Strategy-Proofness and the Core in House Allocation Problems

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July 28, 1999
Revised: December 19, 2000
Working-Paper Version

*I would like to thank Tayfun Sönmez, an anonymous associate editor, and especially an anonymous referee for helpful comments and suggestions. This paper is a revised version of the paper entitled "Collusion-Proof Mechanisms for Matching Problems."

Abstract

We study house allocation problems introduced by Shapley and Scarf (1974). We prove that a mechanism (a social choice function) is individually rational, anonymous, strategy-proof, and non-bossy (but not necessarily Pareto efficient) if and only if it is either the core mechanism or the no-trade mechanism, where the no-trade mechanism is the one that selects the initial allocation for each profile of preferences. This confirms the intuition that even if we are willing to accept inefficiency, there exists no interesting strategy-proof mechanism other than the core mechanism. *Journal of Economic Literature* Classification Numbers: C71, C78, D71, D78, D89.

Key Words: strategy-proofness; non-bossiness; indivisible goods; housing market; mechanism design.

1 Introduction

We consider house allocation problems introduced by Shapley and Scarf (1974).[1] There are a group of agents, each of whom initially owns one indivisible object. A mechanism (a social choice function) reallocates the objects with the condition that no one receives more than one object. House allocation mechanisms are used in real life, for example, to reallocate university apartments. An important theorem, due to Ma (1994), states that the unique mechanism that is Pareto efficient, individually rational, and strategy-proof is the core mechanism. We prove that a similar characterization holds when we replace Pareto efficiency with a collusion-proofness condition.

The collusion-proofness condition that we consider is what is known as non-bossiness in the literature. Roughly speaking, a mechanism is called non-bossy if no one can change the welfare of others without changing his own. If a mechanism is bossy, then at some preference profile, some agent can affect others' welfare with no cost to himself. This implies a possibility of collusion where this agent reports false preferences, in exchange for a transfer from those who gain from his lie. A well-known mechanism that violates this condition is the Vickrey auction, where the second highest bidder can change the price of the object with no cost to himself. Collusion is indeed often observed in auctions (Cassady, 1967).

The formal statement of our result is the following. A mechanism is individually rational, strategy-proof, non-bossy, and anonymous if and only if it is either the core mechanism or the "no-trade mechanism." The no-trade mechanism is the one that selects the initial allocation for each profile of preferences. There exists no mechanism "between" the core mechanism and the no-trade mechanism that satisfies the four axioms.

The major difference between our characterization and Ma's is that we use non-bossiness and do not use Pareto efficiency. While Pareto efficiency is desirable, it constitutes a significant part of the definition of the core, together with individual rationality. For two-person economies, the core is, by definition, equivalent to Pareto efficiency plus individual rationality. On the other hand, the relation between the core and non-bossiness is less trivial. The core is defined for each preference profile (i.e., an intra-profile axiom), while non-bossiness is defined for a set of preference profiles (i.e., an inter-profile axiom).

Our result is similar to that of Roth (1977) for Nash bargaining problems. Roth proves that all of Nash's axioms except for Pareto efficiency are satisfied only by the Nash solution and the "disagreement solution." The disagreement solution is the one that selects the

[1] A partial list of papers that study house allocation is: Abdulkadiroğlu and Sönmez (1998); Hylland and Zeckhauser (1979); Ma (1994); Miyagawa (1999); Pápai (2000); Sönmez (1999); Wako (1991); Zhou (1990).

disagreement point for each bargaining problem.

Finally, we would like to mention that a similar result holds for marriage problems introduced by Gale and Shapley (1962). We can show that a mechanism for marriage problems is individually rational, strategy-proof, non-bossy, and anonymous, if and only if it makes every agent remain single for each profile of preferences. This result is intuitive in light of our result for house allocation problems and the fact that for marriage problems, no selection from the core is strategy-proof for both sides (Roth, 1982). Marriage problems are discussed in Section 6.

2 Preliminaries

2.1 The Model

We consider the model introduced by Shapley and Scarf (1974). We denote the set of agents by $N = \{1, 2, \ldots, n\}$. We often denote $N \setminus \{i\}$ by $-i$. Each agent initially owns one object. The object initially owned by agent i is called "object i," and thus N denotes the set of objects as well. An *allocation* is a list $x = (x_1, x_2, \ldots, x_n)$ such that[2]

$$x_i \in N \text{ for each } i \in N;$$
$$i \neq j \Longrightarrow x_i \neq x_j.$$

Here $x_i \in N$ denotes the object allocated to agent i. The second condition simply states that no two agents receive the same object. An allocation is simply a permutation of N.[3] Let X be the set of allocations. Let the initial allocation be denoted by

$$e = (1, \ldots, n).$$

Each agent $i \in N$ has a complete and transitive preference relation R_i defined over N. The associated strict preference relation is denoted by P_i. We assume, as usual, that R_i is strict, i.e., for any $j \neq k$, either $j \ P_i \ k$ or $k \ P_i \ j$. Thus $j \ R_i \ k$ means that either $j \ P_i \ k$ or $j = k$. Let $\mathcal{R}$ be the set of complete, transitive, and strict preference relations defined over N. A generic preference profile is denoted by $R = (R_1, \ldots, R_n) \in \mathcal{R}^N$. We write R_S for $(R_i)_{i \in S}$.

[2] Throughout the paper, "$A \Longrightarrow B$" means "A implies B," and "$A \Longleftrightarrow B$" means "A if and only if B."

[3] We are assuming that no agent can obtain more than one object. The case in which one can obtain more than one object has been studied (Ehlers and Klaus, 1999; Klaus and Miyagawa, 1999; Pápai, 1998).

We represent R_i by an ordered list of the objects:

$$R_i : k_1, k_2, k_3, i, k_4, \ldots, k_{n-1}.$$

We say that object k is *acceptable* for agent i at R_i if $k\ R_i\ i$. Since the ordering of the objects that are not acceptable is immaterial for our analysis, we will write

$$R_i : k_1, k_2, k_3, i, \ldots$$

We write $\hat{R}_i \sim_i R_i$ if $\hat{R}_i$ and R_i induce the same ranking above the endowment (i.e., $j\ \hat{P}_i\ k\ \hat{R}_i\ i \Longleftrightarrow j\ P_i\ k\ R_i\ i$). For a pair of preference profiles $R, \hat{R} \in \mathcal{R}^N$, we write $\hat{R} \sim R$ if $\hat{R}_i \sim_i R_i$ for all $i \in N$.

We denote by R_i^0 a preference relation for agent i for which the top object is his endowment, i.e.,

$$R_i^0 : i, \ldots$$

2.2 Mechanisms

A *mechanism* is a function $\varphi \colon \mathcal{R}^N \to X$, which associates an allocation $x \in X$ with each preference profile $R \in \mathcal{R}^N$. We denote by $\varphi_i(R)$ the object allocated to agent i.

An important mechanism for Shapley-Scarf economies is the (strict or strong) core mechanism. To define this mechanism, we need an additional piece of notation. Given a coalition $S \subseteq N$, let

$$X_S = \{y \in X : y_i \in S \text{ for all } i \in S\}.$$

This is the set of allocations where agents in S exchange their endowments among themselves.

An allocation $x \in X$ is in the *core* for a preference profile $R \in \mathcal{R}^N$ if there exist no coalition $S \subseteq N$ and no allocation $y \in X_S$ such that $y_i\ R_i\ x_i$ for all $i \in S$, with strict preference holding for some $i \in S$. Let $C(R)$ denote the core for $R \in \mathcal{R}^N$. Since R_i is strict for all $i \in N$, $C(R)$ is a singleton (Roth and Postlewaite, 1977). Thus C is a function. We call the function C the *core mechanism.* The unique core allocation can be computed easily by means of the *top trading cycle algorithm*, introduced by David Gale (Shapley and Scarf, 1974).[4]

[4]The algorithm works as follows. In the first round, each agent "points" to his most preferred object, and then we look for "cycles." A set of agents $\{i_1, i_2, \ldots, i_m\}$ form a cycle if i_1 points to i_2, i_2 points to i_3, ..., and i_m points to i_1. If agents form a cycle, they exchange their endowments according to the cycle, and then leave the economy. In the next round, those who remain in the economy repeat the procedure

The *no-trade mechanism* is the mechanism $NT\colon \mathcal{R}^N \to X$ defined by $NT(R) = e$ for all $R \in \mathcal{R}^N$. That is, the mechanism selects the initial allocation for each preference profile. This is not an interesting mechanism, but we need it to state our main result.

2.3 Axioms

We now introduce a few axioms, which are all standard in the literature.

A mechanism φ is *individually rational* if for all $R \in \mathcal{R}^N$ and all $i \in N$, $\varphi_i(R)\, R_i\, i$.

A mechanism is anonymous if it is defined independently of the names of the agents. To define this axiom formally, let $\pi\colon N \to N$ be a permutation (i.e., bijection), and let $T(R, \pi)$ be the preference profile R' defined by the condition that for all $i, j, k \in N$,

$$j\, R_i\, k \Longleftrightarrow \pi(j)\, R'_{\pi(i)}\, \pi(k).$$

Thus R' is identical to R except that agent i is renamed agent $\pi(i)$ (and hence object i is renamed object $\pi(i)$). For convenience, we write π as a vector in N^N. For example, by $\pi = (2, 3, 1)$, we mean that $\pi(1) = 2$, $\pi(2) = 3$, and $\pi(3) = 1$. For this example, if $R' = T(R, \pi)$, then

$$R_1 : 2, 1, 3 \Longrightarrow R'_2 : 3, 2, 1. \tag{1}$$

A mechanism φ is *anonymous* if for all $R \in \mathcal{R}^N$, all permutations $\pi\colon N \to N$, and all $i \in N$, $\varphi_{\pi(i)}(T(R, \pi)) = \pi(\varphi_i(R))$. For example (1), if φ is anonymous and $\varphi_1(R) = 3$, then $\varphi_2(R') = 1$.

A mechanism φ is *strategy-proof* if in its associated revelation game, it is a dominant strategy for each agent to report his preferences truthfully, i.e., for all $R \in \mathcal{R}^N$, all $i \in N$, and all $R'_i \in \mathcal{R}$, $\varphi_i(R)\, R_i\, \varphi_i(R'_i, R_{-i})$.

Finally, a mechanism φ is *non-bossy* (Satterthwaite and Sonnenschein, 1981) if no agent can change the allocation for others without changing the allocation for himself, i.e., for all $R \in \mathcal{R}^N$, all $i \in N$, and all $R'_i \in \mathcal{R}$,

$$[\varphi_i(R'_i, R_{-i}) = \varphi_i(R)] \Longrightarrow [\varphi(R'_i, R_{-i}) = \varphi(R)].$$

To understand the meaning of this condition, suppose that it is violated, i.e., $\varphi_i(R'_i, R_{-i}) = \varphi_i(R)$ and $\varphi_j(R'_i, R_{-i}) \neq \varphi_j(R)$ for some $j \neq i$. This means that agent i can change the welfare level of agent j with no cost to himself. This suggests a possibility of collusion

among themselves. The algorithm terminates when no agent remains in the economy. Since at least one cycle forms in each round, the algorithm terminates eventually.

where agent i reports a preference relation that is favorable for agent j in exchange for a transfer from agent j. A well-known mechanism that violates non-bossiness is the Vickrey auction, where the second highest bidder can change the price of the object while remaining a loser. Collusion is indeed observed often in auctions. Non-bossiness would not eliminate collusion completely, but non-bossiness is still an appealing property.[5]

We introduce a few facts that will be useful.

Fact 1. Let φ be a mechanism that is strategy-proof and non-bossy. Then for all $R \in \mathcal{R}^N$, all $i \in N$, and all $R'_i \in \mathcal{R}$, if the most preferred object for R'_i is $\varphi_i(R)$, then $\varphi(R'_i, R_{-i}) = \varphi(R)$.

Proof. Strategy-proofness implies $\varphi_i(R'_i, R_{-i})\ R'_i\ \varphi_i(R)$. Since $\varphi_i(R)$ is the most preferred object for R'_i, it follows that $\varphi_i(R'_i, R_{-i}) = \varphi_i(R)$. This together with non-bossiness implies $\varphi(R'_i, R_{-i}) = \varphi(R)$. $\square$

Fact 2. Let φ be a mechanism that is strategy-proof, non-bossy, and individually rational. Then

$$R'_i \sim_i R_i \Longrightarrow \varphi(R'_i, R_{-i}) = \varphi(R_i, R_{-i}).$$

We omit the straightforward proof. Note that by using Fact 2 repeatedly, we obtain

$$R' \sim R \Longrightarrow \varphi(R') = \varphi(R).$$

3 The Result

The axioms defined in the previous section are all satisfied trivially by the no-trade mechanism. The axioms are satisfied also by the core mechanism. It is well-known that the core mechanism is strategy-proof (Roth, 1982; see also Moulin, 1995). To see that the core mechanism is non-bossy, suppose $C_i(R'_i, R_{-i}) = C_i(R) = j$. Let R''_i be such that its top object is j. Then $C(R)$ is a core allocation also for preference profile (R''_i, R_{-i}), since blocking this allocation is even more difficult at this preference profile. This establishes $C(R''_i, R_{-i}) = C(R)$, and a symmetric argument yields $C(R''_i, R_{-i}) = C(R'_i, R_{-i})$, as desired.

We have seen that the no-trade mechanism and the core mechanism satisfy our axioms. Our contribution is to show that no other mechanism does.

[5] A violation of non-bossiness does not mean that collusion must occur, especially since agents may not know each other's preferences. Similarly, a violation of strategy-proofness does not mean that a misrepresentation of preferences must occur.

Theorem 1. *A mechanism is individually rational, anonymous, strategy-proof, and non-bossy, if and only if it is either the core mechanism or the no-trade mechanism.*

This result complements Ma's important theorem (Ma, 1994), which states that a mechanism is individually rational, strategy-proof, and Pareto efficient if and only if it is the core mechanism. The main insight that our result provides is that even if we are willing to accept inefficiency, there is no interesting strategy-proof mechanism other than the core mechanism.[6]

Theorem 1 is similar to a theorem due to Roth (1977) for Nash bargaining. Roth shows that all of Nash's axioms except for Pareto efficiency are satisfied only by the Nash solution and the "disagreement solution." The disagreement solution is the one that selects the disagreement point for each bargaining problem.

4 Proof of Theorem 1

Let φ be a mechanism satisfying all of our axioms. We prove that φ is either the core mechanism (C) or the no-trade mechanism (NT). We use induction on n. The theorem holds trivially when $n = 2$, since when $n = 2$, the core mechanism and the no-trade mechanism are the only mechanisms that are individually rational. Given $k \geq 2$, suppose that the result holds if $n \leq k$, and suppose $n = k + 1$.

An induction argument on n is effective since when agent 1's preferences are fixed at R_1^0, the function $\varphi(R_1^0, \cdot)$ is essentially a mechanism for $(n-1)$-person economies, to which we can apply the induction hypothesis.

We have to be careful since R_j for $j \neq 1$ is defined over N, not $N \setminus \{1\}$. But it is easy to verify that if R_j and R'_j induce the same ranking over $N \setminus \{1\}$, then $\varphi(R_1^0, R_j, R_{N\setminus\{1,j\}}) = \varphi(R_1^0, R'_j, R_{N\setminus\{1,j\}})$ (by strategy-proofness, agent j obtains the same object, and by non-bossiness, so does everyone else). This fact enables us to define functions g_j^1 by

$$g_j^1(R_2|_{N\setminus\{1\}}, \ldots, R_n|_{N\setminus\{1\}}) = \varphi_j(R_1^0, R_{-1}), \quad j \in \{2, \ldots, n\},$$

where $R_j|_{N\setminus\{1\}}$ is agent j's ranking over $N \setminus \{1\}$. Then the function $g^1 = (g_2^1, \ldots, g_n^1)$ is a mechanism for economies in which the set of agents is $N \setminus \{1\}$. The function g^1 satisfies all of our axioms, and thus by our induction hypothesis, g^1 is either the core mechanism

[6]Miyagawa (1999) demonstrates that the core mechanism stands out even when monetary transfers are allowed. Specifically, it is proved that for a mechanism to be strategy-proof, non-bossy, individually rational, onto, and budget balanced, it has to set the price of each object in advance and allocates the objects according to the core of the Shapley-Scarf economy associated with the prices.

or the no-trade mechanism. Then we have

$$[g^1 = NT] \Longrightarrow [\forall R_{-1} \in \mathcal{R}^{N \setminus \{1\}},\ \varphi(R_1^0, R_{-1}) = e]; \tag{2}$$

$$[g^1 = C] \Longrightarrow [\forall R_{-1} \in \mathcal{R}^{N \setminus \{1\}},\ \varphi(R_1^0, R_{-1}) = C(R_1^0, R_{-1})]. \tag{3}$$

We define g^i for $i \neq 1$ similarly. By anonymity, either $g^i = NT$ for all $i \in N$ or $g^i = C$ for all $i \in N$. We prove

$$[g^i = NT \quad \forall i \in N] \Longrightarrow [\varphi = NT];$$
$$[g^i = C \quad \forall i \in N] \Longrightarrow [\varphi = C].$$

4.1 Case 1: $g^i = NT$ for each $i \in N$

We first claim that for all $R \in \mathcal{R}^N$,

$$[\varphi_i(R) = i \text{ for some } i \in N] \Longrightarrow [\varphi(R) = e]. \tag{4}$$

Indeed, (2) for g^i implies $\varphi(R_i^0, R_{-i}) = e$. Thus if $\varphi_i(R) = i$, then non-bossiness implies $\varphi(R) = e$.

Thus it suffices to prove that for all $R \in \mathcal{R}^N$, there exists $i \in N$ such that $\varphi_i(R) = i$. Suppose, to the contrary, that there exists $R \in \mathcal{R}^N$ such that $\varphi_i(R)\ P_i\ i$ for all $i \in N$. Without loss of generality, assume $\varphi_1(R) = 2$. Let $x = \varphi(R)$.

Define preference profiles $\hat{R}$, R', and R'' as follows:

A1. For all $i \in N$, $\hat{R}_i : x_i, i, \ldots$

A2. $R'_1 : 3, 2, 1, \ldots$

A3. For all $i \neq 1$ and all $k \notin \{x_i, i\}$, $x_i\ P'_i\ k\ P'_i\ i$;

A4. $R'' = T((\hat{R}_1, R'_{-1}), \pi)$ where $\pi = (1, 3, 2, 4, \ldots, n)$.

Note that A2 and A4 are well-defined since $n = k + 1 \geq 3$. We will show that

$$\varphi(R) = \varphi(\hat{R}) = \varphi(R'_1, \hat{R}_{-1}) = \varphi(R') = \varphi(\hat{R}_1, R'_{-1}); \tag{5}$$

$$\varphi_1(R'') = 3; \tag{6}$$

$$\varphi(R'') = e. \tag{7}$$

Claims (6) and (7) provide a desired contradiction.

The first equality in (5) follows from Fact 1.

To see the second equality in (5), let $x' = \varphi(R'_1, \hat{R}_{-1})$. By strategy-proofness, $x'_1 \; R'_1 \; \varphi_1(\hat{R}) = x_1 = 2$, which implies $x'_1 \in \{2, 3\}$. We show $x'_1 = 2$. Suppose, to the contrary, that $x'_1 = 3$, and let $k \neq 3$ be the agent such that $x_k = 3$. Individual rationality implies $x'_k \in \{3, k\}$. Since object 3 is assigned to agent 1, we have $x'_k = k$. It then follows from (4) that $x' = e$, which is in contradiction with $x'_1 = 3$. This contradiction establishes $x'_1 = 2 = x_1$. Non-bossiness then implies $x' = x$.

The third equality in (5) follows from Fact 1 applied to agents $N \setminus \{1\}$. The last equality in (5) follows from Fact 1 applied to agent 1.

To see (6), note that anonymity implies $\varphi_1(R'') = \pi(\varphi_1(\hat{R}_1, R'_{-1})) = \pi(x_1) = 3$, where the second equality follows from (5).

Finally, to derive (7), note first that

$$R''_1 : 3, 1, \ldots$$

By strategy-proofness, $2 = \varphi_1(R') \; R'_1 \; \varphi_1(R''_1, R'_{-1})$, which implies $\varphi_1(R''_1, R'_{-1}) = 1$. Then by (4), $\varphi(R''_1, R'_{-1}) = e$. We now change agent 2's preferences to R''_2. By strategy-proofness, $2 = \varphi_2(R''_1, R'_{-1}) \; R'_2 \; \varphi_2(R''_1, R''_2, R'_{N \setminus \{1,2\}})$. Since the endowment is the least preferred object for R'_2, it follows that $\varphi_2(R''_1, R''_2, R'_{N \setminus \{1,2\}}) = 2$. Non-bossiness then implies $\varphi(R''_1, R''_2, R'_{N \setminus \{1,2\}}) = e$. Repeating this argument for agents $\{3, 4, \ldots, n\}$ yields (7).

4.2 Case 2: $g^i = C$ for each $i \in N$

A key to the proof for this case is Lemma 1, which we first describe informally. Suppose that agents $S = \{1, 2, \ldots, m\} \subseteq N$ are sitting around a table in such a way that the right-hand neighbor of agent i is agent $i + 1 \pmod{m}$. Lemma 1 says that if for each agent in S, the endowment of his right-hand neighbor is the only object he prefers to his endowment, then each agent in S obtains the endowment of his right-hand neighbor, i.e., the mechanism does not select the initial allocation for S.

To obtain intuition, consider the case when $S = N$ and $n \geq 3$. Then consider another preference profile such that for each agent, the top choice is the endowment of his right-hand neighbor, the second choice is the endowment of his left-hand neighbor, and the third choice is his own endowment. Anonymity implies that for this preference profile, either (i) each agent obtains the endowment of his right-hand neighbor or (ii) each agent obtains the endowment of his left-hand neighbor or (iii) each agent keeps his endowment. If (i) or (ii) holds, then the desired result follows from Fact 1. If (iii) holds, then suppose that agent 1 reports a false preference relation for which the top choice is his endowment.

He neither gains nor loses. But since $g^1 = C$, the mechanism now lets agents $n-1$ and n exchange their endowments. This means that agent 1's false report makes agents $n-1$ and n better off with no cost to agent 1, in violation of non-bossiness. This means that Case (iii) does not occur.

In the above argument, we assumed that $S = N$ and $n \geq 3$. If $S \subsetneq N$, then not all agents are sitting around the table, and thus we have to be careful in using anonymity. Moreover, the above argument does not work if only two agents are sitting around the table (since then $n-1 = 1$). We deal with the two-agent case using a less transparent argument.

Lemma 1. *Let $S = \{i_1, \ldots, i_m\} \subseteq N$ be such that $m \equiv |S| \geq 2$ and let $R \in \mathcal{R}^N$. Suppose that*

$$\forall k \in \{1, \ldots, m\}, \quad R_{i_k} : i_{k+1}, i_k, \ldots,$$

where $i_{m+1} = i_1$. Then for all $k \in \{1, \ldots, m\}$, $\varphi_{i_k}(R) = i_{k+1}$, where $i_{m+1} = i_1$.

Proof. Without loss of generality, assume $S = \{1, 2, \ldots, m\}$. Denote $x = \varphi(R)$. We show that

$$\forall i \in S, \quad x_i = i + 1 \pmod{m}. \tag{8}$$

Case 1: $m \geq 3$. Define R' by

$$\begin{aligned} &\forall i \in S, \quad R'_i : i+1, i-1, i, \ldots && \pmod{n} \\ &\forall i \notin S, \quad R'_i : x_i, i, \ldots && (\text{if } x_i = i, \text{ then } R'_i : i, \ldots) \end{aligned}$$

Since $m \geq 3$, it follows that for each $i \in S$, $i+1 \pmod{m} \neq i-1 \pmod{m}$, i.e., agent i's right-hand neighbor and left-hand neighbor differ. Applying Fact 1 to $N \setminus S$ yields

$$\varphi(R_S, R'_{N \setminus S}) = x. \tag{9}$$

Observe now that at preference profile R', agents in S are symmetric in the sense that the relation between i and $i+1$ is identical to the relation between $i+1$ and $i+2$. Asymmetry exists if we take into account agents' rankings below their endowments, but this asymmetry is immaterial by Fact 2. Thus anonymity implies that one of the following cases holds:[7]

[7] A formal argument goes as follows. Let $\pi = (2, 3, \ldots, m, 1, m+1, m+2, \ldots, n)$ and $R'' = T(R', \pi)$. Then it can be verified that $R'' \sim R'$. Thus Fact 2 and anonymity imply $\pi(\varphi_i(R')) = \varphi_{\pi(i)}(R'') = \varphi_{\pi(i)}(R')$. This means that if $\varphi_1(R') = 2$, then $\pi(2) = \varphi_{\pi(1)}(R')$ or equivalently $\varphi_2(R') = 3$. Repeating this argument,

B1. For all $i \in S$, $\varphi_i(R') = i + 1 \pmod{m}$;

B2. For all $i \in S$, $\varphi_i(R') = i$;

B3. For all $i \in S$, $\varphi_i(R') = i - 1 \pmod{m}$.

If B1 holds, then we apply Fact 1 to S to obtain $\varphi(R_S, R'_{N\setminus S}) = \varphi(R')$. This together with (9) implies $\varphi(R') = x$, which in turn implies (8).

If B2 holds, then Fact 1 implies $\varphi(R_1^0, R'_{-1}) = \varphi(R')$. But then $\varphi_m(R_1^0, R'_{-1}) = m \neq m - 1 = C_m(R_1^0, R'_{-1})$.[8] This is in contradiction with $g^1 = C$ and (3). This contradiction means that B2 does not hold.

Finally, suppose that B3 holds. Define R''_S by

$$\forall i \in S, \quad R''_i : i-1, i, \ldots \quad (\text{mod } m)$$

By Fact 1,

$$\varphi(R''_S, R'_{N\setminus S}) = \varphi(R'). \tag{10}$$

Now, an important observation is that the preference profile $(R''_S, R'_{N\setminus S})$ is equivalent, in the sense of $\sim$, to the preference profile $(R_S, R'_{N\setminus S})$, except that the agents in S are named in the reversed order. That is, for π defined by

$$\pi = (m, m-1, \ldots, 1, m+1, m+2, \ldots, n),$$

we have $T((R''_S, R'_{N\setminus S}), \pi) \sim (R_S, R'_{N\setminus S})$. Thus, anonymity, Fact 2, and (10) imply that for all $i \in S$, $\varphi_i(R_S, R'_{N\setminus S}) = i + 1$. This together with (9) implies (8).

Case 2: $m = 2$. Thus $S = \{1, 2\}$. Denote $x = \varphi(R)$, and suppose, by way of contradiction, that $x_1 = 1$ and $x_2 = 2$.

If $x_k = k$ for some $k \notin \{1, 2\}$, then Fact 1 and (3) imply $\varphi(R) = \varphi(R_k^0, R_{-k}) = C(R_k^0, R_{-k})$. This implies $\varphi_1(R) = 2$ and $\varphi_2(R) = 1$, and thus we are done. So we assume that $x_k \neq k$ for all $k \notin \{1, 2\}$, and that $x_3 = 4$ without loss of generality.

Define $\hat{R}$ by

$$\forall k \in \{1, 2\}, \quad \hat{R}_k = R_k;$$
$$\forall k \notin \{1, 2\}, \quad \hat{R}_k : x_k, k, \ldots$$

we obtain that if $\varphi_1(R') = 2$, then B1 holds. Similarly, $\varphi_1(R') = 1$ implies B2, and $\varphi_1(R') = m$ implies B3.

[8] The equality does not hold if $m = 2$, since then $m - 1 = 1$.

By Fact 1, $\varphi(\hat{R}) = x$.

Define R'_1 and R'_3 by

$$R'_1 : 2, 3, 1, \ldots$$
$$R'_3 : 1, 4, 3, \ldots$$

Note that $\varphi_1(R'_1, \hat{R}_{-1}) \neq 3$, since objects 1 and 2 are not acceptable for agents $k \notin \{1, 2\}$ at $\hat{R}_k$. And strategy-proofness implies $x_1 = 1\, \hat{R}_1\, \varphi_1(R'_1, \hat{R}_{-1})$, which implies $\varphi_1(R'_1, \hat{R}_{-1}) \neq 2$. Thus $\varphi_1(R'_1, \hat{R}_{-1}) = 1$ and non-bossiness then implies $\varphi(R'_1, \hat{R}_{-1}) = x$.

Let $x' = \varphi(R'_1, R'_3, \hat{R}_{N\setminus\{1,3\}})$. Strategy-proofness implies $x'_3\, R'_3\, x_3 = 4$, which implies $x'_3 \in \{1, 4\}$. We derive a contradiction in each of the cases.

Suppose $x'_3 = 1$. Then one can easily verify that $x'_1 = 3$, $x'_2 = 2$, and $x'_4 = 4$. By applying Fact 1 to agent 4 and then using (3), we obtain $x' = \varphi(R'_1, R'_3, R^0_4, \hat{R}_{N\setminus\{1,3,4\}}) = C(R'_1, R'_3, R^0_4, \hat{R}_{N\setminus\{1,3,4\}})$. But this is a contradiction since agents 1 and 2 block x', i.e., $2\, P'_1\, x'_1 = 3$ and $1\, \hat{P}_2\, x'_2 = 2$.

Suppose $x'_3 = 4$. Non-bossiness then implies $x' = x$. By applying Fact 1 to agent 2 and then using (3), we obtain $x = \varphi(R'_1, R^0_2, R'_3, \hat{R}_{N\setminus\{1,2,3\}}) = C(R'_1, R^0_2, R'_3, \hat{R}_{N\setminus\{1,2,3\}})$. But this is a contradiction since agents 1 and 3 block x, i.e., $3\, P'_1\, x_1 = 1$ and $1\, P'_3\, x_3 = 4$. $\square$

We are now ready to complete the proof of Theorem 1 (Case 2). Take any preference profile $R \in \mathcal{R}^N$, and let $x = C(R)$. We show that $\varphi(R) = x$. Define $\hat{R}$ by

$$\forall i \in N, \quad \hat{R}_i : x_i, i, \ldots \quad (\text{if } x_i = i, \text{ then } \hat{R}_i : i, \ldots)$$

Then $\varphi_i(\hat{R}) = x_i$ for all $i \in N$. This follows from Lemma 1 for $i \in N$ such that $x_i \neq i$. For $i \in N$ such that $x_i = i$, it follows from individual rationality.

For each $i \in N$, let r_i be the round in which agent i obtains an object in the top trading cycle algorithm when the preference profile is R (see Footnote 4 for a description of the algorithm). Without loss of generality, assume that $i < j$ implies $r_i \leq r_j$. For each $i \in \{1, \ldots, n+1\}$, define

$$R^i = (\hat{R}_1, \ldots, \hat{R}_{i-1}, R_i, \ldots, R_n).$$

We use an induction argument to prove that $\varphi(R^i) = x$ for all $i \in \{1, \ldots, n+1\}$. Note that $\varphi(R^{n+1}) = x$, since $R^{n+1} = \hat{R}$. So, suppose that $\varphi(R^{i+1}) = x$ for some $i \in \{1, \ldots, n\}$, and consider the preference profile $R^i = (R_i, R^{i+1}_{-i})$. Let $N_{r_i} = \{j \in N : r_j \geq r_i\}$. The following facts imply $\varphi_i(R^i) = x_i$:

1. Objects in N_{r_i} are not acceptable for agents in $N \setminus N_{r_i}$ at R^i. Thus $\varphi_i(R^i) \in N_{r_i}$;

2. The definition of the top trading cycle algorithm implies that x_i is the most preferred object in N_{r_i} for R_i (since agent i points to x_i in round r_i);

3. Strategy-proofness implies $\varphi_i(R^i)\ R_i\ x_i$.

Non-bossiness then implies $\varphi(R^i) = x$. This completes the induction argument, and it follows that $\varphi(R) = x$, since $R = R^1$. □

5 Independence of the Axioms

We verify that none of the axioms in Theorem 1 is redundant. We exhibit mechanisms that satisfy all but one of the axioms.

Example 1 (Bossy).

$$\varphi(R) = \begin{cases} e & \text{if for some } i \in N,\ R_i : i, \ldots \\ C(R) & \text{otherwise.} \end{cases}$$

That is, the mechanism selects the core allocation except when some agent's top choice is his endowment, in which case it selects the initial allocation. The mechanism is strategy-proof, since no agent wants to lie to induce the initial allocation, and if one's top choice is his endowment, truth-telling gives him his top choice. The mechanism is bossy, since there exists a preference profile R where the endowment is not the top choice for anyone, $C(R) \neq e$, and $C_i(R) = i$ for some agent i. Then agent i can change the selected allocation from $C(R)$ to e by reporting R_i^0.

Example 2 (Not anonymous). Let $y \in X$ be an allocation such that $y_i \neq i$ for all $i \in N$. Then let

$$\varphi^y(R) = \begin{cases} y & \text{if } y_i\ P_i\ i \text{ for all } i \in N; \\ e & \text{otherwise.} \end{cases}$$

That is, the mechanism φ^y selects the initial allocation except when allocation y is unanimously preferred to the initial allocation. This mechanism violates anonymity since allocation y depends on how we assign names to the agents. Non-bossiness is satisfied because of the condition $y_i \neq i$ for all $i \in N$.

Example 3 (Not strategy-proof). Let $b(R_i) \in N$ be the top choice of agent i. Then let

$$\varphi(R) = \begin{cases} (b(R_1), \ldots, b(R_n)) & \text{if } b(R_i) \neq i \text{ for all } i \in N, \\ & \text{and } b(R_i) \neq b(R_j) \text{ for all } i \neq j; \\ e & \text{otherwise.} \end{cases}$$

That is, the mechanism selects the initial allocation except when it is feasible to give each agent his most preferred object and no agent prefers his endowment. This mechanism is not strategy-proof; at some preference profile, an agent can avoid the initial allocation and gain by lying about his top choice. Non-bossiness is satisfied because of the condition $b(R_i) \neq i$ for all $i \in N$.

Example 4 (Not individually rational). Let $\varphi(R) = C(R^*)$ where each R_i^* is identical to R_i except that the endowment is the last choice. That is, if

$$R_i : k_1, k_2, \ldots, k_m, i, k_{m+1}, \ldots, k_{n-1},$$

then

$$R_i^* : k_1, k_2, \ldots, k_m, k_{m+1}, \ldots, k_{n-1}, i.$$

It is easy to see that non-bossiness of C implies that φ is non-bossy as well. To see why φ is strategy-proof, suppose that $C_i(R^*) = k_\ell \neq i$. The strategy-proofness of C implies that for all $\hat{R}_i \in \mathcal{R}$,

$$k_\ell \; R_i^* \; C_i(\hat{R}_i^*, R_{-i}^*) \; \hat{R}_i^* \; k_\ell.$$

The first part implies $C_i(\hat{R}_i^*, R_{-i}^*) \notin \{k_1, \ldots, k_{\ell-1}\}$. The second part together with $k_\ell \neq i$ implies $C_i(\hat{R}_i^*, R_{-i}^*) \neq i$, and hence $C_i(\hat{R}_i^*, R_{-i}^*) \in \{k_\ell, \ldots, k_{n-1}\}$, as desired.

6 Marriage Problems

Matching problems that are closely related to house allocation problems are marriage problems (Gale and Shapley, 1962). We can prove a similar result for marriage problems.

Theorem 2. *Suppose that there are at least two men and two women. Then a mechanism for marriage problems is individually rational, anonymous, strategy-proof, and non-bossy, if and only if it makes every agent remain single for each profile of preferences.*

Proof. See the Appendix. $\square$

Here, anonymity means that agents on the same side are treated symmetrically, but it allows mechanisms that treat the two sides asymmetrically. Thus anonymity is satisfied by the men-optimal stable matching mechanism.

Theorem 2 is intuitive in light of Theorem 1 and the fact that for marriage problems, no selection from the core is strategy-proof for both sides (Roth, 1982). Alcalde and Barberà (1994) prove a similar impossibility result, which states that there exists no mechanism that is Pareto efficient, individually rational, and strategy-proof. Their result does not imply, and is not implied by, Theorem 2. Theorem 2 confirms the intuition that no interesting strategy-proof mechanism exists for marriage problems, even if we are willing to accept inefficiency.

A Appendix: Marriage Problems

A.1 Preliminaries

We consider the model introduced by Gale and Shapley (1962). Let M and W be two non-empty, finite, disjoint sets. We call an element of M a man, and an element of W a woman. We allow $|M| \neq |W|$. Each man $m \in M$ has a (complete and transitive) preference relation R_m defined over $W \cup \{m\}$. The associated strict relation is denoted by P_m. As before, we assume that R_m is strict, i.e., for all $i \neq j$, either $i \, P_m \, j$ or $j \, P_m \, i$. Thus $i \, R_m \, j$ means that either $i \, P_m \, j$ or $i = j$. Similarly, each woman $w \in W$ has a strict (complete and transitive) preference relation R_w defined over $M \cup \{w\}$. Let $\mathcal{R}$ be the set of preference profiles of all agents $R = (R_i)_{i \in M \cup W}$. Given $i \in M \cup W$, let R_i^0 be a preference relation for agent i such that the top choice is i. As before, we represent preferences R_m by:

$$R_m : w_1, w_2, m, w_3, w_4, \ldots, w_{|W|}.$$

Since the precise ordering of women below himself is not important in the following analysis, we write:

$$R_m : w_1, w_2, m, \ldots$$

A **matching** is a function $\mu\colon M \cup W \to M \cup W$ satisfying the following:

1. For all $m \in M$, $\mu(m) \in W \cup \{m\}$.
2. For all $w \in W$, $\mu(w) \in M \cup \{w\}$.
3. For all $m \in M$ and all $w \in W$, $\mu(m) = w$ if and only if $\mu(w) = m$.

The first condition says that a man is matched to either a woman or himself. The second condition is the same condition for women. The last condition simply says that if man m is matched to woman w, then w should be matched to m.

Given $i \in M \cup W$, $\mu(i)$ is called the mate of agent i. When $\mu(i) = i$, we say that agent i remains single.

A **mechanism** is a function φ that associates with each preference profile $R \in \mathcal{R}$ a matching $\varphi(R)$. We denote by $\varphi_i(R)$ the agent matched to agent i.

We now can redefine the axioms introduced in the previous section for marriage problems. Strategy-proofness and non-bossiness are defined without modification. Individual rationality now states that no one should be made worse off than remaining single, i.e., for all $i \in M \cup W$, $\varphi_i(R)\; R_i\; i$.

The meaning of anonymity is the same as before. We just note that the permutations π should satisfy

$$\begin{aligned} \pi(m) \in M \quad & \forall m \in M; \\ \pi(w) \in W \quad & \forall w \in W. \end{aligned}$$

A permutation changes the agents' names, but not their genders. Thus we allow a mechanism that treats the two sides asymmetrically, such as the men-optimal stable matching mechanism (Gale and Shapley, 1962).

The following is the main result of this section.

Theorem 2. *Suppose $|M|, |W| \geq 2$. Then a mechanism is individually rational, anonymous, strategy-proof, and non-bossy, if and only if it makes everyone remain single for each profile of preferences.*

A.2 Proof of Theorem 2

Let μ^0 be the matching in which everyone remains single, and φ^0 be the mechanism that chooses μ^0 for each preference profile.

Let φ be a mechanism satisfying all of our axioms.

A.2.1 Step 1

We first show that $\varphi = \varphi^0$ when $|M| = |W| = 2$. Let $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$. For each m, let

$$R_m^{12} : w_1, w_2, m.$$

Define R_m^{21} and R_w^{ij} similarly. Given $m \in M$ and $i \in \{1, 2\}$, let

$$R_m^i : w_i, m, \ldots$$

Let

$$\mu = \varphi(R_{m_1}^{12}, R_{m_2}^{21}, R_{w_1}^{21}, R_{w_2}^{12}). \tag{11}$$

To complete Step 1, it suffices to prove $\mu = \mu^0$. Suppose, to the contrary, that $\mu \neq \mu^0$. We distinguish two cases.

Case 1: No one remains single at $\boldsymbol{\mu}$. Without loss of generality, assume

$$\mu = \begin{pmatrix} m_1 & m_2 \\ w_1 & w_2 \end{pmatrix}.$$

This means that m_1 is matched to w_1, and m_2 is matched to w_2. The other case can be proved symmetrically. By changing w_1's preferences to $R_{w_1}^1$, we obtain that $\varphi(R_{m_1}^{12}, R_{m_2}^{21}, R_{w_1}^{1}, R_{w_2}^{12}) = \mu$. Now, permute the women's names. Anonymity implies

$$\varphi(R_{m_1}^{21}, R_{m_2}^{12}, R_{w_1}^{12}, R_{w_2}^{1}) = \begin{pmatrix} m_1 & m_2 \\ w_2 & w_1 \end{pmatrix} \equiv \mu'. \tag{12}$$

Changing w_1's preferences to $R_{w_1}^{21}$, we obtain

$$\varphi(R_{m_1}^{21}, R_{m_2}^{12}, R_{w_1}^{21}, R_{w_2}^{1}) = \mu'. \tag{13}$$

Applying strategy-proofness to (11), we obtain $\varphi(R_{m_1}^{12}, R_{m_2}^{21}, R_{w_1}^{21}, R_{w_2}^{1}) = \hat{\mu}$ where $\hat{\mu}(w_2) = w_2$. We distinguish three cases.

Subcase 1: $\hat{\mu} = \mu^0$. Changing the preferences of m_1 and m_2 to $R_{m_1}^{21}$ and $R_{m_2}^{12}$ respectively, we obtain $\varphi(R_{m_1}^{21}, R_{m_2}^{12}, R_{w_1}^{21}, R_{w_2}^{1}) = \mu^0$, in contradiction with (13).

Subcase 2: $\hat{\mu} = \bar{\mu}$ where

$$\bar{\mu} \equiv \begin{pmatrix} m_1 & m_2 & w_2 \\ w_1 & m_2 & w_2 \end{pmatrix}.$$

Changing the preferences of m_2 and w_2 to $R_{m_2}^{12}$ and $R_{w_2}^{0}$ respectively, we obtain

$$\varphi(R_{m_1}^{12}, R_{m_2}^{12}, R_{w_1}^{21}, R_{w_2}^{0}) = \bar{\mu}. \tag{14}$$

Permuting the men's names, we obtain

$$\varphi(R^{12}_{m_1}, R^{12}_{m_2}, R^{12}_{w_1}, R^{0}_{w_2}) = \begin{pmatrix} m_1 & m_2 & w_2 \\ m_1 & w_1 & w_2 \end{pmatrix} \equiv \tilde{\mu}.$$

But by (14), w_1 can gain by reporting $R^{12}_{w_1}$ at profile $(R^{12}_{m_1}, R^{12}_{m_2}, R^{21}_{w_1}, R^{0}_{w_2})$, a contradiction.

Subcase 3: $\hat{\mu} = \tilde{\mu}$. By changing m_1's preferences to $R^{21}_{m_1}$, we obtain that

$$\varphi(R^{21}_{m_1}, R^{21}_{m_2}, R^{21}_{w_1}, R^{1}_{w_2}) = \tilde{\mu}.$$

But then by (13), if m_2 reports $R^{12}_{m_2}$ instead, he will be matched to the same mate while making m_1 and w_2 strictly better off. This is in contradiction with non-bossiness.

Case 2: A man and a woman remain single at μ. Without loss of generality, assume $\mu = \bar{\mu}$, where m_1 and w_1 are matched. Changing w_1's preferences to $R^{12}_{w_1}$, we obtain $\varphi(R^{12}_{m_1}, R^{21}_{m_2}, R^{12}_{w_1}, R^{12}_{w_2}) = \bar{\mu}$. Permuting the names for both sides, we obtain

$$\varphi(R^{12}_{m_1}, R^{21}_{m_2}, R^{21}_{w_1}, R^{21}_{w_2}) = \begin{pmatrix} m_2 & m_1 & w_1 \\ w_2 & m_1 & w_1 \end{pmatrix}.$$

Changing w_2's preferences to $R^{12}_{w_2}$, we obtain that $\varphi_{w_2}(R^{12}_{m_1}, R^{21}_{m_2}, R^{21}_{w_1}, R^{12}_{w_2}) \neq w_2$, in contradiction with $\mu(w_2) = \bar{\mu}(w_2) = w_2$.

A.2.2 Step 2

We complete the proof by induction. Let $k \in \{3, 4, \ldots\}$ and suppose that $\varphi = \varphi^0$ if $\min\{|M|, |W|\} \leq k - 1$. And suppose $\min\{|M|, |W|\} = k$. Without loss of generality, assume $|M| = k$, and let $M = \{m_1, m_2, \ldots, m_k\}$ and $W = \{w_1, w_2, \ldots, w_l\}$ with $l \geq k$. Let $R \in \mathcal{R}$ and $\mu = \varphi(R)$, and we prove $\mu = \mu^0$. We distinguish two cases.

Case 1: There exists a man m such that $\mu(m) = m$. Then $S \equiv \{i \in M \cup W : \mu(i) = i\}$ is non-empty. Applying strategy-proofness and non-bossiness repeatedly, we obtain $\varphi(R^0_S, R_{-S}) = \mu$. Since $\min\{|M \setminus S|, |W \setminus S|\} \leq k - 1$, the induction hypothesis implies $\varphi(R^0_S, R_{-S}) = \mu^0$, and hence $\mu = \mu^0$.

Case 2: For all men m, $\mu(m) \neq m$. We derive a contradiction. Without loss of generality, assume that for all $i \in \{1, \ldots, k\}$, $\mu(m_i) = w_i$. Let $W' = \{w_1, \ldots, w_k\}$. Given

$i \in M \cup W'$, let

$$R'_i : \mu(i), i, \ldots$$

For all $w \in W \setminus W'$, let $R'_w = R^0_w$. Applying strategy-proofness and non-bossiness repeatedly, we obtain $\varphi(R') = \mu$.

Let

$$\hat{R}_{m_1} : w_2, w_1, m_1, \ldots$$

Let $\hat{\mu} = \varphi(\hat{R}_{m_1}, R'_{-m_1})$. Strategy-proofness implies $\hat{\mu}(m_1) \neq m_1$. Since m_1 is not acceptable for w_2, $\hat{\mu}(m_1) = w_1$. Then by non-bossiness, $\hat{\mu} = \mu$.

For each man $m \neq m_1$, let

$$\hat{R}_m : \mu(m), \underbrace{\ldots}_{W\setminus\{\mu(m)\}}, m.$$

That is, m's top choice is $\mu(m)$ and the last choice is to remain single. Similarly, for each $w \in W'$, let

$$\hat{R}_w : \mu(w), \underbrace{\ldots}_{M\setminus\{\mu(w)\}}, w.$$

For each $w \in W \setminus W'$, let $\hat{R}_w = R^0_w$. Then by strategy-proofness and non-bossiness,

$$\varphi(\hat{R}) = \mu; \tag{15}$$

$$\varphi(R'_{m_1}, \hat{R}_{-m_1}) = \mu. \tag{16}$$

Let

$$R''_{m_1} : w_2, m_1, \ldots$$

Let $\mu'' = \varphi(R''_{m_1}, \hat{R}_{-m_1})$. By (15), $\mu''(m_1) = m_1$. Then by Case 1, $\mu'' = \mu^0$. Let $\hat{R}'$ be the preference profile obtained from $\hat{R}$ by permuting the names of w_1 and w_2. Note that R'_{m_1} is obtained from R''_{m_1} by the same permutation. Thus anonymity implies $\varphi(R'_{m_1}, \hat{R}'_{-m_1}) = \mu^0$. Note that at this profile, for each $i \neq m_1$, to remain single is the worst choice. Thus changing the preferences of each $i \neq m_1$ to $\hat{R}_i$, we obtain $\varphi(R'_{m_1}, \hat{R}_{-m_1}) = \mu^0$, in contradiction with (16). □

A.3 Independence of the Axioms

We verify that none of the axioms in Theorem 2 is redundant. We exhibit mechanisms that satisfy all but one of the axioms.

Example 5 (Bossy). Consider the mechanism that selects μ^0 for all profiles R except in the following case: there exist a man m and a woman w such that (i) m and w are acceptable for each other ($w \; P_m \; m$ and $m \; P_w \; w$), and (ii) for all the other agents i, the top choice is to remain single ($i \; P_i \; k$ for all $k \neq i$). If such a pair (m, w) exists, the mechanism assigns the matching where m and w are matched, and all the others remain single. This mechanism violates non-bossiness, since when (i) and (ii) hold for some pair (m, w), another agent can change the matching to μ^0 by reporting that to remain single is not the top choice.

Example 6 (Not individually rational). Consider the mechanism where $\varphi(R) = \mu^0$ for all R except in the following case: there exists exactly one man m whose top choice is not to remain single. In this case, $\varphi(R)$ is the matching where m is matched to his most preferred woman, and the others remain single. This mechanism violates individual rationality, since it ignores the women's preferences.

Example 7 (Not anonymous). Let us denote $M = \{m_1, m_2, \ldots, m_{|M|}\}$ and $W = \{w_1, \ldots, w_{|W|}\}$, and let $n = \min\{|M|, |W|\}$. Then for each $i \in \{1, 2, \ldots, n\}$, m_i and w_i are matched if they are acceptable for each other ($m_i \; P_{w_i} \; w_i$ and $w_i \; P_{m_i} \; m_i$), and otherwise they both remain single. Thus for $i > n$, man m_i (or woman w_i), if exists, remains single independently of R.

Example 8 (Not strategy-proof). If there exists a matching μ such that for each agent i, $\mu(i)$ is the top choice for i, and $\mu(i) \neq i$, then $\mu = \varphi(R)$. Otherwise, $\varphi(R) = \mu^0$. This mechanism is not strategy-proof; at some profile, an agent can gain by lying about his top choice.

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