i_0) if each $\bar{A}^{f_\alpha}$ is an $\mathcal{I}_{f_\alpha}$-system on (I, i_0) and

(V) $\forall\, \alpha < \lambda \colon \bigcap_{\xi < \alpha} A_{i_0}^{f_\xi} \notin \mathcal{I}_{f_\alpha}$.

Existence of systems

Remark 3.11

An $\mathcal{I}_f$-system on (I, i_0) exists iff $\mathrm{cof}(\mathcal{I}_f) \leq |I|$.

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An $\mathcal{I}_f$-system on (I, i_0) exists iff $\mathrm{cof}(\mathcal{I}_f) \leq |I|$.

Lemma 3.12

If $\mathrm{cov}(\mathcal{M}) = \mathfrak{d} = \lambda$, $D \subseteq \omega^{\uparrow\omega}$ *is dominating and* $\mathrm{supcof} \leq |I|$, *then there is a* λ*-dominating system on* (I, i_0).

$\mathrm{supcof} \leq |I| \rightarrow \forall f : \exists \bar{A}^f = \langle A^f_i : i \in I \rangle$ and $\mathcal{I}_f$-system on (I, i_0).

Fix $\{h_\alpha : \alpha < \lambda\}$ dom. family

We define $\{d_\lambda : \alpha < \lambda\} \subseteq D$ dom. family by recursion on α. (to get (V))

$\boxed{\alpha}$ $\xi < \alpha \rightarrow A^{d_\xi}_{i_0}$: dense G_δ, $|\{A^{d_\xi}_i : \xi < \alpha\}| < \lambda = \mathrm{cov}(\mathcal{M})$

by adding a perfect set of Cohen reals. we get a perfect set $P \subseteq \bigcap_{\xi < \alpha} A^{d_\xi}_{i_0}$.

$P \notin \mathcal{SN} = \bigcap_{f \in D} \mathcal{I}_f$ $\therefore$ $\exists d_\alpha \in D$ s.t. $P \notin \mathcal{I}_{d_\alpha}$ and $d_\alpha \geq h_\alpha$

$\therefore \bigcap_{\xi < \alpha} A^{d_\xi}_{i_0} \notin \mathcal{I}_{d_\alpha}$

Main Lemma

Definition 3.13

λ-DS(I, i_0): There is a λ-dominating system $\langle \bar{A}^{d_\alpha} : \alpha < \lambda \rangle$ on (I, i_0).

Main Lemma

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λ-DS(I, i_0): There is a λ-dominating system $\langle \bar{A}^{d_\alpha} : \alpha < \lambda \rangle$ on (I, i_0).

Main Lemma 3.14

Under λ-DS(I, i_0), for any $\langle \mathcal{C}_\alpha : \alpha < \lambda \rangle$ satisfying

$$\mathcal{C}_\alpha \subseteq \mathcal{I}_{d_\alpha} \text{ and } \sum_{\xi < \alpha} |\mathcal{C}_\xi| < \mathrm{non}(\mathcal{SN}) \text{ for all } \alpha < \lambda,$$

there is some $K \in \mathcal{SN}$ such that $K \nsubseteq C$ for all $C \in \bigcup_{\alpha<\lambda} \mathcal{C}_\alpha$.

By recursion on $\alpha < \lambda$, define $G : \lambda \to I$ and $\{ x_c^\alpha : c \in \mathcal{C}_\alpha \}$ as follows.

$\boxed{\text{Step 2}}$ For $C \in \mathcal{C}_\alpha$, $B_C := C \cup \underbrace{\{ x^\xi_E : \xi < \alpha,\ E \in \mathcal{C}_\xi \}}$

size $\leq \sum_{\xi<\alpha} |\mathcal{C}_\xi| < \mathrm{non}(SN) = \mathrm{min\,non}$

$B_C \in \mathcal{I}_{d_\alpha}$.

But $\bigcap_{\xi<\alpha} A^{d_\xi}_{i_0} \subseteq \bigcap_{\xi<\alpha} A^{d_\xi}_{G(\xi)} \notin \mathcal{I}_{d_\alpha}$ $\therefore$ $\bigcap_{\xi<\alpha} A^{d_\xi}_{G(\xi)} \notin B_C$

$\therefore$ $\exists$ $\underline{x^\alpha_C \in \bigcap_{\xi<\alpha} A^{d_\xi}_{G(\xi)}} \setminus B_C$

$\{ x^\xi_E : \xi \leq \alpha,\ E \in \mathcal{C}_\xi \} \subseteq \underline{A^{d_\alpha}_{G(\alpha)}}$.

size $< \mathrm{non}(SN)$

$K = \{ x^\alpha_C : \alpha < \lambda,\ C \in \mathcal{C}_\alpha \} \subseteq \bigcap_{\alpha<\lambda} A^{d_\alpha}_{G(\alpha)} \in SN$

$x^\alpha_C \in K \setminus C$ for $C \in \mathcal{C}_\alpha$ $\therefore$ $K \nsubseteq C$.

Results

$\mathrm{cov}(\mathcal{M}) = \gamma = \lambda \leq \mathrm{non}(\mathcal{SN})$

Theorem 3.15

~~λ-$DS(I, i_0)$~~ *implies* $\lambda < \mathrm{cof}(\mathcal{SN})$.

If in addition $\mathrm{cf}(\mathrm{non}(\mathcal{SN})) = \mathrm{cf}(\lambda)$ *then* $\mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$.

Let $\{C_\alpha : \alpha < \lambda\} \subseteq \mathcal{SN}$

$\therefore\ \mathcal{B}_\alpha = \{C_\alpha\}, \quad \sum_{\xi < \alpha} 1 = \alpha < \lambda \leq \mathrm{non}(\mathcal{SN}),$

Results

Theorem 3.15

λ-*DS*(I, i_0) *implies* $\lambda < \mathrm{cof}(\mathcal{SN})$.

If in addition $\mathrm{cf}(\mathrm{non}(\mathcal{SN})) = \mathrm{cf}(\lambda)$ *then* $\mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$.

Corollary 3.16

If $\mathfrak{d} \leq \mathrm{cof}(\mathcal{SN})$ *then* $\mathrm{cov}(\mathcal{M}) < \mathrm{cof}(\mathcal{SN})$.

Cases:
- $\mathrm{cov}(\mathcal{M}) = \mathfrak{d} = \lambda \xrightarrow[3.15]{} \lambda < \mathrm{cof}(\mathcal{SN})$
- $\mathrm{cov}(\mathcal{M}) < \mathfrak{d}$ $\therefore$ by $\mathfrak{d} \leq \mathrm{cof}(\mathcal{SN})$.

Results

Theorem 3.15

λ-*DS*(I, i_0) *implies* $\lambda < \mathrm{cof}(\mathcal{SN})$.

If in addition $\mathrm{cf}(\mathrm{non}(\mathcal{SN})) = \mathrm{cf}(\lambda)$ *then* $\mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$.

Corollary 3.16

If $\mathfrak{d} \leq \mathrm{cof}(\mathcal{SN})$ *then* $\mathrm{cov}(\mathcal{M}) < \mathrm{cof}(\mathcal{SN})$.

Question

Is $\mathfrak{b} \leq \mathrm{cof}(\mathcal{SN})$? Is $\mathrm{cof}(\mathcal{N}) \leq \mathrm{cof}(\mathcal{SN})$?

Question (Yorioka 2002)

Is $\aleph_1 < \mathrm{cof}(\mathcal{SN})$?

Theorem 3.17

Under λ-DS$(\mu, 0)$, if $\mathrm{non}(\mathcal{SN}) = \mathrm{supcof} = \mu$ *and* $\mathrm{cf}(\mu) = \lambda$ *then* $\lambda^\lambda \preceq_{\mathrm{T}} \mathcal{SN}$*. In particular*

$$\mathrm{add}(\mathcal{SN}) \leq \lambda \text{ and } \mathfrak{d}_\lambda \leq \mathrm{cof}(\mathcal{SN}).$$

Theorem 3.17

Under λ-DS$(\mu, 0)$, if $\mathrm{non}(\mathcal{SN}) = \mathrm{supcof} = \mu$ *and* $\mathrm{cf}(\mu) = \lambda$ *then* $\lambda^\lambda \preceq_{\mathrm{T}} \mathcal{SN}$*. In particular*

$$\mathrm{add}(\mathcal{SN}) \leq \lambda \text{ and } \mathfrak{d}_\lambda \leq \mathrm{cof}(\mathcal{SN}).$$

Yorioka's Characterization Theorem follows.

$\lambda = \mathrm{minadd} = \mathrm{supcof} \longrightarrow \mathrm{cov}(\mu) = \mathfrak{d} = \mathrm{non}(SN) = \mathrm{supcof} = \lambda$ regular.

$\therefore \lambda^\lambda \preceq_T SN \preceq_T \mathbb{C}^{\mathfrak{d}}_{[\mathrm{supcof}]^{<\mathrm{minadd}}} = \mathbb{C}^{\lambda}_{[\lambda]^{<\lambda}} \cong_T \lambda^\lambda$

$\therefore SN \cong_T \lambda^\lambda$

Theorem 3.17

Under λ-DS$(\mu, 0)$, if $\mathrm{non}(\mathcal{SN}) = \mathrm{supcof} = \mu$ *and* $\mathrm{cf}(\mu) = \lambda$ *then* $\lambda^\lambda \preceq_{\mathrm{T}} \mathcal{SN}$. *In particular*

$$\mathrm{add}(\mathcal{SN}) \leq \lambda \text{ and } \mathfrak{d}_\lambda \leq \mathrm{cof}(\mathcal{SN}).$$

Yorioka's Characterization Theorem follows.

Corollary 3.18

After adding λ-many Cohen reals with $\lambda \geq \aleph_1$ regular, $\mathrm{cof}(\mathcal{SN}) = \mathfrak{d}_\lambda$.

More about $\mathrm{add}(\mathcal{SN})$

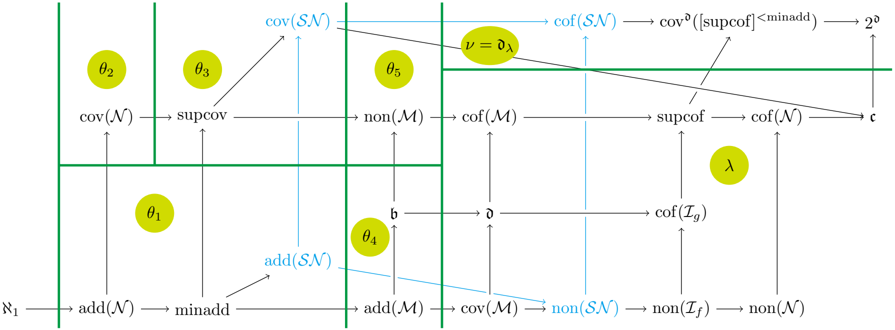
Question (Cardona & M. & Rivera-Madrid)

Is it consistent that $\mathrm{add}(\mathcal{SN}) < \min\{\mathrm{cov}(\mathcal{SN}), \mathrm{non}(\mathcal{SN})\}$?

Can the four cardinal characteristics associated with $\mathcal{SN}$ be pairwise different?

Positive answer (BCM)

We can force:



More questions

Question

Is $\mathrm{minadd} = \mathrm{add}(\mathcal{N})$? Is $\mathrm{supcof} = \mathrm{cof}(\mathcal{N})$?

Question

Is it consistent with ZFC that $\mathrm{add}(\mathcal{N}) < \mathrm{add}(\mathcal{SN}) < \mathfrak{b}$?
Or even $\mathrm{add}(\mathcal{N}) < \mathfrak{b} < \mathrm{add}(\mathcal{SN})$?

$\mathrm{add}(N) = \mathrm{cof}(N) = \lambda \to \lambda < \mathrm{cof}(S\mathcal{M})$

Q $\mathrm{cov}(N) = \mathrm{cof}(N) = \lambda \to \lambda < \mathrm{cof}(S\mathcal{M})$?