House Allocation with Transfers

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May 27, 1999
Final version: October 10, 2000
To appear in *Journal of Economic Theory*

Abstract

We consider the Shapley-Scarf house allocation problem where monetary transfers are allowed. We characterize the class of mechanisms that are strategy-proof, ex post individually rational, ex post budget balanced, and "collusion-proof." In these mechanisms, the price of each object is fixed in advance, and the objects are reallocated according to the (unique) core assignment of the Shapley-Scarf economy associated with the prices. The special case in which all prices are zero is the core mechanism studied by Shapley and Scarf. Our mechanisms are compelling alternatives to the Groves mechanisms, which satisfy neither budget balance nor our condition of collusion-proofness. *Journal of Economic Literature* Classification Numbers: C71, C78, D71, D78, D89.

Key Words: Housing markets; indivisible goods; strategy-proofness; non-bossiness; fixed prices; side-payments; axiomatic analysis.

*The paper was previously entitled "Mechanisms for Multilateral Trading and Fixed Prices". I would like to thank an anonymous associate editor and an anonymous referee for helpful comments and suggestions. I also thank Dilip Abreu, Jean-Pierre Benoit, Prajit Dutta, Chiaki Hara, Tetsuya Kishimoto, Bettina Klaus, Shinji Ohseto, Debraj Ray, James Schummer, Shigehiro Serizawa, Rajiv Sethi, Paolo Siconolfi, and Koichi Tadenuma for helpful comments. Of course, I am solely responsible for the remaining deficiencies of the paper.

1 Introduction

We consider a house allocation problem of Shapley and Scarf [32] where monetary transfers are allowed. There are a group of agents, each of whom initially owns an indivisible object (e.g., a house). The problem is to reallocate the objects in such a way that everyone receives one object. A real-life application of the problem is the reallocation of university apartments. We characterize the class of mechanisms (social choice functions) that are strategy-proof, ex post individually rational, ex post budget balanced, and "collusion-proof". The mechanisms are intuitive, "detail-free", and straightforward to implement in practice.

The economy we study is the Shapley-Scarf economy plus "money". There is a perfectly divisible private good, and preferences are quasi-linear with respect to this good.[1] Competitive equilibrium, core, and no-envy have been studied in similar economies (e.g., [2, 7, 24, 34, 36]). The mechanisms we propose differ from these solution concepts.

The class of mechanisms we introduce is indexed by a vector $p = (p_1, \dots, p_n)$, where p_i works as the price of the object initially owned by agent i. The vector determines the transfer scheme in a natural way: each agent pays the price of the object he receives to the initial owner of the object. The transfer scheme in effect reduces our economy to a Shapley-Scarf economy without money, where an agent's valuation for an object is equal to his true valuation minus the price of the object. The induced Shapley-Scarf economy has a unique core assignment for generic type profiles, and this is what we select. We call this mechanism the *fixed-price core mechanism* associated with the price vector p. This is a straightforward extension of Shapley and Scarf's core mechanism.

Why do we have to care about the fixed-price core mechanisms? First, they satisfy a number of desirable properties. They satisfy individual rationality and strategy-proofness as Shapley and Scarf's core mechanism does. The mechanisms are also budget balanced, since no one pays money to the planner. Budget imbalance is a major drawback of the Groves mechanisms, and in many cases, budget balance is important for the planner as well as for the agents. The mechanisms are also "collusion-proof" to some degree. Specifically, the mechanisms satisfy what is called *non-bossiness*, introduced by Satterthwaite and Sonnenschein [30]. Non-bossiness says that no one can change the allocation for others without changing the allocation for himself. Non-bossiness is violated, for example, by the Vickrey auction, since the second highest bidder can change the price of the object with no cost to himself. Indeed, collusion is observed often in auctions. Non-bossiness alone would not eliminate collusion completely, but non-bossiness is

[1] Our setting differs from auction settings, where buyers initially do not own the objects. In our setting, everyone is a buyer as well as a seller. Our setting differs from the one in Myerson and Satterthwaite [20], who consider only bilateral trades. Our setting differs also from the one in Cramton et al. [6], since there are more than one object in our economy.

still an appealing property.

The fact that the fixed-price core mechanisms satisfy these desirable properties is straightforward to prove. Our main contribution is to show that no other mechanism does. This axiomatic characterization provides a compelling theoretical foundation for the fixed-price core mechanisms. The axiomatization is similar to Ma's [16], but Ma does not allow monetary transfers. What we show is that immunity to strategic manipulation implies that we have to transfer money according to fixed prices.

More precisely, we proved that if a mechanism is strategy-proof, individually rational, nonbossy, and onto (but not necessarily budget balanced), then it is identical to a mechanism described above, except that the transfer scheme is more general and based on a price *matrix* $P = (p_{ij})$. The element p_{ij} is the amount that agent i has to pay whenever he receives the endowment of agent j. Individual rationality requires non-positive diagonal elements, but this is the only condition that must be satisfied by P. The mechanism based on P satisfies budget balance if and only if $p_{ij} = p_j - p_i$ for some vector $p = (p_1, \ldots, p_n)$.

An option for the planner is to set all prices equal to zero, in which case the fixed-price core mechanism is equivalent to Shapley and Scarf's core mechanism. But the planner may want to set non-zero prices to accommodate, for example, the asymmetry of the participants due to seniority and hierarchy (e.g., for university apartments, asymmetry exists between students and faculty as well as between junior faculty and senior faculty).

Our result is similar to that of Barberà and Jackson [3], who consider classical exchange economies. They consider a similar set of axioms[2], and show that it implies what they call fixed-price mechanisms. Prices are fixed in advance in their mechanisms as well (although some flexibility is allowed). Our contribution is, perhaps, to confirm the intuition that we have to fix prices if we want to avoid strategic manipulation.

A closely related work is Schummer [31]. He characterizes the strategy-proof and budget balanced mechanisms in our setting for the *two-person* case, but his characterization for the n-person case is partial (one of his results is stated below as Lemma 1). We use his result and provide a complete characterization for the n-person case. We should mention that his results apply also to mechanisms that are not individually rational, since in his setup no one initially owns an object.[3] Schummer also proves that no strategy-proof mechanism is Pareto efficient in our setup.[4]

[2] Barberà and Jackson use anonymity in addition, while we do not.

[3] For the two-person case, Schummer shows that a mechanism is strategy-proof and budget balanced if and only if it is constant, dictatorial, or "status-quo-preserving." A fixed-price core mechanism is a status-quo-preserving mechanism where the "status-quo" is the initial allocation.

[4] Hagerty and Rogerson [11] consider the two-person case, allowing mechanisms to be stochastic. They show that all the planner can do is to announce a price randomly before agents reveal their types. It is not straightfor-

One of the most important mechanisms in quasi-linear environments is the Vickrey-Clarke-Groves (VCG) mechanism [5, 10, 37]. The VCG mechanism reallocates the objects to maximize the sum of valuations, but it does not balance the budget. We believe that there are situations where budget balance is more important for the planner than selecting an assignment that maximizes the sum of valuations. In such a case, the fixed-price core mechanisms are compelling options for the planner.

The outcomes of the fixed-price core mechanisms are not necessarily Pareto efficient. How inefficient are they? We are currently investigating this question by numerically computing the total expected surplus of the fixed-price core mechanisms and the VCG mechanism (Miyagawa [18]). Our simulation results suggest that the fixed-price core mechanisms are on average more efficient than the VCG mechanism when the correlation among the valuations is not large. In this case, the welfare loss of a fixed-price core mechanism due to price rigidity is on average smaller than the size of budget imbalance of the VCG mechanism.

We conclude this section by emphasizing that our mechanisms satisfy incentive compatibility in *dominant strategies*, which is considerably stronger than the Bayesian counterpart. The Bayesian approach relies on a number of strong assumptions: the common knowledge of priors, common knowledge of rationality, and expected utility hypothesis. While these assumptions are standard in economic theory, they may not be "good" assumptions in some applications. This suggests that some Bayesian mechanisms may not be "robust" in that whether they produce the desired outcome is sensitive to the environment [11]. The dominant-strategy approach does not depend on any of the assumptions.[5]

2 Model

There are $n \geq 2$ agents, and the set of agents is denoted by $N = \{1, 2, \ldots, n\}$. There are n objects, and each agent $i \in N$ initially owns one object. The object initially owned by agent i is called "object i", and thus the set of objects is also denoted by N. There is a divisible private good, called "money", and each agent's preferences are quasi-linear with respect to this good. Agent i's utility level when he consumes object $k \in N$ and pays $t_i \in \mathbb{R}$ units of money is

$$\theta_i(k) - t_i.$$

ward to extend their analysis to the n-person case. Ohseto [21, 22] studies the n-person case, but focuses on the case when there is only one indivisible object.

[5] Of course, it would be interesting to characterize Bayesian mechanisms in our setup. It poses a technical difficulty, however, since our type space is multi-dimensional [15, 25].

Note that t_i may be negative, in which case agent i receives $-t_i$ units of money. The number $\theta_i(k) \in \mathbb{R}$ is agent i's valuation for object k, and this may be negative as well. Agent i's *type* is a vector $\theta_i = (\theta_i(k))_{k \in N}$. Without loss of generality, we set $\theta_i(i) = 0$. The set of types for agent i is $\Theta_i = \{\theta_i = (\theta_i(k))_{k \in N} \in \mathbb{R}^N : \theta_i(i) = 0\}$.

Since the mechanisms that we will propose are incentive compatible in dominant strategies, it is immaterial how much each agent knows about other agents' types. It suffices that each agent knows his own type. Note that while it is standard to assume that each agent knows his own type, it is a demanding requirement in practice, particularly when the number of objects is large and the objects are considerably heterogeneous in a multi-dimensional way.[6] But our mechanisms do not require that each agent's knowledge about his own type be complete. This point will be discussed in Section 7.

We denote a generic type profile by $\theta = (\theta_i)_{i \in N}$. The set of type profiles is denoted by $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$. We often denote $N \setminus \{i\}$ by "$-i$," and $N \setminus \{i, j\}$ by "$-i, j$." With this notation, (θ'_i, θ_{-i}) is the type profile where agent i's type is θ'_i, and the type of agent $j \neq i$ is θ_j. We define $(\theta'_i, \theta'_j, \theta_{-i,j})$ similarly.

Our collective choice problem is to reallocate objects N and transfer money among agents. An *assignment* is a list $x = (x_i)_{i \in N}$ such that

$$x_i \in N \text{ for each } i \in N;$$
$$i \neq j \text{ implies } x_i \neq x_j.$$

Here, x_i is the object allocated to agent i, and $x_i = j$ means that agent i receives the object initially owned by agent j. The second condition above simply states that no two agents receive the same object. We denote the set of assignments by X.

An *allocation* is a list $(x, t) \in X \times \mathbb{R}^N$ such that x is an assignment and for each $i \in N$, $t_i \in \mathbb{R}$.

A (direct) *mechanism* is a function $\varphi(\cdot) = (x(\cdot), t(\cdot)) \colon \Theta \to X \times \mathbb{R}^N$ such that for each type profile $\theta \in \Theta$, $\varphi(\theta) = (x(\theta), t(\theta))$ is an allocation. At this point, we impose no further restriction on the function $\varphi = (x, t)$. Let $\varphi_i(\theta) = (x_i(\theta), t_i(\theta))$.

3 Axioms

This section introduces six properties that mechanisms may satisfy.

A mechanism (x, t) is *strategy-proof* if in its associated revelation game, truthful revelation

[6] For example, Columbia University has approximately 5,700 apartments available for faculty, staff, and students, and it does not seem possible to classify the apartments into a small number of types.

is a dominant strategy for each agent; that is, for all $\theta \in \Theta$, all $i \in N$, and all $\theta'_i \in \Theta_i$, $\theta_i(x_i(\theta)) - t_i(\theta) \geq \theta_i(x_i(\theta'_i, \theta_{-i})) - t_i(\theta'_i, \theta_{-i})$.

A mechanism (x, t) is (ex post) *individually rational* if for all $\theta \in \Theta$ and all $i \in N$, $\theta_i(x_i(\theta)) - t_i(\theta) \geq 0 \equiv \theta_i(i)$.

A mechanism (x, t) is (ex post) *budget balanced* if for all $\theta \in \Theta$, $\sum_{i \in N} t_i(\theta) = 0$.

A mechanism $\varphi = (x, t)$ is *non-bossy* (Satterthwaite and Sonnenschein [30]) if for all $\theta \in \Theta$, all $i \in N$, and all $\theta'_i \in \Theta_i$,

$$\varphi_i(\theta'_i, \theta_{-i}) = \varphi_i(\theta) \Rightarrow \varphi(\theta'_i, \theta_{-i}) = \varphi(\theta).$$

In words, a mechanism is non-bossy if no agent can change the allocation for others without changing the allocation for himself. Non-bossiness is desirable; a violation of it implies a possibility of collusion where an agent lies about his type in exchange for a transfer from those who benefit from his lie. A well-known bossy mechanism is the Vickrey auction, where the second highest bidder can change the price for the winner while remaining a loser. Indeed, collusion is often observed in auctions [4].

A mechanism (x, t) is *decision-efficient* if for all $\theta \in \Theta$,

$$x(\theta) \in \arg\max_{x \in X} \sum_{i \in N} \theta_i(x_i).$$

Finally, a mechanism (x, t) is *onto* if the function $x \colon \Theta \to X$ is onto.[7] This is a minimal condition of flexibility, and it is satisfied, for example, if the mechanism is decision-efficient.

The mechanisms we propose do not satisfy decision-efficiency, and hence the outcomes of our mechanisms may be Pareto inefficient. Since we assume quasi-linear preferences, Pareto efficiency is equivalent to decision-efficiency plus budget balance. A few remarks follow.

First, Pareto efficiency and strategy-proofness are incompatible in our environment [31] as well as in generic quasi-linear environments [9, 12, 39]. The Groves mechanisms are decision-efficient, but not budget balanced.

A traditional approach is to characterize the strategy-proof mechanisms that are decision-efficient, and see if any of them is budget balanced. However, since Pareto efficiency is not attainable, it is not clear whether we should give priority to decision-efficiency [14]. There are many situations where budget balance is more important than choosing an assignment that maximizes the sum of valuations. Some of our mechanisms balance the budget, and thus they are appealing options for the planner when budget balance is important.

[7] Ontoness is also called "citizen sovereignty" or "no-imposition" in the literature.

4 Results

4.1 Fixed-Price Core Mechanisms

We introduce a family of mechanisms that we propose. These mechanisms, which we call the *fixed-price core mechanisms*, are parameterized by a $n \times n$ "price matrix" $P = (p_{ij})$. The entry p_{ij} is the amount that agent i pays whenever he receives object j. The price matrix P has to satisfy

$$p_{ii} \leq 0 \quad \forall i \in N. \tag{1}$$

That is, an agent should not pay a positive amount if he receives his endowment. This ensures individual rationality, as we will see. Condition (1) alone does not ensure budget balance. For budget to be balanced, the price matrix has to satisfy additional conditions, which will be discussed in Section 4.2.

To complete the description of our mechanisms, it remains to specify $x(\theta)$ for a given P that satisfies (1). Note first that P determines the ordinal ranking of the objects for each agent. The induced ordinal ranking over N for agent i is denoted by $R_i(\theta_i; P)$ and defined by

$$k \; R_i(\theta_i; P) \; j \iff \theta_i(k) - p_{ik} \geq \theta_i(j) - p_{ij}. \tag{2}$$

Then $(R_1(\theta_1; P), \ldots, R_n(\theta_n; P))$ is what is called a *Shapley-Scarf economy* [32]. An assignment $x \in X$ is in the *core* of the Shapley-Scarf economy $(R_1, \ldots, R_n)$ if there do not exist a coalition $S \subseteq N$ and an assignment $y \in X$ such that $\cup_{i \in S}\{y_i\} = S$, and for all $i \in S$, $y_i \; R_i \; x_i$, with strict preference holding for some $i \in S$. The core of Shapley-Scarf economy $R = (R_1, \ldots, R_n)$ is denoted by $C(R)$. Roth and Postlewaite [26] prove that $C(R)$ is a singleton when for all $i \in N$, the preference relation R_i is strict over the set $\{k \in N : k \; R_i \; i\}$ of *acceptable* objects. When everyone's preferences are strict over acceptable objects in the induced Shapley-Scarf economy, the economy has a unique core assignment, and this is what we select. The unique core assignment can be easily computed with the top trading cycle algorithm due to David Gale (see Shapley and Scarf [32]).[8]

The ordering $R_i(\theta_i; P)$ may contain indifference among acceptable objects. An easy way to handle indifference is to break ties with a fixed rule. Formally, a *tie-breaking rule* is a collection of strict orderings $\succ = (\succ_i)_{i \in N}$ defined over N. Given a tie-breaking rule $\succ$, we define an ordering

[8] Other papers studying the Shapley-Scarf economy or its variants without transfers include [1, 13, 16, 19, 23, 33, 38, 40].

$R_i(\theta_i; P, \succ_i)$ by

$$k \; P_i(\theta_i; P, \succ_i) \; j \iff \begin{cases} \text{either} & \theta_i(k) - p_{ik} > \theta_i(j) - p_{ij} \\ \text{or} & [\theta_i(k) - p_{ik} = \theta_i(j) - p_{ij} \text{ and } k \succ_i j], \end{cases} \tag{3}$$

where $P_i(\theta_i; P, \succ_i)$ is the strict preference relation associated with $R_i(\theta_i; P, \succ_i)$. That is, when agent i is indifferent between two objects, we break the tie in favor of the object that is ranked higher by $\succ_i$. The tie-breaking rule eliminates indifference and the resulting Shapley-Scarf economy has a unique core assignment.

Definition 1. Given a $n \times n$ matrix P satisfying (1) and a tie-breaking rule $\succ$, the *fixed-price core mechanism based on P and $\succ$* is the mechanism $\varphi = (x, t)$ defined by:

1. $x(\theta) = C(R_1(\theta_1; P, \succ_1), \ldots, R_n(\theta_n; P, \succ_n))$;

2. For all $i \in N$, $t_i(\theta) = p_{ij}$ where $j = x_i(\theta)$.

The mechanism is denoted by $\varphi^{P,\succ}$.

The mechanism $\varphi^{P,\succ}$ is strategy-proof simply because prices are fixed in advance and the Shapley-Scarf core mechanism is strategy-proof, even with a tie-breaking rule (Roth [27]). $\varphi^{P,\succ}$ is individually rational simply because the Shapley-Scarf core mechanism is individually rational, and P satisfies (1). It is easy to verify that $\varphi^{P,\succ}$ is also onto and non-bossy.

The fact that the fixed-price core mechanisms are strategy-proof, individually rational, non-bossy, and onto is basic. Our main contribution is to show that no other mechanism satisfies these axioms simultaneously.

Specifically, we show that if a mechanism is strategy-proof, individually rational, non-bossy, and onto, then the mechanism coincides with a fixed-price core mechanism on the set of type profiles where no one is indifferent between acceptable objects. This result does not tell us the outcome of the mechanism when there is indifference. But the issue of indifference is somewhat minor, since indifference occurs rarely when types are drawn from atomless distributions.

Definition 2. Given a $n \times n$ matrix P satisfying (1), a *fixed-price core mechanism based on* $\boldsymbol{P}$ is a mechanism (x, t) such that for all $\theta \in \Theta$:

1. If the preference relation $R_i(\theta_i; P)$, defined by (2), is strict over acceptable objects for all $i \in N$, then $x(\theta) = C(R_1(\theta_1; P), \ldots, R_n(\theta_n; P))$;

2. For all $i \in N$, $t_i(\theta) = p_{ij}$ where $j = x_i(\theta)$.

A mechanism is a *fixed-price core mechanism* if it is a fixed-price core mechanism based on some $n \times n$ matrix P that satisfies (1).

The following is our main result.

Theorem 1. *If a mechanism is strategy-proof, individually rational, non-bossy, and onto, then it is a fixed-price core mechanism.*

Proof. See Section 5 and Appendix. □

4.2 Budget Balance

For a generic price matrix satisfying (1), the associated fixed-price core mechanism does not satisfies budget balance. Budget balance holds, however, for certain price matrices.

Proposition 1. *A fixed-price core mechanism based on P is budget balanced if and only if there exists a vector $\rho = (\rho_1, \ldots, \rho_n) \in \mathbb{R}^n$ such that*

$$p_{ij} = -\rho_i + \rho_j \quad \forall i, j \in N. \tag{4}$$

Proof. (Only if) Suppose that a fixed-price core mechanism based on some P is budget balanced. Let $\rho = (p_{11}, p_{12}, \ldots, p_{1n})$, and we show that (4) holds. By budget balance and individual rationality, $p_{ii} = 0$ for all $i \in N$. Thus $\rho_1 = 0$, and (4) holds trivially for $i = 1$. (4) holds also when $j = 1$, since budget balance holds when i and 1 exchange their endowments and the other agents keep their endowments, and hence

$$p_{i1} = -p_{1i} = -\rho_i = -\rho_i + \rho_1.$$

Finally, to see that (4) holds also when $i \neq 1$ and $j \neq 1$, note that budget balance has to hold when agents $\{i, j, 1\}$ form a trading cycle, and the other agents keep their endowment, and hence

$$p_{ij} = -p_{j1} - p_{1i} = \rho_j - \rho_i.$$

(If) If P satisfies (4) for some vector ρ, then budget balance holds, since budget is balanced within *any* trading cycle. Indeed, for any set $\{i_1, i_2, \ldots, i_m\} \subseteq N$,

$$\begin{aligned}
&p_{i_1 i_2} + p_{i_2 i_3} + \cdots + p_{i_m i_1} \\
&= (-\rho_{i_1} + \rho_{i_2}) + (-\rho_{i_2} + \rho_{i_3}) + \cdots + (-\rho_{i_m} + \rho_{i_1}) = 0.
\end{aligned}$$

□

The number ρ_i is naturally interpreted as the price of object i. Then this proposition says that in a fixed-price core mechanism that balances the budget, there is a unique price for each object, and each agent pays the price of the object he obtains *to* the initial owner of the object. This transfer scheme is nothing but the one used most frequently in markets. The difference is that in fixed-price core mechanisms, prices are determined exogenously. Note that since one of the prices is redundant, all that the planner has to set is $n-1$ numbers. Proposition 1 together with Theorem 1 implies

Corollary 1. *If a mechanism is strategy-proof, individually rational, non-bossy, onto, and budget balanced, then it is a fixed-price core mechanism based on a price matrix P that satisfies* (4) *for some vector $\rho \in \mathbb{R}^n$.*

4.3 Indifference

As mentioned above, an easy way to handle indifference is to use a tie-breaking rule, but it is not the only option for the planner. Can we say anything general about $x(\theta)$ when some agents have indifference? First, we show that $x(\theta)$ need not be in the core of the associated Shapley-Scarf economy. To see this, consider a three-agent case with $N = \{1, 2, 3\}$. Suppose that we use a fixed-price core mechanism with some P. Consider a type profile θ such that

$$R_1(\theta_1; P) = [2, 3], 1;$$
$$R_2(\theta_2; P) = R_3(\theta_3; P) = 1, 2, 3.$$

That is, agent 1 is indifferent between 2 and 3, but he prefers each of them to his endowment. Agents 2 and 3 prefer 1 to 2, and 2 to 3. Suppose we use a tie-breaking rule such that $2 \succ_1 3$. Since agent 1's tie is broken in favor of 2, the mechanism selects assignment $x = (2, 1, 3)$. But this is not in the core, since it is Pareto dominated by assignment $(3, 1, 2)$. It is in the core *after* we break ties, but not in terms of the actual preferences.

We have shown that when there is indifference, $x(\theta)$ need not be in the core of the associated Shapley-Scarf economy. But it turns out that $x(\theta)$ has to be in the *weak* core. An assignment x is in the *weak core* of Shapley-Scarf economy R if there exists no coalition $S \subseteq N$ and no assignment $y \in X$ such that $\cup_{i \in S}\{y_i\} = S$ and for all $i \in S$, $y_i \, P_i \, x_i$.

Proposition 2. *Let (x, t) be a mechanism that is strategy-proof, individually rational, non-bossy, and onto, and let P be the associated price matrix. Then for any type profile $\theta \in \Theta$, $x(\theta)$ is in the weak core of the Shapley-Scarf economy $(R_1(\theta_1; P), \ldots, R_n(\theta_n; P))$.*

Proof. Suppose, by contradiction, that some coalition S can block $x(\theta)$ by means of an assignment x'; i.e., $\cup_{i \in S}\{x'_i\} = S$, and $x'_i \, P_i(\theta_i; P) \, x_i(\theta)$ for all $i \in S$. We construct a type profile

θ' such that

$$\forall i \in S : R_i(\theta_i'; P) = x_i', x_i(\theta), i, \ldots \quad (\text{or } x_i', x_i(\theta) = i, \ldots)$$
$$\forall i \notin S : R_i(\theta_i'; P) = x_i(\theta), i, \ldots \quad (\text{or } x_i(\theta) = i, \ldots)$$

That is, for each $i \notin S$, $R_i(\theta_i'; P)$ is such that $x_i(\theta)$ is the unique top object, and if $x_i(\theta) \neq i$, then i is the unique second top object. For each $i \in S$, $R_i(\theta_i'; P)$ is such that x_i' is the unique top object, $x_i(\theta)$ is the unique second top object, and if $x_i(\theta) \neq i$, then i is the unique third top object.

Then, assignment $x(\theta)$ is not in the (weak) core of this Shapley-Scarf economy, since it is blocked by coalition S. Since each agent's preferences are strict over acceptable objects, Theorem 3 implies that $x(\theta')$ is the (unique) core assignment of this economy. Thus we can derive a desired contradiction by showing $x(\theta') = x(\theta)$. To show this, consider type profile (θ_i', θ_{-i}) for some agent i. Assume $i \in S$ for a moment. By individual rationality, $x_i(\theta_i', \theta_{-i}) \in \{x_i', x_i(\theta), i\}$. But $x_i(\theta_i', \theta_{-i}) \neq x_i'$; otherwise, when agent i is of type θ_i, he gains by reporting θ_i'. Furthermore, if $x_i(\theta) \neq i$, then $x_i(\theta_i', \theta_{-i}) \neq i$; otherwise, when agent i is of type θ_i', he gains by reporting θ_i obtaining $x_i(\theta)$. Thus we obtain $x_i(\theta_i', \theta_{-i}) = x_i(\theta)$. Non-bossiness then implies $x(\theta_i, \theta_{-i}') = x(\theta)$. The argument is the same (even simpler) when $i \notin S$. Repeating this argument for the other agents sequentially, we obtain $x(\theta') = x(\theta)$. □

By this proposition, we can refine the definition of fixed-price core mechanisms (Definition 2) by adding the third condition: $x(\theta)$ is in the weak core of the Shapley-Scarf economy $(R_1(\theta_1; P), \ldots, R_n(\theta_n; P))$.

5 Proof of Theorem 1

We give a proof of Theorem 1. Our starting point is the following result.

Lemma 1 (Schummer [31]). *If a mechanism (x, t) is strategy-proof and non-bossy, then for all $\theta, \theta' \in \Theta$, if $x(\theta) = x(\theta')$, then $t(\theta) = t(\theta')$.*

This result tells us that the selected assignment uniquely determines the transfers. We denote by $p_i(x)$ the amount that agent i pays whenever the selected assignment is x. By ontoness, $p_i(x)$ is well-defined for all $x \in X$. By individual rationality, $p_i(x) \leq 0$ if $x_i = i$.

Lemma 1 already implies that in a mechanism that is strategy-proof and non-bossy, the way in which transfers can depend on preferences is considerably limited. Our main contribution is to show that the transfer function is more rigid than Lemma 1 implies, if individual rationality and ontoness are also satisfied. Specifically, if the mechanism is only strategy-proof and non-bossy, an agent may have to pay different amounts for the same object depending on the object assignment for the other agents. We show that this does not occur if the mechanism is also individually rational and onto (the proof is in the Appendix).

Theorem 2 (Price Independence). *In a mechanism that is strategy-proof, individually rational, non-bossy, and onto, the amount that an agent pays or receives depends only on the object he receives, and not on the object assignment for the other agents. That is, for all $\theta, \theta' \in \Theta$ and all $i \in N$, if $x_i(\theta') = x_i(\theta)$, then $t_i(\theta') = t_i(\theta)$.*

This theorem implies that the price functions $(p_i(\cdot))_{i \in N}$ can be represented by a $n \times n$ price matrix $P = (p_{ij})$. Individual rationality implies (1).

To derive Theorem 1, it remains to show that $x(\theta)$ has to be the core assignment in the associated Shapley-Scarf economy when no one is indifferent between acceptable objects:

Theorem 3. *Let (x, t) be a mechanism that is strategy-proof, individually rational, non-bossy, and onto, and let P be the associated price matrix. Let $\theta \in \Theta$ be a type profile such that for each $i \in N$, the preference relation $R_i(\theta_i; P)$ is strict over the acceptable objects. Then*

$$x(\theta) = C(R_1(\theta_1; P), \ldots, R_n(\theta_n; P)).$$

The proof can be found in Svensson [35] but we provide it for completeness. We first introduce a useful concept. The *object option set* for agent i given θ_{-i} is the set $K_i(\theta_{-i})$ defined by

$$K_i(\theta_{-i}) = \{x_i \in N : x = x(\theta_i, \theta_{-i}) \text{ for some } \theta_i \in \Theta_i\}.$$

Strategy-proofness implies that $x_i(\theta)$ is a most preferred object in $K_i(\theta_{-i})$ for $R_i(\theta_i; P)$.

Proof of Theorem 3. Let $x = C(R_1(\theta_1; P), \ldots, R_n(\theta_n; P))$. Let θ' be a type profile where x_i is the unique top object for $R_i(\theta'_i; P)$, and if $x_i \neq i$, then i is the unique second object for $R_i(\theta'_i; P)$.

We first show $x(\theta') = x$. By ontoness, there exists a type profile $\hat{\theta} \in \Theta$ such that $x(\hat{\theta}) = x$. Strategy-proofness implies

$$x_1(\theta'_1, \hat{\theta}_{-1}) \; R_1(\theta'_1; P) \; x_1.$$

Since x_1 is the unique top object for $R_1(\theta'_1; P)$, this implies $x_1(\theta'_1, \hat{\theta}_{-1}) = x_1$. Non-bossiness then implies $x(\theta'_1, \hat{\theta}_{-1}) = x$. Repeating the same argument for the other agents, we obtain $x(\theta') = x$.

Recall that x can be computed by means of the top trading cycle algorithm. Let $m \in \{1, 2, \ldots\}$ be the number of rounds that the algorithm needs to compute x (when the type profile is θ). Let N_m be the set of agents who form cycles in the final round, and pick any agent $i \in N_m$. We show $x(\theta_i, \theta'_{-i}) = x$.

Since $x(\theta') = x$, we have $x_i \in K_i(\theta'_{-i})$. Since i points to x_i in round m, and $R_i(\theta_i; P)$ is strict over acceptable objects, x_i is i's unique top object in N_m for $R_i(\theta_i; P)$. Furthermore, $K_i(\theta'_{-i})$ contains no object outside N_m, since agents outside N_m prefer their endowments to objects in N_m. Thus x_i is the unique top object in $K_i(\theta'_{-i})$ for $R_i(\theta_i; P)$, which implies $x_i(\theta_i, \theta'_{-i}) = x_i$. Non-bossiness then implies $x(\theta'_i, \theta_{-i}) = x$. Repeating this argument for the other agents in N_m, we obtain $x(\theta_{N_m}, \theta'_{N \setminus N_m}) = x$. Repeating the same argument for the remaining rounds $m-1, m-2, \ldots, 1$, we obtain $x(\theta) = x$. □

6 Independence of the Axioms

We verify that none of the axioms in Theorem 2 is redundant. In what follows, we exhibit a mechanism that violates price independence and satisfies all of the axioms except for one.

Example 1 (Not individually rational). We consider a 3-agent economy. First, agent 1 picks any object and pays nothing. Then agent 2 picks from the remaining objects, but the price schedule for agent 2 depends on the object picked by agent 1. Specifically, the prices that agent 2 faces are:

$$\begin{cases} (p_{22}, p_{23}) & \text{if agent 1 picks object 1;} \\ (p'_{21}, p'_{23}) & \text{if agent 1 picks object 2;} \\ (p''_{21}, p''_{22}) & \text{if agent 1 picks object 3.} \end{cases}$$

For example, p_{23} is the price of object 3 for agent 2 when agent 1 picks object 1. Agent 3 receives the remaining object, and $t_3(\theta) = -t_2(\theta)$. Then this mechanism is strategy-proof, non-bossy, onto, and budget balanced, but it is not individually rational for agents 2 and 3. The mechanism violates price independence for agents 2 and 3.

Example 2 (Not onto). We consider the 4-agent economy. The planner first announces three matrices:

$$P_{12} = \begin{bmatrix} 0 & p_{12} \\ p_{21} & 0 \end{bmatrix}, \quad P_{34} = \begin{bmatrix} 0 & p_{34} \\ p_{43} & 0 \end{bmatrix}, \quad P'_{34} = \begin{bmatrix} 0 & p'_{34} \\ p'_{43} & 0 \end{bmatrix}.$$

We divide the economy into two subeconomies, one with agents 1 and 2, and the other with agents 3 and 4. We allow trades only within subeconomies. We determine the allocation in the subeconomy of agents 1 and 2 according to the fixed-price core mechanism based on price matrix P_{12} and some tie-breaking rule. We then determine the allocation in the subeconomy of agents 3 and 4 similarly, but the price matrix to be used is P_{34} if there is a trade between agents 1 and 2, and P'_{34} otherwise. This mechanism is not onto, but it is strategy-proof, non-bossy, and individually rational. Price independence is violated for agents 3 and 4.

Example 3 (Not strategy-proof). For a given type profile θ, consider the Shapley-Scarf economy induced by some tie-breaking rule $\succ$ together with a price matrix P such that for all $i, j \in N$,

$$\begin{aligned} p_{ij} &= 10 \quad \text{if } i \neq j; \\ p_{ii} &= 0. \end{aligned}$$

We then perform the *first round* of the top trading cycle algorithm. Agents who form a cycle in the first round exchange their endowments according to the cycle. Agents who do not form cycles in the first round keep their endowments. Let the generated assignment be denoted by $x^*(\theta)$. Agent i pays $t_i = f_i(x^*(\theta))$, where the functions $f_i \colon X \to \mathbb{R}$ satisfy:

$$f_i(x) < 10 \quad \text{if } x_i \neq i; \tag{5}$$

$$f_i(x) \leq 0 \quad \text{if } x_i = i; \tag{6}$$

$$\sum_{i \in N} f_i(x) = 0. \tag{7}$$

Then it is easy to verify that the assignment function x^* is non-bossy. Since f_i is a function of $x \in X$, the mechanism as a whole is non-bossy. It is easy to see that the mechanism is also onto, individually rational (by (5) and (6)), and budget balanced (by (7)). Price independence is violated when $f_i(x) \neq f_i(x')$ for x and x' such that $x_i = x'_i$.

Example 4 (Bossy). The mechanism we consider is equivalent to a fixed-price core mechanism with a tie-breaking rule, except that the price matrix to be used is a function of θ_1. Specifically, the price matrix, denoted by $P(\theta_1) = (p_{ij}(\theta_1))_{i,j \in N}$, is such that $p_{ij}(\theta_1) = 0$ if any of the following conditions holds:

1. $i = 1$ or $j = 1$;

2. $i = j$;

3. $\theta_1(k) \geq 0$ for some $k \neq 1$.

By Condition 1, all prices associated with agent 1 and object 1 are zero. Condition 2 ensures individual rationality. Condition 3 implies that the submatrix obtained by eliminating the first row and the first column of $P(\theta_1)$ can differ from zero only when the most highly valued object for agent 1 is his endowment. Note that when θ_1 satisfies Condition 3, all prices are zero.

This mechanism is strategy-proof, especially for agent 1. Indeed, when $\theta_1(k) < 0$ for all $k \neq 1$, if agent 1 tells the truth, he obtains his endowment and pays nothing, which is the best outcome for him given that prices are always zero for him. When $\theta_1(k) \geq 0$ for some $k \neq 1$, agent 1 can change the price matrix only by announcing a θ_1' such that $\theta_1'(k) < 0$ for all $k \neq 1$. But if he announces such a type, he receives his endowment and pays zero, which does not make him better off (by individual rationality of the Shapley-Scarf core).

The mechanism is budget balanced if for each $\theta_1 \in \Theta_1$, the submatrix obtained by eliminating the first row and the first column of $P(\theta_1)$ satisfies (4) for some vector $(\rho_2, \dots, \rho_n) \in \mathbb{R}^{n-1}$. Note that agent 1 receives object $k \neq 1$ only when Condition 3 holds and all prices are zero.

7 Concluding Remarks

The desirability of the fixed-price core mechanisms is obvious from the axioms. Another appealing aspect of the mechanisms is that they are straightforward to implement in practice. In these mechanisms, what participants have to do is to report their *ordinal* rankings over the objects given the prices, not the numerical values they attach to the objects. As long as participants know the price matrix that is being used, announcing one's type is equivalent to announcing one's ordinal ranking induced by the price matrix (i.e., $R_i(\theta_i; P)$). Moreover, participants do not have to rank all objects; they only have to rank the objects that they are interested in (i.e., acceptable objects). These properties are satisfied also by the Gale-Shapley [8] mechanisms for two-sided matching, which are being used in the American entry-level market for physicians [28, 29]. On the other hand, the Vickrey-Clarke-Groves (VCG) mechanism is difficult to implement in practice since it asks participants to reveal their numerical valuations for all objects.

We left many interesting issues for future research. A particularly important issue to be addressed is a characterization of the prices that are optimal in some sense for the planner. In practice, the planner has to decide which prices to use, and this paper does not provide a guideline for the decision.

A biggest drawback of the fixed-price core mechanisms is Pareto inefficiency due to fixed prices. But as we noted above, inefficiency is not avoidable since Pareto efficiency and strategy-proofness are incompatible in our setting. Furthermore, inefficiency may not be a serious problem in practice, especially when the mechanisms are used repeatedly, say, once a year. By individual rationality, the mechanisms achieve a Pareto improvement every year, provided that participants treat a mechanism as a one-shot game every year. The repetition would be more effective in realizing gains from trade if different prices are used.

A Appendix: Proof of Theorem 2

This section proves Theorem 2. So, let $\varphi = (x, t)$ be a mechanism that is strategy-proof, individually rational, non-bossy, and onto.

We first introduce additional notation. It will be useful to represent an assignment by means of a collection of "cycles." A *cycle* is a list $C = (C_i)_{i \in S}$ for some non-empty subset $S = \{i_1, \ldots, i_m\} \subseteq N$ such that

$$C_{i_1} = i_2,\ C_{i_2} = i_3, \ldots, C_{i_{m-1}} = i_m,\ C_{i_m} = i_1.$$

We denote S, the set of agents involved in C, by $N(C)$. A cycle C is *non-trivial* if $|N(C)| \geq 2$. We denote by ij the cycle C consisting of agents i and j such that $C_i = j$ and $C_j = i$.

We can represent any assignment by a collection of *non-trivial* cycles $(C^1, \ldots, C^L)$ such that the sets $(N(C^\ell))_{\ell=1}^L$ are pairwise disjoint. We then assume that an agent who belongs to none of these cycles keeps his endowment. For example, (C) represents the assignment where agent i in cycle C receives object C_i and the agents outside C keep their endowments.

We denote by x^0 the initial assignment.

Given an assignment $x = (C^\ell)_{\ell \in L}$, an assignment z is called a *subassignment* of x if $z = (C^\ell)_{\ell \in L'}$ for some subset $L' \subseteq L$. That is, z is obtained by "dissolving" some or none of the non-trivial cycles in x. We denote by $SA(x)$ the set of subassignments of x. We allow $L' = \emptyset$ in the above definition, and hence $x^0 \in SA(x)$ for all $x \in X$.

Let $I(\theta) = \{x \in X : \theta_i(x_i) - p_i(x) \geq 0\}$, which is the set of assignments that are "individually rational" for type profile θ.

We denote by $\underline{p}$ the minimal amount that an agent may have to pay:

$$\underline{p} = \min_{i \in N} \min_{x \in X} p_i(x).$$

Since $p_i(x^0) \leq 0$, $\underline{p} \leq 0$. Individual rationality implies

$$\theta_i(k) < \underline{p} \Rightarrow x_i(\theta_i, \theta_{-i}) \neq k. \tag{8}$$

Given $S \subseteq N$ and $i \notin S$, let

$$D_i(S) = \{\theta_i \in \Theta_i : \text{ for each } k \notin S \cup \{i\}, \theta_i(k) < \underline{p}\}.$$

When $\theta_i \in D_i(S)$, (8) implies that i receives an object in $S \cup \{i\}$. Note that $\theta_i \in D_i(S)$ does not tell us anything about $\theta_i(k)$ for objects $k \in S$; they may also be less than $\underline{p}$. To simplify notation, when $S = \{k\}$, we write $D_i(k)$ instead of $D_i(\{k\})$. Similarly, when $S = \{k, \ell\}$, we write $D_i(k, \ell)$. We allow $S = \emptyset$; $\theta_i \in D_i(\emptyset)$ means that $\theta_i(k) < \underline{p}$ for all $k \neq i$. It will be convenient to define

$$D_i(i) = D_i(\emptyset).$$

Given a type $\theta_i \in \Theta_i$ and an object $k \in N$, a type θ'_i is called a *strict Maskin monotonic transformation* of θ_i at k if for all $j \neq k$, $\theta'_i(k) - \theta'_i(j) > \theta_i(k) - \theta_i(j)$. A strict Maskin monotonic transformation is a special case of a Maskin monotonic transformation [17]; the latter allows equality in the above condition. The following lemma is standard in the literature and thus we omit its proof.

Lemma 2. *For all $\theta \in \Theta$, all $i \in N$, and all $\theta'_i \in \Theta_i$, if θ'_i is a strict Maskin monotonic transformation of θ_i at $x_i(\theta)$, then $\varphi(\theta'_i, \theta_{-i}) = \varphi(\theta)$.*

The (assignment) *option set* for agent i given θ_{-i} is the set $O_i(\theta_{-i})$ defined by

$$O_i(\theta_{-i}) = \{x \in X : x = x(\theta_i, \theta_{-i}) \text{ for some } \theta_i \in \Theta_i\}.$$

This is the set of *assignments* that i can induce by reporting some type $\theta_i \in \Theta_i$ given that the other agents report θ_{-i}. By strategy-proofness, $x(\theta)$ is a most preferred assignment for i in his option set. That is, $x(\theta) \in \arg\max_{x \in O_i(\theta_{-i})}[\theta_i(x_i) - p_i(x)]$.

Given an object $k \in N$, let

$$O_i^k(\theta_{-i}) = \{x \in O_i(\theta_{-i}) : x_i = k\}.$$

This is the set of assignments in agent i's option set where agent i receives object k. Then $|O_i^k(\theta_{-i})| \leq 1$. Indeed, let $x(\theta_i, \theta_{-i}) = x$ and $x(\theta'_i, \theta_{-i}) = x'$, and if $x'_i = x_i = k$, then by strategy-proofness, $p_i(x') = p_i(x)$. Non-bossiness then implies $x(\theta_i, \theta_{-i}) = x(\theta'_i, \theta_{-i})$. Thus, if an agent i can obtain object k given the other agents' types, then there exists only one assignment in his option set that gives object k to him. Note that $|O_i^i(\theta_{-i})| = 1$ by individual rationality.

Given an assignment $x \in X$, let

$$T(x) = \{i \in N : x_i \neq i\}.$$

Lemma 3. *For all $x \in X$ and all $\theta \in \Theta$, if $I(\theta) \subseteq SA(x)$ and for all $i \in T(x)$,*

$$\theta_i(x_i) - p_i(x) > \pi_i(x) \equiv \max\{-p_i(z) : z_i = i \text{ and } z \in SA(x)\}, \tag{9}$$

then $x(\theta) = x$.

Proof. We first note that $\pi_i(x) \geq -p_i(x^0) \geq 0$, which implies $x \in I(\theta)$. Without loss of generality, assume $T(x) = \{1, 2, \ldots, m\} \equiv S$. Assume $S \neq \emptyset$; the result is trivial otherwise. For each $i \in S$, let $\theta'_i \in D_i(x_i)$ such that

$$\theta'_i(x_i) - p_i(x) > \pi_i(x); \tag{10}$$

$$\theta'_i \leq \theta_i. \tag{11}$$

That is, we decrease the valuations for all objects, but we do it by a small amount for x_i and by sufficiently large amounts for all $k \notin \{x_i, i\}$.

Ontoness implies that there exists a type profile $\bar{\theta} \in \Theta$ such that $x(\bar{\theta}) = x$. For each $i \in N$, let $\hat{\theta}_i \in D_i(x_i)$ be a strict Maskin monotonic transformation of $\bar{\theta}_i$ at x_i. If $x_i = i$, $\hat{\theta}_i$ is obtained by decreasing the valuations for all $k \neq i$ by large amounts. If $x_i \neq i$, it is obtained by increasing the valuation for object x_i and decreasing the valuations for all $k \notin \{x_i, i\}$ by large amounts. By Lemma 2, $x(\hat{\theta}) = x$.

Let $z = x(\theta'_1, \hat{\theta}_{-1})$. Since $\theta'_1 \in D_1(x_1)$, z_1 is either 1 or x_1. We claim $z_1 = x_1$, which in turn implies $z = x$ by non-bossiness. Suppose to the contrary that $z_1 = 1$. Since $\hat{\theta}_i \in D_i(x_i)$ for all $i \in N$, it follows that $z \in SA(x)$. Hence if z is selected, 1's utility level is

$$u_1(z) = -p_1(z) \leq \pi_1(x), \tag{12}$$

where the inequality follows from $z \in SA(x)$ and the definition of $\pi_1(x)$. But $x \in O_1(\hat{\theta}_{-1})$, and if x is selected, 1's utility level is

$$u_1(x) = \theta'_1(x_1) - p_1(x) > \pi_1(x),$$

where the inequality follows from (10). This and (12) imply that 1 prefers x to z. This is in contradiction with strategy-proofness.

We can repeat the above argument for agents $\{2, 3, \ldots, m\}$ sequentially, and we obtain $x(\theta'_1, \ldots, \theta'_m, \hat{\theta}_{m+1}, \ldots, \hat{\theta}_n) = x$. For convenience, we denote this type profile by θ'.

Now, consider profile (θ_1, θ'_{-1}), and let $z = x(\theta_1, \theta'_{-1})$. Then we can show

$$\theta_i(z_i) \geq p_i(z) \quad \forall i \in N. \tag{13}$$

To see this, note:

1. If $i = 1$, it follows from $z \in I(\theta_1, \theta'_{-1})$;

2. If $i \in S \setminus \{1\}$, then since $z \in I(\theta_1, \theta'_{-1})$, we have $\theta'_i(z_i) \geq p_i(z)$. The desired result then follows from (11);

3. If $i \notin S$, then since $\theta'_i = \hat{\theta}_i \in D_i(i)$, we have $z_i = i$, and hence $p_i(z) \leq 0 = \theta_i(z_i)$.

Inequalities (13) mean $z \in I(\theta)$, and so the hypothesis of the lemma implies $z \in SA(x)$. This means that z_1 is either 1 or x_1. We show $z_1 = x_1$, which in turn implies $z = x$ by non-bossiness. Suppose to the contrary that $z_1 = 1$. Then 1's utility level is

$$u_1(z) = -p_1(z) \leq \pi_1(x), \tag{14}$$

where the inequality follows from $z \in SA(x)$ and the definition of $\pi_1(x)$. But $x(\theta') = x$ implies $x \in O_1(\theta'_{-1})$, and 1's utility level if x is selected is

$$u_1(x) = \theta_1(x_1) - p_1(x) > \pi_1(x),$$

where the inequality follows from the hypothesis of the lemma. This and (14) imply that 1 prefers x to z, in contradiction with strategy-proofness. This contradiction establishes $x(\theta_1, \theta'_{-1}) = x$.

We can repeat the above argument for agents $\{2, \ldots, m\}$ sequentially, and obtain

$$x(\theta_1, \ldots, \theta_m, \hat{\theta}_{m+1}, \ldots, \hat{\theta}_n) = x.$$

Finally, change agent $m + 1$'s type to θ_{m+1}, and let

$$z = x(\theta_1, \ldots, \theta_m, \theta_{m+1}, \hat{\theta}_{m+2}, \ldots, \hat{\theta}_n).$$

Note that for all $i \geq m + 2$, $\hat{\theta}_i \in D_i(i)$ and so $z_i = i$. This implies $z \in I(\theta) \subseteq SA(x)$, and hence $z_{m+1} = m + 1$. Non-bossiness then implies $z = x$. Repeating this argument for agents $\{m + 3, \ldots, n\}$, we obtain $x(\theta) = x$. □

We next prove that the amount that an agent receives when he is allocated his endowment is independent of the object allocation for the other agents.

Lemma 4. *For all $x \in X$ and all $i \notin T(x)$, $p_i(x) = p_i(x^0)$.*

Proof. Our proof is by induction on the number of non-trivial cycles in x. The lemma holds trivially when x has no non-trivial cycle. Suppose that the lemma holds for all assignments x that have at most $m \in \{0, 1, \ldots\}$ non-trivial cycles. We show that the lemma holds also for all assignments that have $m + 1$ non-trivial cycles. So, let $x \in X$ be an assignment that has $m + 1$

non-trivial cycles, and denote

$$x = (C, C^1, \ldots, C^m).$$

Without loss of generality, assume $1 \notin T(x)$ and $C_2 = 3$, and let

$$y = (12, C^1, \ldots, C^m).$$

We show $p_1(x) = p_1(x^0)$. Let θ be a type profile such that:

D1. $\theta_1 \in D_1(2)$ and $\theta_1(2) - p_1(y) > -p_1(x^0)$;

D2. $\theta_2 \in D_2(1,3)$ and $\theta_2(1) - p_2(y) > \theta_2(3) - p_2(x) > -p_2(x^0)$;

D3. For all $i \notin \{1,2\}$, $\theta_i \in D_i(x_i)$ and if $i \in T(x)$ in addition, then $\theta_i(x_i) - p_i(x) > -p_i(x^0)$ and $\theta_i(y_i) - p_i(y) > -p_i(x^0)$.

We first claim

$$y \in O_2(\theta_{-2}). \tag{15}$$

To see this, let $\theta_2' \in D_2(1)$ such that

$$\theta_2'(1) - p_2(y) > -p_2(x^0). \tag{16}$$

This together with D1 and D3 implies $I(\theta_2', \theta_{-2}) \subseteq SA(y)$. Moreover, for all $i \in T(y)$, any assignment $y' \in SA(y)$ such that $y_i' = i$ contains at most m non-trivial cycles, and our induction hypothesis then implies $p_i(y') = p_i(x^0)$. This implies that $\pi_i(y) = -p_i(x^0)$ for all $i \in T(y)$. Then Lemma 3, together with D1, D3, and (16), implies $x(\theta_2', \theta_{-2}) = y$, which establishes (15). Similar arguments establish[9]

$$x \in O_2(\theta_{-2}) \cap O_1(\theta_{-1}). \tag{17}$$

Since $\theta_2 \in D_2(1,3)$, $x_2(\theta) \in \{1,2,3\}$. (15) and (17) imply that $O_2^1(\theta_{-2}) = y$ and $O_2^3(\theta_{-2}) = x$. And by D2, agent 2 prefers y to x. Note also that the assignment $z = O_2^2(\theta_{-2})$ has at most m non-trivial cycles, and thus our induction hypothesis implies $p_2(z) = p_2(x^0)$. This and D2 imply that agent 2 prefers y to z as well, which establishes

$$x(\theta) = y. \tag{18}$$

[9]For $x \in O_2(\theta_{-2})$, one can show that if $\theta_2'' \in D_2(3)$ is such that $\theta_2''(3) - p_2(x) > -p_2(x^0)$, then $x(\theta_2'', \theta_{-2}) = x$. For $x \in O_1(\theta_{-1})$, one can show that if $\theta_1'' \in D_1(1)$, then $x(\theta_1'', \theta_{-1}) = x$.

This and (17) then imply that it should not be the case that agent 1 prefers x to y, and thus $\theta_1(2) - p_1(y) \geq -p_1(x)$. Since this holds whenever the inequality in D1 holds, it follows that $p_1(x^0) \leq p_1(x)$. This establishes a half of the lemma.

It remains to show $p_1(x^0) \geq p_1(x)$, and so suppose otherwise. Then we can find a $\theta_1' \in D_1(2)$ such that

$$-p_1(x) < \theta_1'(2) - p_1(y) < -p_1(x^0). \tag{19}$$

Since $\theta_1' \in D_1(2)$, $x_1(\theta_1', \theta_{-1}) \in \{1, 2\}$. (17) and (18) imply $O_1^1(\theta_{-1}) = x$ and $O_1^2(\theta_{-1}) = y$. Since the first inequality in (19) implies that agent 1 prefers y to x, it follows that $x(\theta_1', \theta_{-1}) = y$. Thus $O_2^1(\theta_1', \theta_{-1,2}) = y$, and hence for a $\theta_2' \in D_2(1)$ with sufficiently large $\theta_2'(1)$, we have

$$x(\theta_1', \theta_2', \theta_{-1,2}) = y. \tag{20}$$

But note that the assignment $z' = O_1^1(\theta_2', \theta_{-1,2})$ contains at most m non-trivial cycles, and thus our induction hypothesis together with the second inequality in (19) implies

$$-p_1(z') = -p_1(x^0) > \theta_1'(2) - p_1(y).$$

This means that agent 1 of type θ_1' prefers z' to y, which is in contradiction with (20). □

This lemma implies that there exists a number p_{ii} for each $i \in N$ such that for all assignments $x \in X$, if $x_i = i$, then $p_i(x) = p_{ii}$. To prove price independence, it remains to show the existence of p_{ij} for all $i, j \in N$.

Lemma 5. *For any assignment of the form $(C, C^1, \ldots, C^m)$, where $m \in \{0, 1, \ldots\}$, and for any $i \in N(C)$, we have*

$$p_i(C, C^1, \ldots, C^m) = p_i(ij, C^1, \ldots, C^m),$$

where $j = C_i$. When $m = 0$, this means $p_i(C) = p_i(ij)$.

Proof. Without loss of generality, assume $i = 1$, $j = 2$, and $C_2 = 3$. Let

$$x = (C, C^1, \ldots, C^m), \quad y = (12, C^1, \ldots, C^m).$$

We prove $p_1(x) = p_1(y)$.

We first prove $p_1(x) \geq p_1(y)$. So, suppose otherwise. Let θ be a type profile such that:

E1. $\theta_2 \in D_2(1)$. For all $i \neq 2$, $\theta_i \in D_i(x_i)$;

E2. For all $i \in T(y)$, $\theta_i(y_i) - p_i(y) > -p_{ii}$;

E3. For all $i \in T(x) \setminus \{1,2\}$, $\theta_i(x_i) - p_i(x) > -p_{ii}$.

E1 implies $I(\theta) \subseteq SA(y)$. E2 together with Lemma 3 then implies $x(\theta) = y$.

Let $\theta'_2 \in D_2(1,3)$ such that

$$\theta'_2(3) - p_2(x) > -p_{22}, \tag{21}$$

and $\theta'_2(1)$ is large. Since $O^1_2(\theta_{-2}) = y$, it follows that if $\theta'_2(1)$ is sufficiently large, $x(\theta'_2, \theta_{-2}) = y$.

Since we are assuming $p_1(x) < p_1(y)$, we can find a $\theta'_1 \in D_1(2)$ such that

$$p_1(x) < \theta'_1(2) + p_{11} < p_1(y). \tag{22}$$

Since $\theta'_1 \in D_1(2)$, $x_1(\theta'_1, \theta'_2, \theta_{-1,2}) \in \{1,2\}$. Since $x(\theta'_2, \theta_{-2}) = y$, $O^2_1(\theta'_2, \theta_{-1,2}) = y$. The second inequality in (22) implies that agent 1 prefers $O^1_1(\theta'_2, \theta_{-1,2})$ to y, which establishes $x(\theta'_1, \theta'_2, \theta_{-1,2}) = O^1_1(\theta'_2, \theta_{-1,2}) \equiv x'$. But since $x'_2 = 2$, (21) implies that agent 2 prefers x to x'. We can then derive a desired contradiction by showing $x \in O_2(\theta'_1, \theta_{-1,2})$. Indeed, we can show that $x = x(\theta'_1, \theta''_2, \theta_{-1,2})$ for $\theta''_2 \in D_2(3)$ such that

$$\theta''_2(3) - p_2(x) > -p_{22}. \tag{23}$$

To see this, note first that since $\theta''_2 \in D_2(3)$, $I(\theta'_1, \theta''_2, \theta_{-1,2}) \subseteq SA(x)$. Moreover, (9) holds for assignment x and type profile $(\theta'_1, \theta''_2, \theta_{-1,2})$, by (23), the first inequality in (22), and E3.

The proof for the other inequality $p_1(x) \le p_1(y)$ is similar. We first let θ be a type profile such that

F1. For all $i \in N$, $\theta_i \in D_i(x_i)$;

F2. For all $i \in T(x)$, $\theta_i(x_i) - p_i(x) > -p_{ii}$;

F3. For all $i \in T(y) \setminus \{1,2\}$, $\theta_i(y_i) - p_i(y) > -p_{ii}$.

Then F1 and F2 together with Lemma 3 imply $x(\theta) = x$. Let $\theta'_2 \in D_2(1,3)$ such that

$$\theta'_2(1) - p_2(y) > -p_{22}. \tag{24}$$

Then if $\theta'_2(3)$ is sufficiently large, $x(\theta'_2, \theta_{-2}) = x$. If $p_1(y) < p_1(x)$, then we can find a $\theta'_1 \in D_1(2)$ such that

$$p_1(y) < \theta'_1(2) + p_{11} < p_1(x). \tag{25}$$

Then the second inequality in (25) implies that agent 1 prefers $O_1^1(\theta_2', \theta_{-1,2}) \equiv x'$ to $x = O_1^2(\theta_2', \theta_{-1,2})$, and thus $x(\theta_1', \theta_2', \theta_{-1,2}) = x'$. But $x_2' = 2$ and by (24), agent 2 prefers y to x'. This is a contradiction since $y \in O_2(\theta_1', \theta_{-1,2})$. Indeed, $y = x(\theta_1', \theta_2'', \theta_{-1,2})$ for $\theta_2'' \in D_2(1)$ such that $\theta_2''(1) - p_2(y) > -p_{22}$, because of the first inequality in (25) and F3. $\square$

Lemma 6. *For any assignment of the form $(ij, C^1, \ldots, C^m)$, where $m \in \{1, 2, \ldots\}$, we have*

$$p_i(ij, C^1, \ldots, C^m) = p_i(ij, C^2, \ldots, C^m).$$

When $m = 1$, this means $p_i(ij, C^1) = p_i(ij)$.

Proof. Without loss of generality, assume $i = 1$ and $j = 2$. Let

$$x = (12, C^1, \ldots, C^m), \quad y = (12, C^2, \ldots, C^m).$$

We show $p_1(x) = p_1(y)$.

Assume, without loss of generality, that $N(C^1) = \{3, 4, \ldots, q\}$ and $C_3^1 = 4, C_4^1 = 5, \ldots, C_q^1 = 3$. And let T be a cycle defined by

$$T_3 = 1,\ T_1 = 2,\ T_2 = 4,\ T_4 = 5, \ldots, T_q = 3.$$

And let $z = (T, C^2, \ldots, C^m)$. Then Lemma 5 implies $p_1(z) = p_1(y)$.

(Part 1) We first show $p_1(x) \geq p_1(y)$. So, suppose otherwise. Let θ be a type profile such that:

Z1. For all $i \in N$, $\theta_i \in D_i(z_i)$;

Z2. For all $i \in T(z)$, $\theta_i(z_i) - p_i(z) > -p_{ii}$;

Z3. For all $i \in T(x) \setminus \{1, 2, 3\}$, $\theta_i(x_i) - p_i(x) > -p_{ii}$.

Z1 implies $I(\theta) \subseteq SA(z)$, and Z2 and Lemma 3 then imply $x(\theta) = z$.

Let $\theta_2' \in D_2(1, 4)$ such that $\theta_2'(4)$ is large. Since $z \in O_2(\theta_{-2})$, it follows that if $\theta_2'(4)$ is sufficiently large, $x(\theta_2', \theta_{-2}) = z$.

Similarly, let $\theta_3' \in D_3(1, 4)$ such that

$$\theta_3'(4) - p_3(x) > -p_{33}, \tag{26}$$

and $\theta_3'(1)$ is large. Since $z \in O_3(\theta_2', \theta_{-2,3})$, it follows that if $\theta_3'(1)$ is sufficiently large,

$$x(\theta_2', \theta_3', \theta_{-2,3}) = z. \tag{27}$$

Since we are assuming $p_1(x) < p_1(y)$, there is a $\theta'_1 \in D_1(2)$ such that

$$p_1(x) < \theta'_1(2) + p_{11} < p_1(y) = p_1(z). \tag{28}$$

To simplify notation, let $\theta' = (\theta'_{\{1,2,3\}}, \theta_{N\setminus\{1,2,3\}})$. By (27), $O_1^2(\theta'_{-1}) = z$, and by the second inequality in (28), agent 1 prefers $O_1^1(\theta'_{-1})$ to z. Since $\theta'_1 \in D_1(2)$, this implies $x_1(\theta') = 1$, which in turn implies $x_2(\theta') = 2$.

Let $\theta''_2 \in D_2(1)$ such that

$$\theta''_2(1) - p_2(x) > -p_{22}. \tag{29}$$

Note that $x_2(\theta') = 2$ even when $\theta'_2(1)$ is large. This implies $x_2(\theta''_2, \theta'_{-2}) = 2$. We can now derive a desired contradiction by showing that $x(\theta''_2, \theta'_{-2}) = x$. To see this, note first $I(\theta''_2, \theta'_{-2}) \subseteq SA(x)$. Moreover, (9) holds for assignment x and the type profile $(\theta''_2, \theta'_{-2})$, by (29), the first inequality in (28), (26), and Z3.

(Part 2) We now prove $p_1(x) \leq p_1(y)$. Suppose otherwise. Let θ be such that:

X1. For all $i \in N$, $\theta_i \in D_i(x_i)$;

X2. For all $i \in T(x)$, $\theta_i(x_i) - p_i(x) > -p_{ii}$;

X3. For all $i \in T(z) \setminus \{1,2,3\}$, $\theta_i(z_i) - p_i(z) > -p_{ii}$.

X1 and X2 together with Lemma 3 imply $x(\theta) = x$.

Let $\theta'_3 \in D_3(1,4)$ such that

$$\theta'_3(4) - p_3(x) > -p_{33}. \tag{30}$$

Since $\theta'_3 \in D_3(1,4)$, $x_3(\theta'_3, \theta_{-3}) \in \{1,3,4\}$. By the construction of θ_{-3}, $x_3(\theta'_3, \theta_{-3}) \neq 1$. By (30), agent 3 prefers $x = O_3^4(\theta_{-3})$ to $O_3^3(\theta_{-3})$, which establishes $x(\theta'_3, \theta_{-3}) = x$.

Let $\theta'_2 \in D_2(1,4)$ such that $\theta'_2(1)$ is large. Since $x \in O_2(\theta'_3, \theta_{-2,3})$, it follows that if $\theta'_2(1)$ is sufficiently large,

$$x(\theta'_2, \theta'_3, \theta_{-2,3}) = x. \tag{31}$$

Since we are assuming $p_1(y) < p_1(x)$, there is a $\theta'_1 \in D_1(2)$ such that

$$p_1(z) = p_1(y) < \theta'_1(2) + p_{11} < p_1(x). \tag{32}$$

To simplify notation, let $\theta' = (\theta'_{\{1,2,3\}}, \theta_{N\setminus\{1,2,3\}})$. Since $\theta'_1 \in D_1(2)$, $x_1(\theta') \in \{1,2\}$. By the second inequality in (32), agent 1 prefers $O_1^1(\theta'_{-1})$ to $x = O_1^2(\theta'_{-1})$. Thus $x_1(\theta') = 1$, which in turn implies $x(\theta') \in SA(C^1, \ldots, C^m)$. Let $x' = x(\theta')$.

Let $\theta''_2 \in D_2(4)$ such that

$$\theta''_2(4) - p_2(z) > -p_{22}. \tag{33}$$

Since $\theta''_2 \in D_2(4)$, $x_2(\theta''_2, \theta'_{-2}) \in \{2,4\}$. Note that $x_2(\theta') = 2$, and this holds even when $\theta'_2(4)$ is large. This means $x_2(\theta''_2, \theta'_{-2}) \neq 4$, and thus $x(\theta''_2, \theta'_{-2}) = O_2^2(\theta'_{-2}) = x'$.

Let $\theta''_3 \in D_3(1)$ such that

$$\theta''_3(1) - p_3(z) > -p_{33}. \tag{34}$$

Note that $x_3(\theta''_2, \theta'_{-2}) = x'_3 \neq 1$, and this holds even when $\theta'_3(1)$ is large. This means that $x_3(\theta''_2, \theta''_3, \theta'_{-2,3}) \neq 1$. We can now derive a desired contradiction by showing $x(\theta''_2, \theta''_3, \theta'_{-2,3}) = z$. To see this, note first that $I(\theta''_2, \theta''_3, \theta'_{-2,3}) \subseteq SA(z)$, since $\theta''_3 \in D_3(1)$, $\theta'_1 \in D_1(2)$, and $\theta''_2 \in D_2(4)$. Moreover, (9) holds for z at the type profile $(\theta''_2, \theta''_3, \theta'_{-2,3})$, by (34), (33), the first inequality in (32), and X3. $\square$

Proof of Theorem 2. Let $x, x' \in X$ such that $x'_i = x_i$. If $x_i = i$, then by Lemma 4, $p_i(x') = p_i(x^0) = p_i(x)$. So suppose $x_i = j \neq i$. If we denote $x = (C, C^1, \ldots, C^m)$ with $i \in N(C)$, then

$$\begin{aligned} p_i(C, C^1, \ldots, C^m) &= p_i(ij, C^1, \ldots, C^m) && \text{by Lemma 5} \\ &= p_i(ij). && \text{by Lemma 6} \end{aligned}$$

This establishes $p_i(x) = p_i(ij) = p_i(x')$. $\square$

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