Decentralized Matching: The Role of Commitment*

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Abstract

The two-sided matching literature has focused on static and centralized games. However, in many markets, the matching is determined in decentralized fashion and continues to change. This paper considers infinitely-repeated matching games, where firms whose positions become vacant make offers to workers, who then decide which offers to accept and the game continues. We study how the stationary-equilibrium outcome depends on whether players commit to their employment relationships. We show that, without commitment from either side of the market (i.e., each contract expires in a period), the equilibrium matching is stable in all periods. With one-sided commitment (where firms offer tenured jobs) or two-sided commitment, the final matching may be unstable. With one-sided commitment, the final matching may be one where all workers are worse off and all firms are better off than in every stable matching, implying that the workers are made worse off by job protection.

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1 Introduction

In a two-sided matching market (with, e.g., firms and workers), a matching (a collection of firm-worker pairs) is not stable if there exists a pair of agents who prefer each other to their current partners. Such a pair is called a "blocking pair," and matchings with no blocking pair are called "stable matchings." For the standard matching problem, a stable matching exists for any number of agents and any profile of preferences, as shown by Gale and Shapley (1962). Gale and Shapley also provide an algorithm that finds a stable matching. The equivalent algorithm has been used since early 1950s by a centralized matchmaking mechanism to assign new American physicians to hospitals (Roth, 1984). The hypothesis of Roth (1991) is that the success of a centralized labor market depends on whether the matchmaking mechanism generates a stable matching.

The focus of the literature has been matching markets that are centralized, where participants submit preferences to a matchmaker that computes a matching using the Gale–Shapley algorithm. The algorithm has stages of firms making offers and workers replying to offers, but they are done within a computer of the matchmaker. On the other hand, many markets—including the market for economists—are decentralized, where a matching is determined by offers and replies exchanged directly between market participants. How does a decentralized market compare with a centralized one? Does a decentralized market generate a stable matching? Is the answer sensitive to the way in which the decentralized market operates?

To address these questions, this paper studies equilibria in a dynamic game of matching. We extend the original Gale–Shapley model to a dynamic and noncooperative setting where firms and workers interact repeatedly in a decentralized manner. In every period, firms with vacant positions make offers to workers, who then choose individually which offers to accept. The market takes place every period and all agents derive utility from their matching in each period. We focus on stationary equilibria, where actions vary only with the payoff-relevant state of the game.

Our dynamic matching model makes it possible to analyze the role of commitment, in particular, how equilibrium matching depends on whether agents commit themselves to their employment relationships. In our stylized model, three interesting possibilities are considered. The first possibility is that agents make no commitment beyond one period. That is, employees can accept new job offers but may also be dismissed. The second possibility is that, once a pair of agents are matched, they withdraw from the market and stay together permanently. The third possibility is that firms make commitments while workers do not. That is, workers remain active in the job market but are protected from dismissal, as in the case of tenured professors and government employees.

In the absence of commitment, we show that every stationary equilibrium yields a stable matching and, conversely, any stable matching can be sustained in a stationary equilibrium. That is, the game (form) implements the core in stationary equilibrium. This result is a dynamic extension of Alcalde and Romero-Medina

(2000), whose game is the same as our period-game. In the case of no commitment, our game is therefore an infinitely repeated version of the game of Alcalde and Romero-Medina (2000). Given that we focus on stationary equilibrium ruling out punishment schemes in repeated games, if all contracts expire in a single period, the equilibrium behaves as in the static model.

If both sides of the market commit, we show that the final matching of a stationary equilibrium may be unstable. In such an equilibrium, an unstable matching is realized in the first period and no pair is matched thereafter. In the equilibrium, a firm f makes an offer to a worker w although f prefers another worker w' and knows that w' also prefers f to the firm that w' accepts. The firm f does not deviate by making an offer to w' since the offer would be rejected. The offer is rejected by w' since worker w' knows that rejecting the offer, together with the reaction of w, is enough to change the continuation dynamics of the market and an even better offer will arrive in the future. With two-sided commitment, we also show that the matching may not be determined completely in the first period: some pairs may be matched in the second period or later. The equilibrium with delay and instability highlights the fact that a stationary equilibrium puts no a priori restriction on the way in which continuation equilibrium varies with the state. By restricting the relationship between continuation equilibria across different states, we can select equilibrium with neither delay nor instability. We shall show that this can be achieved by imposing a restriction of consistency that captures the inertia of the equilibrium strategies.

The last case we study is one-sided commitment, where only firms commit. In this case too, stationary equilibria may yield unstable matchings. If some commitments are already in place before the game starts, the result is trivial. What we show is that an equilibrium may yield an unstable matching even if no commitment is made before the game starts. Without any prior commitment and without any random shock or mistake, it is possible that agents knowingly and willingly reach an unstable matching and stay there. A blocking pair exists in the final matching, yet it does not form. If the firm in the blocking pair makes an offer to its blocking partner, the offer may be accepted in this case, but since the worker does not commit, the firm may lose him in the future. In the equilibrium, actually, the blocking triggers a chain reaction in which a firm that lost a worker takes another worker from another firm. In the end, the firm who started the process ends up losing the blocking partner. Anticipating this, the firm in the blocking pair does not make an offer to its blocking partner.

It is even possible that the equilibrium outcome in one-sided commitment makes all workers worse off than in any stable matching. This is interesting since job protection is presumably intended to benefit workers. If job protection is lifted, every equilibrium yields a stable matching and all workers can be made better off. At the same time that job protection can make workers worse off, firms can in fact benefit from job protection. One possible explanation is that job protection makes a position more attractive if workers display strong aversion to unemployment (even for some periods).

There are several papers that also study decentralized matching markets. Haeringer and Wooders (2011) study a dynamic game with a few critical differences from ours. In their model, the payoffs depend only on the matching that obtains at the end. This assumption has two elements. First, there are no time preferences: the time at which the final matching is determined does not affect the payoff. Second, the matchings that are formed temporarily during the game do not affect the payoff directly. By contrast, in the current paper, a matching is formed in every period and the agents accumulate payoffs in every period. Moreover, an agent evaluates a (infinite) stream of payoffs using the discounted sum. Our model is natural for describing the dynamics of matching markets over time, where workers and firms care about not only with whom they are matched but also when and for how long.

Konishi and Sapozhnikov (2005) and Niederle and Yariv (2007) also consider similar models with realistic details on salary or the length of offers. But, as in Haeringer and Wooders, agents who get matched exit the game.

There is also a literature of search models of matching. Our model does not belong to this category: we assume that market participants know each other well and do not have to rely on random encounters. Within the search-model literature, Adachi (2003) is particularly close to our paper since it is also based on the Gale–Shapley model. But, as in the above papers, he also deals only with the case where agents exit as soon as they get matched. Further, to make the distribution of agents stationary, he uses a "replacement assumption": when agents exit, their clones take their places. He shows that, as the discount factor goes to one, the set of equilibrium outcomes converges to the set of stable matchings. As we show, the result does not hold if agents know each other and no replacement arrives, even if the discount factor is close to one.

Blum, Roth, and Rothblum (1997) consider an algorithm that finds a stable matching when some of the agents are initially matched. Towards the end of the paper, the authors study a game similar to ours with one-sided commitment. However, the payoff realizes only once and depends only on the final matching. The paper characterizes Nash equilibria in "preference strategies," where each agent uses a single (possibly false) preference ordering for decisions at all nodes [see also Pais (2008) for an extension].

Alcalde and Romero-Medina (2000), in the context of mechanism design theory, study a game where only one round of offers and replies takes place. That is, firms make offers, workers reply, and the game ends. They show that this game achieves (or implements) stable matchings [see also Alcalde, Pérez-Castrillo, and Romero-Medina (1998)].

There is also a strand of literature that studies, within the original static model, whether a myopic adjustment process based on pairwise blocking converges to a stable matching. Knuth (1976) gives an example showing that a sequence of successive myopic blockings may form a cycle and never reach a stable matching. Roth and Vande Vate (1990) show that, if a blocking pair is chosen randomly at each step of the process, the process reaches a stable matching with probability one. The sequential blocking process implicitly assumes that no commitment is made by the agents. The critical feature of the process is that the agents are myopic: in each step, the acting blocking pair behaves as if the process terminates in their turn.

Our dynamic matching game can also be related to the literature of dynamic coalition-formation games [see, e.g., Chatterjee, Dutta, Ray, and Sengupta (1993), Bloch (1996), Ray and Vohra (1999), Bloch and Diamantoudi (2011)] to the study of decentralized matching market. With a dynamic game of coalition formation, in each period a proposer is selected among the set of active players using a protocol (often exogenously given) to make a proposal of forming a coalition and prospective members respond sequentially to such a proposal. This coalition is formed if all prospective members accept the proposal. Therefore, in each period, at most one coalition can form. In contrast, in our dynamic matching game, all active firms can simultaneously make offers and several firm-worker pairs can form in a single period. Another feature of our game is the different commitment structures that naturally arise in a labor market.

In the next section, we introduce a standard static model of matching. Section 3 defines our dynamic game of matching. Sections 4–6 study respectively the three aforementioned commitment structures. A short conclusion follows. The appendix contains some details omitted from the main text.

2 Static Matching Problem

In a static *matching problem*, introduced by Gale and Shapley (1962), there are two disjoint finite sets F of *firms* and W of *workers*. An *agent* refers to either a firm or a worker. Each agent i has a utility function u_i such that

$$u_f \colon W \cup \{f\} \to \mathbb{R} \quad \text{for all } f \in F,$$
$$u_w \colon F \cup \{w\} \to \mathbb{R} \quad \text{for all } w \in W.$$

Here, $u_i(j)$ denotes agent i's utility of being matched with agent j; $u_i(i)$ is the agent's utility of being unmatched. We normalize utilities so that $u_i(i) = 0$ for all i. If $u_i(j) \geq 0$, we say j is *acceptable* to i. We assume strict preferences: $u_i(j) = u_i(k)$ only if $j = k$.

For simplicity, we assume that each firm has only one position. A *matching* is then a function $\mu \colon F \cup W \to F \cup W$ such that (i) for all $f \in F$, $\mu(f) \in W \cup \{f\}$, (ii) for all $w \in W$, $\mu(w) \in F \cup \{w\}$, and (iii) for all $i, j \in F \cup W$, if $\mu(i) = j$ then $\mu(j) = i$. Here, $\mu(i)$ denotes the agent with whom i is matched. If $\mu(i) = i$, then i is not matched with anyone. Let $\mu_\emptyset$ denote the matching in which no one is matched.

A matching μ is *individually rational* if, for all $i \in F \cup W$, $\mu(i)$ is acceptable to i. A matching μ is *blocked* by a pair $(f, w) \in F \times W$ if

$$u_f(w) > u_f(\mu(f)),$$
$$u_w(f) > u_w(\mu(w)).$$

That is, f and w both prefer each other to their partners under μ. A matching μ is *stable* if it is individually rational and has no blocking pair. By Gale and Shapley (1962), a stable matching exists for any matching problem.

3 Dynamic Matching Game

We consider a situation where agents are matched every period in a decentralized fashion. A *dynamic matching game*, parameterized by a list $(F, W, (u_i, \delta_i)_{i\in F\cup W}, F_c, W_c)$, is defined as follows.

3.1 Periods and Payoffs

Time periods are discrete and indexed by $t = 1, 2, 3, \ldots$. In each period, agents derive a payoff from the realized matching. The period-payoff function for agent i is u_i introduced above and is time-invariant. Each agent i maximizes the discounted sum of period-payoffs,

$$\sum_{t=1}^{\infty} \delta_i^{t-1} u_i(\mu^t(i)),$$

where $\delta_i \in (0, 1)$ is the discount factor and μ^t is the realized matching in period t. The way in which μ^t is determined is described in the following sections.

3.2 Active Agents

At the beginning of each period $t = 1, 2, \ldots$, all agents observe the matching realized in the previous period, denoted by μ^{t-1}. We assume $\mu^0 = \mu_\emptyset$, i.e., no one is matched before the initial period.

The matching μ^{t-1} determines the set of firms and workers who are not able to move in period t. The set of *inactive firms* in period t is given by $F_c(\mu^{t-1}) \subseteq F$. These firms have committed themselves to their employees in μ^{t-1}. During period t, therefore, they can neither dismiss their employees nor hire new ones. That is, their current employees have tenure and their jobs are protected. $F \setminus F_c(\mu^{t-1})$ is the set of *active firms*, which have not made any commitment and therefore retain the right to dismiss their current employees if they have any. Similarly, let $W_c(\mu^{t-1})$ denote the set of *inactive workers* in period t, who cannot switch their employers in period t. Its complement, $W \setminus W_c(\mu^{t-1})$, is the set of *active workers*, who have no commitment and can leave their current employers if they are employed.

We consider the following three specifications for F_c and W_c.

Case 1: *No commitment*. All firms and workers are active regardless of the previous matching: $F_c(\mu^{t-1}) = W_c(\mu^{t-1}) = \emptyset$. Thus, firms can dismiss their employees, and workers can leave their current employers. That is, all labor contracts expire in one period.

Case 2: *Two-sided commitment*. All matched agents are inactive:

$$F_c(\mu^{t-1}) = \{f \in F : \mu^{t-1}(f) \neq f\},$$
$$W_c(\mu^{t-1}) = \{w \in W : \mu^{t-1}(w) \neq w\}.$$

Thus, once a firm and a worker are matched, they stay so permanently.

Case 3: *One-sided commitment*. All the matched firms are inactive, while all workers are active:

$$F_c(\mu^{t-1}) = \{f \in F : \mu^{t-1}(f) \neq f\},$$
$$W_c(\mu^{t-1}) = \emptyset.$$

Thus, workers cannot be dismissed but they may switch to other firms.

3.3 Period-Game

In every period, the agents play the following two-stage game.

In the first stage, every firm simultaneously makes an offer to at most one worker. An active firm can make an offer to any worker while an inactive firm has no option but to keep its employee under μ^{t-1}. For convenience, we treat inactive firms as if they make new offers to their current employees (i.e., renewal offers). Thus, firm f's action, denoted by o_f, is constrained by

$$\begin{cases} o_f \in W \cup \{f\} & \text{if } f \notin F_c(\mu^{t-1}), \\ o_f = \mu^{t-1}(f) & \text{if } f \in F_c(\mu^{t-1}), \end{cases}$$

where $o_f = f$ means that f makes no offer to any worker. Let $O_f(\mu^{t-1})$ denote the set of admissible actions for f.

In the second stage, each worker w privately observes the offers made to her in the first stage, denoted $O_w \equiv \{f \in F : o_f = w\}$. As noted above, O_w includes the renewal offer from the current employer if w has tenure. Workers do not observe any offer made to other workers in the current period. As noted above, each worker observes the entire matching realized in previous periods. Given these observations, each worker simultaneously accepts at most one offer. An active worker w can accept any offer or reject all offers. Inactive workers have no choice but to accept the renewal offers from their current employers. Thus, worker w's response, denoted by r_w, is constrained by

$$\begin{cases} r_w \in O_w \cup \{w\} & \text{if } w \notin W_c(\mu^{t-1}), \\ r_w = \mu^{t-1}(w) & \text{if } w \in W_c(\mu^{t-1}). \end{cases}$$

Let $R_w(\mu^{t-1}, O_w)$ denote the set of admissible responses for w.

Given the actions of firms and workers, the matching in period t, denoted μ^t, is determined by

$$\mu^t(w) = r_w \quad \text{for all } w \in W.$$

3.4 Histories and Strategies

A *history* at the beginning of period t is an ordered list of past actions, given by

$$h^t = \Big((o_f^\tau)_{f\in F}, (r_w^\tau)_{w\in W}\Big)_{\tau=1}^{t-1},$$

where o_f^τ is the offer made by firm f in period $\tau = 1, \ldots, t-1$ and r_w^τ is the reply of worker w in period τ. After the first stage of period t, a history is given by $(h^t, (o_f^t)_{f\in F})$, where h^t is a history at the beginning of this period and $(o_f^t)_{f\in F}$ is the profile of offers made in this period.

The profile of replies $(r_w^\tau)_{w\in W}$ in h^t contains the same information as the realized matching μ^τ, which becomes public information. Since offers are private information, players do not have complete information about the history. Each player observes only his private history, which we now define.

A *private history for firm* f in period t is an ordered list

$$h_f^t = (\mu^0 = \mu_\emptyset,\ o_f^1, \mu^1,\ o_f^2, \mu^2, \ldots, o_f^{t-1}, \mu^{t-1}),$$

where μ^τ is the matching realized in period τ. While μ^τ is public information, o_f^τ is private information. Let H_f^t denote the set of private histories for f in period t. Let $H_f \equiv \cup_{t=1}^\infty H_f^t$ denote the set of all private histories for f.

A (pure) strategy of firm f is a function $\sigma_f\colon H_f \to W \cup \{f\}$ such that for all $h_f^t \in H_f$, $\sigma_f(h_f^t) \in O_f(\mu^{t-1})$, where μ^{t-1} is the last entry of h_f^t.

Similarly, a *private history for worker* w in the middle of period t (when she makes a decision) is an ordered list

$$h_w^t = (\mu^0 = \mu_\emptyset,\ O_w^1, r_w^1, \mu^1,\ O_w^2, r_w^2, \mu^2, \ldots, O_w^{t-1}, r_w^{t-1}, \mu^{t-1},\ O_w^t),$$

where O_w^τ is the set of offers made to w in period τ (including a renewal offer if any) and r_w^τ is her reply in that period. Let H_w^t denote the set of all private histories for w in period t. Let $H_w \equiv \cup_{t=1}^\infty H_w^t$ denote the set of all private histories for w. A strategy of worker w is then a function $\sigma_w\colon H_w \to F \cup \{w\}$ such that, for all $h_w^t \in H_w$, $\sigma_w(h_w^t) \in R_w(\mu^{t-1}, O_w^t)$, where μ^{t-1} and O_w^t are the last two entries of h_w^t.

A strategy profile $\sigma = (\sigma_i)_{i\in F\cup W}$ determines the payoff for each agent in the dynamic game. We limit ourselves to (pure-strategy) sequential equilibria in stationary strategies, where each agent's strategy depends only on the payoff-relevant state of the game, as we now define formally.

3.5 Stationary Strategies

In our dynamic matching game, the state variable is the matching realized in the previous period. That is, given that the realized matching in the previous period is the same, any continuation game from the beginning of any period is equivalent, in the payoff-relevant aspects, regardless of the calendar time and the exact sequence of past actions.

In what follows, we rarely mention continuation games from the middle (i.e., the second stage) of a period. Thus, when we say a *continuation game*, it starts from the beginning of a period, unless we state otherwise.

Depending on the commitment structure, distinct matchings may induce equivalent continuation games. We write $\mu \sim \mu'$ if μ and μ' induce equivalent continuation games, and we say that μ and μ' are *continuation equivalent*. The equivalence relation $\sim$ depends on the commitment structure of the game as follows.

In the no-commitment case, all matchings are continuation equivalent: $\mu \sim \mu'$ for all μ, μ'. In the absence of commitment, the continuation game is the same regardless of what happened in the previous periods.

In the two-sided commitment case, two matchings are continuation equivalent if and only if the set of unmatched agents is identical: $\mu \sim \mu'$ if and only if $\{i \in F \cup W : \mu(i) = i\} = \{i \in F \cup W : \mu'(i) = i\}$. The agents who have been matched cannot change their partner in the rest of the game. So what matters for the remaining agents is the set of remaining agents. How the matched agents are matched is irrelevant.[1]

In the one-sided commitment case, no two matchings are continuation equivalent: $\mu \sim \mu$ if and only if $\mu = \mu'$. Even if the set of matched agents is the same, the continuation game depends on how the agents are currently matched.

With the equivalence relation, we can define stationary strategies as follows. A firm f's strategy σ_f is *stationary* if for any two private histories $h_f = (\ldots, \mu)$ and $h'_f = (\ldots, \mu')$ (possibly with different lengths), if $\mu \sim \mu'$ then $\sigma_f(h_f) = \sigma_f(h'_f)$. For workers' strategies, there is another requirement saying that the set of offers received in the current period is also identical. That is, a worker w's strategy σ_w is *stationary* if for any two private histories $h_w = (\ldots, \mu, O_w)$ and $h'_w = (\ldots, \mu', O'_w)$, if $\mu \sim \mu'$ and $O_w = O'_w$ then $\sigma_w(h_w) = \sigma_w(h'_w)$. A *stationary equilibrium* is a sequential equilibrium in which everyone's strategy is stationary.[2]

4 No Commitment

We first consider the case where no one makes any commitment. The following result states that, in the absence of commitment, the static notion of stability captures the outcomes of stationary equilibria.

Proposition 1 *Consider any dynamic matching game with no commitment. The realized matching in any stationary equilibrium is identical in all periods and is stable. Conversely, for any stable matching, there exists a stationary equilibrium*

[1]Technically speaking, two distinct matchings with the same set of unmatched agents induce different continuation games, since the matched agents' unique admissible action is labeled differently; where each matched firm has to make a renewal offer depends on the matching. But since the matched agents have no choice, we focus on the continuation game among the unmatched agents.

[2]A stationary equilibrium has a property that, at any history at the beginning of a period, on or off the equilibrium path, the remaining strategy profile is a stationary equilibrium of the continuation game.

that yields this matching in every period.

Proof. See the appendix.

With no commitment, the history leading to the current period does not change the continuation game and hence is ignored by the agents in stationary equilibria. Stated differently, what happens in the current period does not affect the outcome in the future. Because of this independence, agents disregard the future and behave as in the static model.

A useful fact is that, for a fixed preference profile, the set of unmatched agents is identical in all stable matchings (Roth and Sotomayor, 1990). Thus, workers and firms that are unmatched in one stationary equilibrium are unmatched in all stationary equilibria.

5 Two-Sided Commitment

5.1 Richness of Equilibria

We now consider the case where both sides of the market commit to their employment relationships. We first show that every stable matching is the outcome of some stationary equilibrium.

Proposition 2 *Consider any dynamic matching game with two-sided commitment. (i) For any stable matching, there exists a stationary equilibrium that yields it in every period. (ii) For each subset $S \subseteq F \cup W$, choose any stable matching μ^S within S. Then there exists a stationary equilibrium such that, for any history at the beginning of a period, if S denotes the set of active agents, the continuation equilibrium yields μ^S in every period.*

Proof. See the appendix.

In the equilibria of Proposition 2, the matching is determined completely in the initial period. The next result says that this is not always the case.

Proposition 3 *There exists a dynamic matching game with two-sided commitment that admits a stationary equilibrium where some pairs are matched in period 2 or later.*

To show why delay is possible in equilibrium, consider the following example with 4 firms and 4 workers, whose ordinal preferences are given by

f_1	f_2	f_3	f_4	w_1	w_2	w_3	w_4
w_1	w_3	w_3	w_4	f_4	f_3	f_1	f_2
w_3	w_2	w_4	w_3	f_1	f_2	f_4	f_3
$\vdots$	w_4	w_1	w_2	f_3	f_4	f_2	f_1
	$\vdots$	w_2	w_1	f_2	f_1	f_3	f_4
		f_3	f_4	w_1	w_2	w_3	w_4

If the agents are sufficiently patient, there exists a stationary equilibrium in which $\{f_1, w_1\}$ and $\{f_2, w_2\}$ are matched in period 1 while $\{f_3, w_3\}$ and $\{f_4, w_4\}$ are matched in period 2. The complete description of the equilibrium is given in the appendix. Here we describe informally how the equilibrium works. In the initial period, every firm makes an offer. Firms f_1 and f_2 make offers to w_1 and w_2, respectively, and get accepted. Firms f_3 and f_4 also make offers to w_1 and w_2, respectively, and get rejected. In the second period, f_3 and f_4 make offers to w_3 and w_4, respectively, and get accepted.

The offers made by f_3 and f_4 in the first period are rejected in equilibrium. Why do not f_3 and f_4 make offers to w_3 and w_4 directly in the first period? The answer is simply that if f_3, for example, deviates in the first period and makes an offer to w_3, the offer will be rejected. As we will see shortly, the continuation play will match f_3 with w_4. Since f_3 prefers w_3, the deviation does not make it better off.

Why does w_3 reject f_3 when f_3 deviates? The key is that, as the recipient of f_3's offer, w_3 knows that w_1 did not receive an offer from f_3. Without an offer from f_3, the equilibrium prescribes w_1 to reject f_1. We shall show why w_1 behaves in this way, but first let us see why w_3 rejects f_3. If w_3 rejects f_3, the behavior of w_1 implies that the only $\{f_2, w_2\}$ is matched in the current period. In the next period, the set of active agents is $F \cup W \setminus \{f_2, w_2\}$ and the continuation strategies prescribe them to be matched immediately as $[\{f_1, w_1\}, \{f_3, w_4\}, \{f_4, w_3\}]$, which is a stable matching for the 6 agents. In this matching, w_3 gets her second choice, while f_3 is only her fourth choice. If w_3 is patient enough, therefore, she prefers to wait for her second choice.

We now explain why w_1 rejects f_1 in the event that w_1 did not receive an offer from f_3. The key to the answer is that, in this particular event, w_1 believes that f_3 made an offer to w_4. Since offers are private information, w_1 does not know to whom f_3 has made an offer or whether f_3 has made an offer at all. In the equilibrium we construct, w_1 holds the particular belief we described. The belief is not unreasonable since, for f_3, making an offer to w_4 is the second best response. The action is actually the best response if f_3 is sufficiently impatient. If f_3 made an offer to w_4 indeed, the equilibrium prescribes w_4 to accept the offer. Therefore w_1 believes that if she rejects f_1, the set of active agents in the next period will be $\{f_1, f_4, w_1, w_3\}$ and the continuation strategy will prescribe the agents to be matched immediately as $[\{f_1, w_3\}, \{f_4, w_1\}]$. The outcome is a stable matching for the 4 agents, where both workers get their first choice. Thus, by rejecting f_1 in the initial period, w_1 can get her first choice in the next period. If w_1 is patient enough, therefore, she prefers to wait for her first choice.

The equilibrium outcome, where $\{f_i, w_i\}$ is matched for all i, is not a stable matching, being blocked by f_2 and w_3. This brings us to

Proposition 4 *There exists a dynamic matching game with two-sided commitment that admits a stationary equilibrium whose final matching is not stable.*

The question is why the blocking pair does not form. Why does not f_2 make an offer to w_3 in the first period? They prefer each other, and once they are matched, they commit to each other. The answer is that, if f_2 makes an offer to w_3, the offer will be rejected. In the continuation play, f_2 will be matched with w_2. Thus the deviation only delays the matching with the same worker. The question is then: why does w_3 reject f_2? The answer is that if w_3 rejects f_2, the set of active agents in the next period is $F \cup W \setminus \{f_1, w_1\}$ and the continuation strategy prescribes them to be matched as $[\{f_2, w_2\}, \{f_3, w_4\}, \{f_4, w_3\}]$, which is a stable matching for the 6 agents. In this matching, w_3 gets her second choice, while her blocking partner, f_2, is only her third choice. If w_3 is patient enough, therefore, she rejects the blocking partner in order to be matched with an even more desirable firm in the next period.

The above description shows that the incentives that support delay and instability in the equilibrium have a similar structure. Delay and instability prevail in equilibrium because a firm's attempt to deviate from the equilibrium to either get the same worker earlier or get together with the blocking partner is thwarted by a rejection from the worker. The worker rejects the firm since doing so affects the continuation play in the worker's favor. This is the case even though offers are observed privately and the rejected offer is a deviation from an equilibrium. Rejecting a private out-of-equilibrium offer can change the continuation play since the deviation by the firm also affects the behavior of the worker who did not receive an offer from this firm.

These results of delay and instability rely on the standard assumption that the continuation equilibrium can vary with the state in any way. There is no a priori restriction on the relationship between the current state and the equilibrium selection of the continuation game. While the assumption is standard, it may be allowing for too much freedom. To identify realistic equilibria, we may need to impose restrictions on the relation between the continuation equilibrium and the state. While a general form of restrictions is beyond the scope of this paper, the next section offers a restriction that appears natural in our specific model.

5.2 Consistency

Consistency is a restriction on the relation between the state and the continuation play. To see the idea, suppose that there are 3 firms and 3 workers and the equilibrium prescribes them to be matched as $[\{f_1, w_1\}, \{f_2, w_2\}, \{f_3, w_3\}]$. Consider a history where only the pair $\{f_1, w_1\}$ has been formed. Stationarity alone imposes no restriction on how the remaining four agents will be matched in the continuation play. However, a natural expectation is that the remaining agents will be matched as $[\{f_2, w_2\}, \{f_3, w_3\}]$.

Formally, for any stationary strategy profile σ and any subset $S \subseteq F \cup W$, let $m(\sigma, S)$ denote the final matching within S under σ after a history at the beginning of a period when S is the set of active agents.

Definition. A stationary strategy profile σ is *consistent* if for any subset $E \subseteq F$ of firms that are matched with workers under $\mu \equiv m(\sigma, F \cup W)$, i.e., $\mu(f) \in W$ for all $f \in E$, if we define

$$S \equiv (F \setminus E) \cup (W \setminus \{\mu(f) : f \in E\}),$$

which is just the set of remaining agents when the firms in E and their partners under μ exit, then $m(\sigma, S) = \mu|_S$, where $\mu|_S$ denotes the restriction of μ on S, i.e., $\mu|_S(i) = \mu(i)$ for all $i \in S$.

Thus, if μ is the final matching realized in the entire game, then for any history (on or off the path) at the beginning of a period, if all the pairs that have been matched in the previous periods are those matched in μ, then the remaining agents are matched according to μ in the continuation game.[3]

We define a *consistent stationary equilibrium* to be a stationary equilibrium in which the strategy profile is consistent.

Proposition 5 *In any dynamic matching game with two-sided commitment, the final matching in any consistent stationary equilibrium is stable.*

Proof. Let μ be the final matching (namely, $\mu = m(\sigma, F \cup W)$) and suppose that it is unstable. Let (f, w) be a blocking pair for μ. We choose a pair so that f is the most preferred firm to form a blocking pair with w. Then consider a subset $T = \{f, \mu(f), w, \mu(w)\} \cup \{i \in F \cup W : \mu(i) = i\}$. Within this set T, f is the first choice for w among the firms for which w is acceptable. Therefore, if T is the set of active agents and f makes an offer to w, w will accept. Thus, in the continuation game, f is matched with w or someone better. However, consistency requires f to be matched with $\mu(f)$ in the continuation game. This is a contradiction since f prefers w to $\mu(f)$. □

The concept of consistency is closely related to what is called the "reduced-game property" (also called "consistency") in social choice theory; see, e.g., Thomson (2011). The reduced-game property is satisfied by stable matchings: if μ is a stable matching and some of the matched pairs in μ are removed from the market, the restriction of μ on the remaining agents is also a stable matching for the remaining agents (or the reduced game).[4] This property of stable matchings, together with Proposition 2 (ii), implies that there exists a stationary equilibrium that is consistent.

Proposition 6 *In any dynamic matching game with two-sided commitment, for any stable matching μ, there exists a consistent stationary equilibrium that yields μ in every period.*

[3]Note that consistency is imposed only between the entire set $F \cup W$ and subsets S. A natural extension is to impose consistency between any two subsets $S, T \subseteq F \cup W$, but such an extension leads to the non-existence of consistent equilibrium: the proof is given in Appendix A.4.

[4]For the static marriage problem, the reduced-game property is studied by Sasaki and Toda (1992).

Proof. Let μ be any stable matching. For any proper subset $S \subsetneq F \cup W$ that is obtained by removing some of the matched pairs under μ, let $\mu^S \equiv \mu|_S$, which is a stable matching within S. For all other subsets $S \subsetneq F \cup W$, let μ^S be any stable matching within S. Given the list of matchings $\{\mu^S\}_{S \subseteq F \cup W}$ with $\mu^{F \cup W} \equiv \mu$, we consider a stationary equilibrium given in Proposition 2 (ii). By the construction of μ^S, the equilibrium is consistent. □

The last two results yield the following characterization of the outcomes of consistent equilibria:

Corollary 1 *In any dynamic matching game with two-sided commitment, a matching μ is the final matching of a consistent stationary equilibrium if and only if μ is a stable matching.*

Now that the final matchings are characterized, we next consider the timing of matching. We can show that if agents are sufficiently patient, they are matched immediately in consistent stationary equilibria.

Proposition 7 *For any dynamic matching game with two-sided commitment, there exists $\underline{\delta} \in (0, 1)$ such that for any $\delta \in [\underline{\delta}, 1)$ and any consistent stationary equilibrium, no pair is formed in period 2 or later.*

Proof. See the appendix.

To summarize, we obtained two conclusions for the case of two-sided commitment. If we consider all stationary equilibria, they may involve delay, and the set of equilibrium matchings is a superset of the set of stable matchings and may contain unstable matchings. On the other hand, if we consider only stationary equilibria that are consistent, they do not involve delay if players are patient, and the set of equilibrium outcomes is equivalent to the set of stable matchings.

6 One-Sided Commitment

We now turn to the one-sided commitment case. As in academic job markets for seniors, workers are offered tenured positions but do not commit themselves to their employers. In stationary equilibria, the payoff-relevant state is the matching in the previous period.

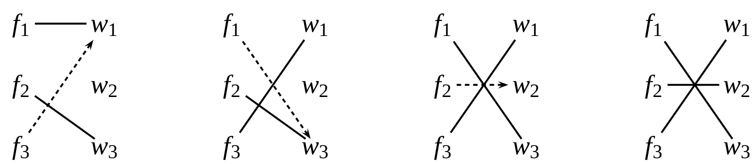
Proposition 8 *There exists a dynamic matching game with one-sided commitment that admits a stationary equilibrium in which, in every period, the realized matching is unstable and such that every stable matching is a Pareto improvement for the workers.*

Proof. To prove our result, consider the following market with 3 firms and 3 workers, whose ordinal preferences are given by

f_1	f_2	f_3	w_1	w_2	w_3
w_1	w_3	w_3	f_3	f_2	f_1
w_3	w_2	w_1	f_1	f_3	f_2
w_2	w_1	w_2	f_2	f_1	f_3
f_1	f_2	f_3	w_1	w_2	w_3

To simplify the exposition, we here assume that all workers are myopic: $\delta_w = 0$ for all w. The result itself also holds if workers are very patient, as we shall discuss. By the assumption of myopia, workers simply take the best acceptable offers every period. We show that there exists a stationary equilibrium in which each f_i makes an offer to w_i in the initial period and the offers are accepted immediately. Thus, no firm can move thereafter. The outcome is not a stable matching since it is blocked by f_2 and w_3. The complete description of the equilibrium strategy profile is given in Figure 1. Here we highlight why the (unique) blocking pair does not form.

To see why, suppose that f_2 makes an offer to w_3 in the first period. Since (f_2, w_3) is a blocking pair and workers are myopic, w_3 accepts. Therefore, in contrast to the two-sided commitment case, a firm's offer to its blocking partner does not receive an immediate rejection. However, since workers make no commitment, f_2 may lose w_3 later. The continuation play proceeds as depicted in the following figure.



Period 2 Period 3 Period 4 Periods 5–

The firm that can move in period 2 is f_3. Having failed to get its first choice, f_3 makes an offer to its second choice, w_1. The offer is accepted since f_3 is the first choice for w_1. In the next period, the only active firm is f_1. Having lost its first choice, f_1 makes an offer to its second choice, w_3. The offer is accepted since f_1 is the first choice for w_3. In period 4, f_2 has no choice but to make an offer to w_2 since the other workers are with their first choices. The matching is then completed. The result is actually the unique stable matching. The deviation by f_2 therefore helps the market reach a stable matching.

The question is whether f_2 gains from the deviation. While f_2 ends up with the same worker as in equilibrium, the deviation induces a different sequence of matchings. After the deviation, f_2 is matched with its first choice for two periods and has no worker for one period. The subsequent periods are not affected. Thus, f_2 does not gain from the deviation if and only if

$$(1 + \delta_{f_2}) u_{f_2}(w_3) \leq (1 + \delta_{f_2} + \delta_{f_2}^2) u_{f_2}(w_2).$$

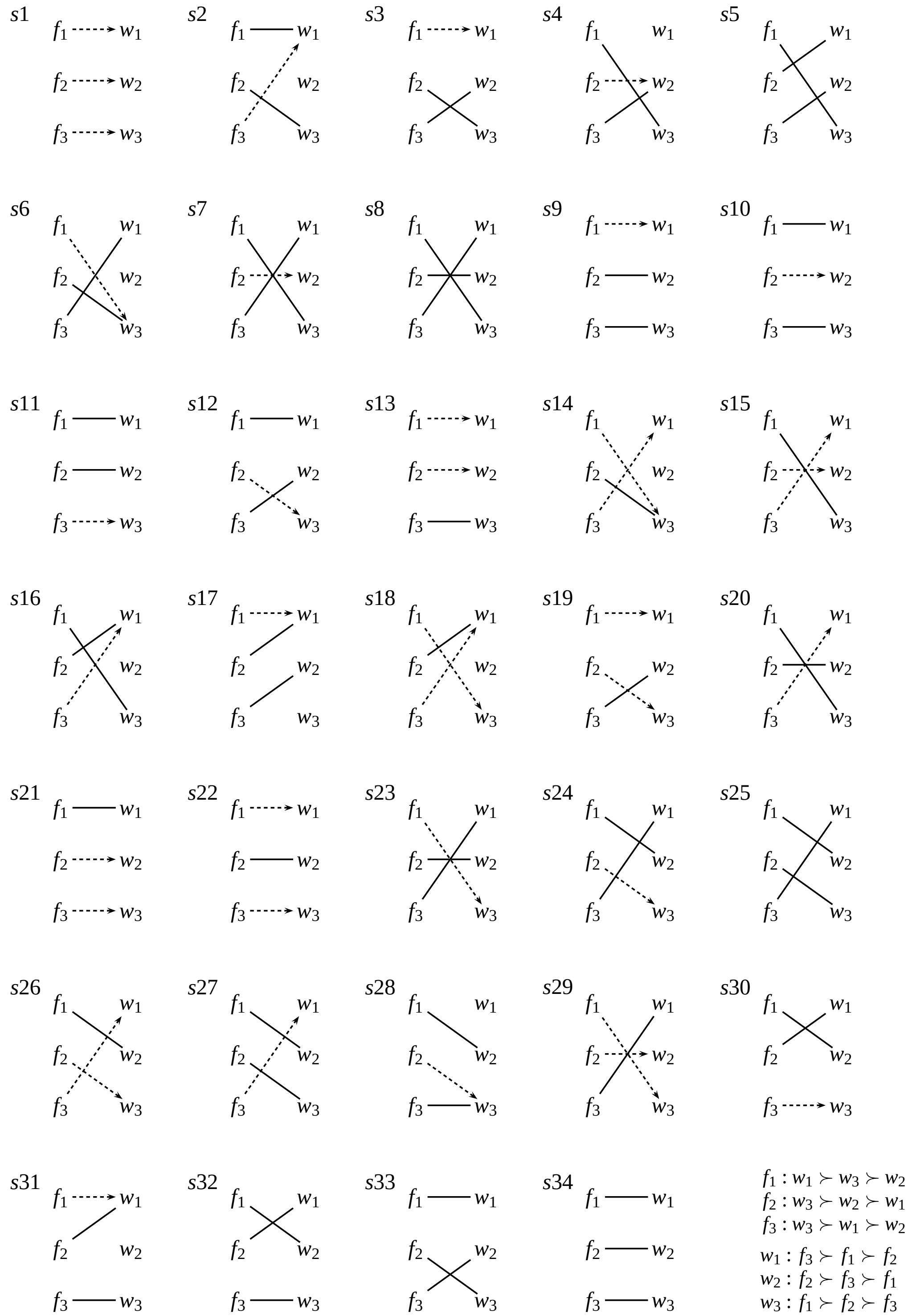


Figure 1: Firms' strategies in the proof of Proposition 8. The solid lines denote the matching at the beginning of the period and the dashed arrows denote the offers prescribed by our equilibrium strategies.

The inequality holds if $u_{f_2}(w_3) - u_{f_2}(w_2)$ is sufficiently small. Thus the deviation does not make the firm better off if the marginal gain from getting a better worker is sufficiently small.

The same argument explains the incentives of f_2 in states $\{s10, s13, s21\}$ (see Figure 1). In this example, the deviation affects f_2 only in the short run: f_2 gets the same worker eventually. As we shall discuss shortly, we can also construct a similar example where the deviation makes f_2 worse off permanently: after the deviation, f_2 ends up with a worker who is less desirable than the worker it gets in equilibrium.

The incentives of f_2 in the other parts of the strategy are simple. The firm makes an offer to w_2 in $\{s4, s7, s15, s29\}$, but the reason is simply that w_3—the firm's first choice—has been or is being approached by her first choice. In the other states where f_2 can move, it makes an offer to the first choice and gets accepted.

The incentives of f_1 and f_3 are straightforward and no condition is necessary on their patience or payoff function. First, f_1 can always secure its second choice (w_3) since it is the first choice for that worker. In the equilibrium, f_1 does not make an offer to its first choice (w_1) only when the worker has been or is being approached by her first choice ($s6$, $s14$, $s18$, $s23$, and $s29$). Similarly, f_3 can always secure its second choice (w_1), but it is not liked by its first choice (w_3). In the equilibrium, f_3 does not make an offer to its first choice only when the first choice has been or is being approached by other firms ($s2$, $s14$–$s16$, $s18$, $s20$, $s26$, and $s27$). □

Intuitively, the unstable matching is sustained as equilibrium outcome since any attempt by f_2 to get the blocking partner succeeds only temporarily and backfires eventually. The temporal success for f_2 in getting w_3 intensifies the competition among the firms. In the end, f_2 loses w_3 to f_1. If the loss from having no worker is large relative to the marginal gain from the better worker, the net effect of initiating a recruiting war is negative.

A comparison between the equilibrium and the unique stable matching ($s8$) reveals that none of the workers prefers the equilibrium outcome. While w_2 is indifferent, the other workers prefer the stable matching. This is interesting since workers are the ones who are protected by tenure. Without job protection, the stable matching is the unique equilibrium outcome. In this example, the workers are better off if job protections are removed. Note that the equilibrium matching Pareto dominates the stable matching for firms so firms are those who benefit from job protection. In fact, the equilibrium matching is the matching given by the firm-side top trading cycle and hence Pareto efficient for firms.[5]

The conclusion of Proposition 8 remains if workers and firms are very patient. In Appendix A.6, we construct an example showing that the proposition holds when $\delta_i \to 1$ for all $i \in F \cup W$. However, the construction is considerably more involved since we need to specify each worker's action contingent on every possible set of offers. Even with the small number of agents, constructing an equilibrium strategy

[5]We are grateful to a referee for this observation.

profile is not easy since there are 34 states. The example is therefore relegated to the appendix. In this example, if a firm deviates by making an offer to the blocking partner, it gets the worker only temporarily and ends up with a worker who is strictly less desirable than the permanent employee in the equilibrium.

7 Conclusion

We considered a dynamic matching game in which firms and workers interact repeatedly in a decentralized job market. The main question was whether a decentralized matching market generates a stable matching of Gale and Shapley (1962) and how the answer varies with the commitment structure of the market. Without commitment, we show that every stationary equilibrium matching is stable and every stable matching can be sustained by a stationary equilibrium. Once commitment is introduced, this equivalence breaks down. With two-sided commitment, there exists a preference profile such that, at a stationary equilibrium, the final matching is unstable and is not formed in the first period. However, for all stationary equilibria that satisfy consistency, the final matching is stable; and any stable matching is the outcome of a stationary equilibrium that is consistent. When only firms commit (i.e., there is job security), the final matching at a stationary equilibrium may be unstable and there even exists a case where the equilibrium matching makes every worker worse off and, at the same time, every firm better off than any stable matching.

These results depend on our assumptions. Let us mention a few of them here. First, we considered only pure actions. Once randomization is allowed, the results of equivalence between equilibrium matchings and stable matchings break down. With no commitment, there exists a preference profile where firms randomize in a stationary equilibrium and the realized matching may not be even Pareto efficient with positive probability. When both sides commit, we showed that the equivalence between equilibrium matchings and stable matchings holds for pure-strategy equilibria that are consistent. But once randomization is allowed, it is not clear how to extend the definition of consistency.

Second, we assumed that a player's period-utility function $u_i(\cdot)$ stays the same over time. If a player's period-utility function changes over time, our definition of consistency, which ignores timing, is not very reasonable. On the other hand, the equivalence without commitment extends easily to time-varying utilities if the profile of utility functions in the current period is public information.

Third, our dynamic matching game builds on the classical one-to-one matching model without monetary transfers (i.e., salary negotiations). Extension to a many-to-one matching model with monetary transfers such as the one in Crawford and Knoer (1981) and Kelso and Crawford (1982) would certainly bring us closer to modeling real labor market dynamics. An employment relationship in such a setting necessarily specifies the salary level. Recently, salary competition in matching model has been brought into focus by, for example, Bulow and Levin (2006) and

Crawford (2008). It would be very interesting to study how commitment structure affects both the matchings and the salaries in equilibrium.

Finally, our comparison of commitment structures is an exercise of comparative statics. It would be interesting to study a model in which each player chooses whether or not to commit, that is, the commitment structure is determined endogenously.

A Appendix: Omitted Proofs

A.1 Proof of Proposition 1

To prove the first statement, consider any stationary equilibrium. First, suppose, by way of contradiction, that the equilibrium yields an unstable matching μ^t in some period t. Let $\{f, w\}$ be a blocking pair for μ^t. Since the firms' strategies are stationary, w's action in period t does not affect the offers she will receive in the subsequent periods. This implies that, in period t, worker w's best action is to accept the most preferred acceptable offer. Since w prefers f to $\mu^t(w)$, it follows that w does not receive an offer from f in this period. Suppose then that, in period t, firm f deviates from the equilibrium and makes an offer to w. By the observation above, w will accept the offer and this has no influence over the other agents' strategies in the continuation game. Therefore, f gains from the deviation, a contradiction.

The above paragraph shows that the realized matching is stable in every period. Since firms' strategies are stationary and pure, the realized matching is the same in every period.

To prove the converse, choose any stable matching μ and consider the following strategy profile: in every period, each firm f makes an offer to $\mu(f)$ and each worker w accepts the most preferred acceptable offer. The workers' strategies are optimal given that the firms' strategies are stationary. The firms' strategies are also optimal since making an offer to a preferred worker will be rejected since μ is stable, and making an offer to a less preferred worker will have no influence over the subsequent periods.

A.2 Proof of Proposition 2

It suffices to prove (ii). For any subset $S \subseteq F \cup W$, let μ^S be any stable matching among S. Let σ be the strategy profile defined as follows. Consider any period and let S be the set of active agents. Each active firm f makes an offer to $\mu^S(f)$. Each active worker w who received an offer from $\mu^S(w)$ (and possibly others) accepts the most preferred offer. For each active worker w who did not receive an offer from $\mu^S(w)$, let T be defined by

$$T \equiv \{w, \mu^S(w)\} \cup O_w \cup \{i \in S : \mu^S(i) \in O_w \cup \{i\}\}. \tag{1}$$

Then w accepts the most preferred offer if

$$\max_{i \in O_w} u_w(i) > \delta_w u_w(\mu^T(w)). \tag{2}$$

Otherwise, w rejects all offers.

Let β be the belief system derived from the strategy profile above, with the following additional rule: in every period, if the set of active agents is S and an active worker w did not receive an offer from $\mu^S(w)$, then w believes that $\mu^S(w)$ did not make an offer to any worker. It is straightforward to construct a sequence of completely mixed perturbations of σ yielding β at the limit.

We claim that (σ, β) is an equilibrium. To see this, take any period and let S be the set of active agents.

We first examine firms' incentives. Let f be an active firm. If the firm follows the equilibrium strategy, it will be matched with $\mu^S(f) \in W \cup \{f\}$ in the current period. Let $w \equiv \mu^S(f)$. If f makes an offer to any worker w' that f prefers to w, then since μ^S is a stable matching, the offer will be rejected. In the next period, therefore, the set of active agents will be $\{f, w\} \cup \{i \in S : \mu^S(i) = i\}$ and the best outcome for f is to get w. Since f gets w earlier in equilibrium, the firm does not gain from the deviation. Similarly, the firm does not gain by not making any offer.

Now, consider workers' incentives. Let w be an active worker and consider any set O_w of offers that w may receive. There are two cases. First, suppose that w receives an offer from $\mu^S(w)$, i.e., $\mu^S(w) \in O_w$. In this event, if w rejects all offers, the set of active firms in the next period will be

$$O_w \cup \{f \in F : \mu^S(f) = f\}.$$

Since μ^S is stable, w prefers $\mu^S(w)$ to any f such that $\mu^S(f) = f$ and for which w is acceptable. Therefore, if $\mu^S(w) \in O_w$, worker w gains nothing by waiting: the optimal action is to take the best offer in O_w.

Suppose $\mu^S(w) \notin O_w$. By the definition of β, worker w believes that $\mu^S(w)$ did not make an offer to any worker. According to the belief, if w rejects all offers, the set of active agents in the next period will be T in (1) and hence the expected (average) utility is the right-hand side of (2). On the other hand, the maximum utility from accepting an offer in the current period is given by the left-hand side. Thus an optimal action is to take the best offer in O_w if (2) holds and rejects all offers otherwise.

A.3 Proof of Propositions 3 and 4

Suppose that there are 4 firms and 4 workers and their ordinal preferences are given by

f_1	f_2	f_3	f_4	w_1	w_2	w_3	w_4
w_1	w_3	w_3	w_4	f_4	f_3	f_1	f_2
w_3	w_2	w_4	w_3	f_1	f_2	f_4	f_3
$\vdots$	w_4	w_1	w_2	f_3	f_4	f_2	f_1
	$\vdots$	w_2	w_1	f_2	f_1	f_3	f_4
		f_3	f_4	w_1	w_2	w_3	w_4

We now construct a stationary equilibrium (σ, β) under which $\{f_1, w_1\}$ and $\{f_2, w_2\}$ are matched in period 1 and $\{f_3, w_3\}$ and $\{f_4, w_4\}$ are matched in period 2. The final matching is blocked by f_2 and w_3.

In all periods where everyone is active (including the first period), firms make offers as follows:

$$f_1 \to w_1, \quad f_2 \to w_2, \quad f_3 \to w_1, \quad f_4 \to w_2.$$

Each worker w simply accepts the most preferred acceptable offer if the following condition holds:

$$\begin{array}{ll} O_{w_1} \cap \{f_1, f_4\} \neq \emptyset \text{ and } O_{w_1} \neq \{f_1\} & \text{if } w = w_1, \\ O_{w_2} \cap \{f_2, f_3\} \neq \emptyset \text{ and } O_{w_2} \neq \{f_2\} & \text{if } w = w_2, \\ O_{w_3} \neq \{f_2\}, \{f_3\} & \text{if } w = w_3, \\ O_{w_4} \neq \{f_1\}, \{f_4\} & \text{if } w = w_4. \end{array}$$

Otherwise, w rejects all firms.

For any proper subset $S \subsetneq F \cup W$, let μ^S be a stable matching within S. The choice of μ^S is arbitrary with the following exceptions:

S	μ^S
$\{f_3, f_4, w_3, w_4\}$	$[\{f_3, w_3\}, \{f_4, w_4\}]$
$\{f_1, f_4, w_1, w_3\}$	$[\{f_1, w_3\}, \{f_4, w_1\}]$
$\{f_2, f_3, w_2, w_4\}$	$[\{f_2, w_4\}, \{f_3, w_2\}]$
$F \cup W \setminus \{f_2, w_2\}$	$[\{f_1, w_1\}, \{f_3, w_4\}, \{f_4, w_3\}]$
$F \cup W \setminus \{f_1, w_1\}$	$[\{f_2, w_2\}, \{f_3, w_4\}, \{f_4, w_3\}]$

With a collection $\{\mu^S\}_{S \subsetneq F \cup W}$, we now specify the strategies in periods where the set of active agents is a proper subset $S \subsetneq F \cup W$, in the same way as in Proposition 2. So, firms make offers to their partners in μ^S. A worker w accepts the most preferred offer if the offer from the expected firm ($\mu^S(w)$) arrived. If the expected offer did not arrive to worker w, then let

$$T \equiv \{w, \mu^S(w)\} \cup O_w \cup \{i \in S : \mu^S(i) \in O_w \cup \{i\}\}.$$

Then w accepts the most preferred offer if $\max_{i \in O_w} u_w(i) > \delta_w u_w(\mu^T(w))$, and rejects all offers otherwise.

Let β be the belief system derived from the strategy profile defined above, with the following two additional rules. First, in periods where everyone is active and the set of offers made to worker w_1 is $O_{w_1} = \{f_1\}$ (thus the expected offer from f_3 did not arrive), then w_1 believes that f_3 made an offer to w_4. Similarly, in periods where everyone is active and the set of offers made to worker w_2 is $O_{w_2} = \{f_2\}$, then w_2 believes that f_4 made an offer to w_3. Second, in all other cases where a worker w was expecting an offer from a firm f but the offer did not come, the worker believes that f did not make an offer to any worker. It is straightforward to perturb σ to generate β in the limit.

We now show that (σ, β) is an equilibrium. Since the only difference from Proposition 2 is when everyone is active, we only check incentives in this state.

Firm f_1 has no incentive to deviate since it gets its first choice in equilibrium. Firm f_2 gets only its second choice (w_2) but does not gain by deviating. Indeed, if f_2 makes an offer to w_3 or w_1, this offer will be rejected and the firm will be matched with w_2 in the next period.[6] Similarly, by not making any offer, f_2 only delays its matching with w_2. Finally, if f_2 makes an offer to w_4, this offer will be accepted, which is not good for f_2 since it prefers w_2.

Firm f_3, on the other hand, gets his first choice (w_3) only in the next period. If this firm is patient enough (i.e., $\delta_{f_3} > u_{f_3}(w_4)/u_{f_3}(w_3)$), therefore, the only possible reason to deviate is to get the first choice in the current period. However, if f_3 makes an offer to w_3, then $O_{w_3} = \{f_3\}$ and hence the offer will be rejected. A symmetric argument applies to f_4.

For workers' incentives, we start with w_1 and w_2. Since they are symmetric, we need only to consider w_1. If she receives an offer from f_4, her optimal action is to accept the offer since f_4 is her top choice. So, in what follows, suppose that w_1 did not receive an offer from f_4. We divide the remaining case into two.

Suppose $O_{w_1} \neq \{f_1\}$. Then it can be checked that w_1 believes that if she rejects all offers, she will be matched with her second choice (f_1) in the next period.[7] So, the optimal choice for w_1 depends on whether $f_1 \in O_{w_1}$. If $f_1 \in O_{w_1}$, then w_1 should accept f_1 in the current period since it is the best offer at hand and rejecting all offers will only delay the matching with the same firm. If $f_1 \notin O_{w_1}$, on the other hand, the optimal reply depends on the worker's patience. If w_1 is sufficiently patient (i.e., $\delta_{w_1} > u_{w_1}(f_3)/u_{w_1}(f_1)$), the optimal reply is to reject all offers now and get f_1 in the next period.

Now, suppose $O_{w_1} = \{f_1\}$. By the construction of β, w_1 believes that f_3 made an offer to w_4 and this offer will be accepted. Thus, w_1 believes that if she rejects f_1, the set of active agents in the next period will be $\{f_1, f_4, w_1, w_3\}$ and hence she will get f_4, which is her first choice. If w_1 is patient enough, therefore, she prefers to wait for her first choice.

Finally, consider w_3 and w_4. Since they are symmetric, we only discuss w_3. If she receives an offer from her top choice (f_1), she obviously accepts it. So, suppose that she did not receive an offer from f_1. If $O_{w_3} = \{f_2\}$ and w_3 rejects the offer, the set of active agents in the next period will be $F \cup W \setminus \{f_1, w_1\}$ and w_3 will get her second choice (f_4). Since the offer at hand is her third choice, if w_3 is patient enough, she prefers to wait for her second choice. Similarly, if $O_{w_3} = \{f_3\}$ (which

[6] The set of active players in the next period will be $F \cup W \setminus \{f_1, w_1\}$.

[7] To see this, note that since $O_{w_1} \neq \{f_1\}$, w_1 believes that the firms in $\{f_1, f_3\} \setminus O_{w_1}$ did not make any offer. Therefore she believes that the only other worker who receives any offer is w_2 and the set of offers received by w_2 is a subset of $\{f_2, f_4\}$. According to w_2's strategy, w_2 accepts f_2 if both f_2 and f_4 made her an offer, but if only one of them did, she accepts none. Therefore, w_1 believes that if she rejects all offers, the set of active players in the next period is $F \cup W \setminus \{f_2, w_2\}$ or $F \cup W$. In either case, w_1 will be matched with f_1 in the next period.

is the fourth choice for w_3) and w_3 rejects the offer, the set of active agents in the next period will be $F \cup W \setminus \{f_2, w_2\}$ and so w_3 will get her second choice (f_4).[8] So if w_3 is patient enough, she prefers to wait. If $f_4 \in O_{w_3}$, it can be checked that if w_3 rejects all offers, she will get either f_4 in the next period or f_3 in the following period.[9] Since she prefers f_4 to f_3, she prefers to accept f_4 in the current period. Similarly, if $O_{w_3} = \{f_2, f_3\}$, rejecting all offers will give her f_3 in two periods, so she should accept f_2 in the current period.

A.4 Strong Consistency

This section considers a stronger version of consistency mentioned in footnote 3. The stronger version is obtained by replacing $F \cup W$ in the definition of consistency by any subset $T \subseteq F \cup W$.

Formally, we say that a stationary strategy profile σ is *strongly consistent* if for any subset $T \subseteq F \cup W$ and any subset $S \subseteq T$, if S is obtained from T by removing some of the matched pairs in $\mu \equiv m(\sigma, T)$, then $m(\sigma, S) = \mu|_S$.

As mentioned before, strong consistency is not compatible with existence.

Proposition 9 *There exists a dynamic matching game with two-sided commitment that admits no stationary equilibrium satisfying strong consistency.*

Proof. Consider a 5×5 matching problem with the following ordinal preferences:

f_1	f_2	f_3	f_4	f_5	w_1	w_2	w_3	w_4	w_5
w_1	w_2	w_3	w_4	w_5	f_2	f_1	f_1	f_4	f_3
w_3	w_1	w_5	$\vdots$	$\vdots$	f_3	f_2	f_3	f_3	f_5
w_2	$\vdots$	w_1			f_1	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$		w_4			$\vdots$				
		f_3							

There is a unique stable matching, which is $\mu_1 \equiv [\{f_1, w_1\}, \{f_2, w_2\}, \ldots, \{f_5, w_5\}]$. Suppose, toward a contradiction, that there exists a strongly consistent stationary equilibrium. Then, by Proposition 5, μ_1 is the final matching in the equilibrium. Consider a subset $S \equiv \{f_1, f_2, f_3, w_1, w_2, w_4\}$. Within the subset, there is a unique stable matching, which is $\mu_2 \equiv [\{f_1, w_2\}, \{f_2, w_1\}, \{f_3, w_4\}]$. Consider a continuation game where S is the initial set of active agents. Observe that the continuation strategy profile remains a stationary equilibrium and is strongly consistent. Therefore, by Proposition 5, the final result of the continuation equilibrium

[8] Note that w_1 will reject f_1.

[9] To see this, note that if w_2 receives an offer, it is from f_2 and she rejects it. On the other hand, w_1 may receive an offer from f_1 or f_3 and she accepts f_1 if she receives an offer from both but accepts none otherwise. Thus, if w_3 rejects all offers, the set of active players in the next period is either $F \cup W \setminus \{f_1, w_1\}$ or $F \cup W$. In the former case, w_3 will be matched with f_4. In the latter case, she will be matched with f_3 in two periods.

is μ_2 (i.e., $m(\sigma, S) = \mu_2$). Now, consider a subset $T \equiv \{f_1, f_2, w_1, w_2\}$. Since T is obtained from S by removing a matched pair in μ_2, strong consistency implies that in the continuation game where T is the initial set of active agents, the final result is $\mu_2|_T = [\{f_1, w_2\}, \{f_2, w_1\}]$. On the other hand, T is also obtained from the entire set of $F \cup W$ by removing three matched pairs in μ_1. Therefore, (strong) consistency also implies that the same continuation game for T results in $\mu_1|_T = [\{f_1, w_1\}, \{f_2, w_2\}] \neq \mu_2|_T$, a contradiction.

A.5 Proof of Proposition 7

Let $n \equiv \min\{|F|, |W|\}$. We first note that, in any stationary equilibrium (of any continuation game), the final matching is determined in n periods. This is the case since, if no pair is formed at a state in equilibrium, the state does not change and stationarity implies that the state remains the same thereafter. Thus, until the final matching is determined, at least one pair is formed every period.

Since the number of agents is finite, there exists $\underline{\delta} \in (0, 1)$ such that for all $i, j, k \in F \cup W$, $u_i(j) > u_i(k)$ implies $\underline{\delta}^n u_i(j) > u_i(k)$.

Let $\delta \in [\underline{\delta}, 1)$ and consider any consistent stationary equilibrium. Suppose, toward a contradiction, that the final matching μ is determined in period 2 or later. Then there exists a pair $(f_1, w_1) \in F \times W$ that is matched in period 2 or later in equilibrium. We consider what happens if f_1 deviates by making an offer to $w_1 = \mu(f_1)$ in the first period. Since the deviation does not make f_1 better off, w_1 is prescribed to reject the offer.

Case 1: The deviation by f_1 does not affect the reply of any worker $w \neq w_1$. In this case, if w_1 rejects all offers (including f_1), the outcome in the first period is the same as in equilibrium, and hence w_1 will be matched with f_1 eventually. But then w_1 is better off by accepting f_1 right away, a contradiction.

Case 2: There exists a worker $w_2 \in W \setminus \{w_1\}$ whose reply is affected by the deviation of f_1. This is possible only if f_1 is prescribed to make an offer to w_2 in the first period. There are two ways in which w_2's reply is affected by the deviation.

Case 2a: While w_2 accepts an offer from $\mu(w_2)$ without f_1's deviation, she rejects it (and any other offer) if f_1 deviates. In this case, after f_1's deviation, if w_2 rejects all offers, she will be matched with $\mu(w_2)$ eventually since f_1's deviation offer goes to $w_1 = \mu(f_1)$ and hence only pairs in μ are formed in this period (and since the equilibrium is consistent). But then w_2 is strictly better off by accepting $\mu(w_2)$ right away, a contradiction.

Case 2b: While w_2 rejects all offers without f_1's deviation, she accepts some offer if f_1 deviates. Let f_2 denote the offer w_2 accepts when f_1 deviates. After f_1's deviation, if w_2 rejects all offers, she will be matched with $\mu(w_2)$ eventually, as in the previous case. Let $t \geq 2$ denote the period in which she is matched with $\mu(w_2)$. Since it takes at most n periods, $t \leq n + 1$. Since w_2 is prescribed to accept f_2, we have $u_{w_2}(f_2) \geq \delta^{t-1} u_{w_2}(\mu(w_2))$. Since $\delta \geq \underline{\delta}$ and $t - 1 \leq n$, it follows that $u_{w_2}(f_2) \geq u_{w_2}(\mu(w_2))$. But then, without f_1's deviation, w_2 is strictly better off by accepting f_2 rather than rejecting all offers to be matched with $\mu(w_2)$ in the future.

A.6 Proof of Proposition 8 for Patient Workers

The proof of Proposition 8 in the main text relies on an example where $\delta_w = 0$ for all workers and therefore poses a question whether the result extends if the workers are patient. This section gives an example showing that the result does extend even if δ_w is close to 1 for all workers.

We consider a 3×3 matching problem with the following ordinal preferences:

f_1	f_2	f_3	w_1	w_2	w_3
w_1	w_3	w_3	f_2	f_3	f_1
w_3	w_2	w_1	f_3	f_2	f_2
w_2	w_1	w_2	f_1	f_1	f_3
f_1	f_2	f_3	w_1	w_2	w_3

Each agent's utility function is given by

$$u_i(j) = \begin{cases} 100 & \text{if } j \text{ is } i\text{'s first choice,} \\ 70 & \text{if } j \text{ is } i\text{'s second choice,} \\ 40 & \text{if } j \text{ is } i\text{'s third choice,} \\ 0 & \text{if } j \text{ is } i\text{'s last choice.} \end{cases}$$

There are two stable matchings:

$$[\{f_2, w_1\}, \{f_3, w_2\}, \{f_1, w_3\}],$$
$$[\{f_3, w_1\}, \{f_2, w_2\}, \{f_1, w_3\}].$$

In the equilibrium we construct, each firm f_i makes an offer to w_i, respectively, in the first period and they are all accepted. The realized matching, i.e., $[\{f_1, w_1\}, \{f_2, w_2\}, \{f_3, w_3\}]$, is not stable since it is blocked by $\{f_2, w_3\}$. Note that each of the stable matchings is a Pareto improvement for the workers.

Figure 2 describes the equilibrium strategy profile. For firms, the dashed arrows specify to whom each active firm makes an offer in the state. For workers, it is more complicated since a worker's response depends on not only the state but also the set of offers made to the worker. In the particular equilibrium constructed here, each worker's strategy in a given state s can be summarized by a *cutoff* denoted by $c_w(s, O) \in F \cup \{w\}$, where O is the set of offers made to the worker. The worker w simply chooses the most preferred offer that is at least as good as the cutoff $c_w(s, O)$. In most cases, the cutoff is the worker herself, i.e., $c_w(s, O) = w$, which means that the worker chooses the most preferred acceptable firm. In the several cases where $c_w(s, O) \neq w$, the cutoffs are specified in Figure 2 in square brackets attached to the worker (e.g., $[f_2]$). Nothing is attached if the cutoff is oneself.

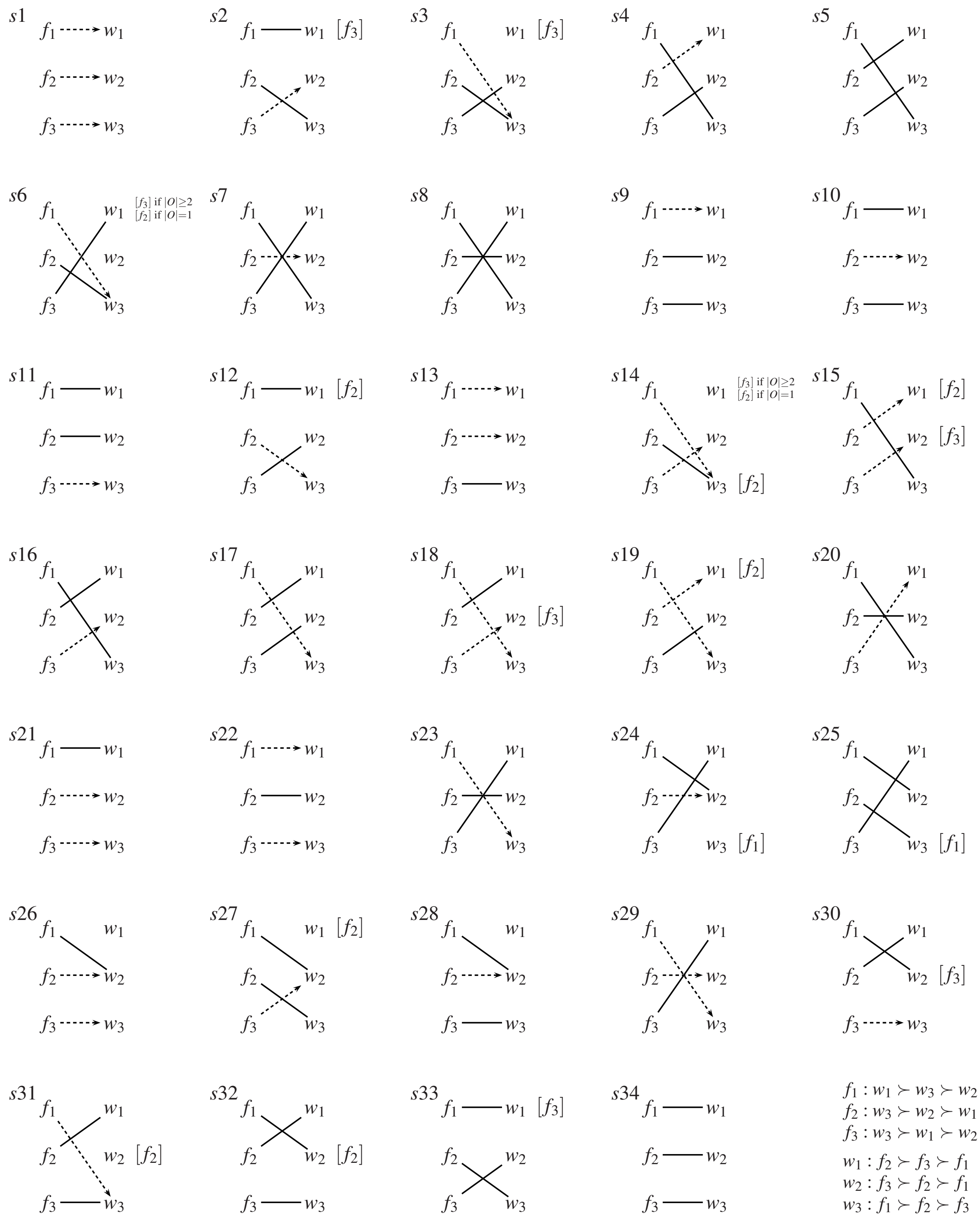


Figure 2: A stationary equilibrium that yields an unstable matching. The square brackets denote the cutoffs used by the workers.

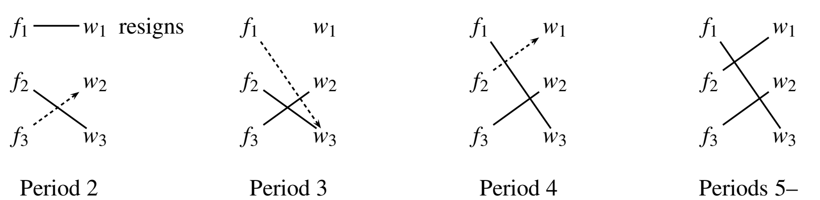
If the worker is currently employed, her current job is included in the offer set. Therefore, if the current job is less preferred to the cutoff, the worker resigns from the current job. For example, w_3 in $s25$ resigns from f_2 since f_2 is less preferred to f_1.

Except for worker 1 in states 6 and 14, the cutoffs are independent of the set of offers. In states 6 and 14, worker 1's cutoff depends on the number of offers (including the renewal offer).

To complete the description of the equilibrium, we need to specify workers' out-of-equilibrium beliefs since offers are private information. If a firm is prescribed to make an offer to a worker but deviates, this worker observes only the fact that the firm makes no offer to her. She does not observe where the firm makes an offer. In the particular equilibrium we construct, the worker in this situation is assumed to believe that the firm does not make any offer to any worker. The particular belief was chosen to simplify our construction of an equilibrium.

The strategy profile together with the belief system is an equilibrium if the agents are sufficiently patient. A sufficient condition is that $\delta_i \geq \sqrt[3]{0.7} \approx 0.89$ for all agents. Verifying sequential rationality is extremely tedious. Below we informally discuss why the blocking pair does not form. A detailed proof can be obtained from the authors upon request.

If f_2 deviates in the first period and makes an offer to its blocking partner, w_3, then the offer would be accepted but trigger a chain of movements in subsequent periods as depicted in the following figure.



In the next period, the firm that moves is f_3. Having failed to get w_3, firm f_3 makes an offer to w_2 and gets accepted.[10] At the same time, w_1 resigns from f_1. This move by w_1 enables f_1 to make an offer to w_3 in period 3. The offer is accepted since f_1 is the first choice for w_3. In period 4, f_2 has no choice but to make an offer to w_1 since the other workers are with their first choice. The matching is then completed. Note that f_2, who initiates the process, ends up with a worker who is less desirable than the one the firm gets in equilibrium. Therefore, if f_2 is sufficiently patient, the deviation makes the firm worse off.

[10] If f_3 deviates by making an offer to w_1, the offer will be accepted but the worker stays with the firm only for one period. After losing w_1, the firm will be eventually matched with w_2. The state transition is $s2 \to s6 \to s15 \to s5$. Given the specific utility function and discount rate, f_3 does not gain from the deviation.

References

Adachi, H. (2003): "A Search Model of Two-Sided Matching Under Nontransferable Utility," *Journal of Economic Theory*, 113, 182–198.

Alcalde, J., D. Pérez-Castrillo, and A. Romero-Medina (1998): "Hiring Procedures to Implement Stable Allocations," *Journal of Economic Theory*, 82, 469–480.

Alcalde, J., and A. Romero-Medina (2000): "Simple Mechanisms to Implement the Core of College Admissions Problems," *Games and Economic Behavior*, 31, 294–302.

Bloch, F. (1996): "Sequential Formation of Coalitions in Games with Externalities and Fixed Payoff Division," *Games and Economic Behavior*, 14, 90–123.

Bloch, F., and E. Diamantoudi (2011): "Noncooperative formation of coalitions in hedonic games," *International Journal of Game Theory*, 40(2), 263–280.

Blum, Y., A. E. Roth, and U. G. Rothblum (1997): "Vacancy chains and equilibration in senior-level labor markets," *Journal of Economic Theory*, 76, 362–411.

Bulow, J., and J. Levin (2006): "Matching and Price Competition," *American Economic Review*, 96, 652–668.

Chatterjee, K., B. Dutta, D. Ray, and K. Sengupta (1993): "A Noncooperative Theory of Coalitional Bargaining," *Review of Economic Studies*, 60, 463–477.

Crawford, V. P. (2008): "The Flexible-Salary Match: A Proposal to Increase the Salary Flexibility of the National Resident Matching Program," *Journal of Economic Behavior and Organization*, 66(2), 149–160.

Crawford, V. P., and E. M. Knoer (1981): "Job Matching with Heterogeneous Firms and Workers," *Econometrica*, 49, 437–450.

Gale, D., and L. Shapley (1962): "College admissions and the stability of marriage," *American Mathematical Monthly*, 69, 9–15.

Haeringer, G., and M. Wooders (2011): "Decentralized Job Matching," *International Journal of Game Theory*, 40(1), 1–28.

Kelso, A. S., and V. P. Crawford (1982): "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica*, 50, 1483–1504.

Knuth, D. E. (1976): *Mariages Stables et leurs relations avec d'autres problèmes combinatoires*. Les Presses de l'Université de Montréal, Montréal.

Konishi, H., and M. Sapozhnikov (2005): "Decentralized Matching Markets with Endogeneous Salaries," Discussion paper, Boston College.

Niederle, M., and L. Yariv (2007): "Matching Through Decentralized Markets," Discussion paper, Stanford University.

Pais, J. (2008): "Incentives in Decentralized Random Matching Markets," *Games and Economic Behavior*, 64, 632–649.

Ray, D., and R. Vohra (1999): "A Theory of Endogenous Coalition Structures," *Games and Economic Behavior*, 26, 286–336.

Roth, A. E. (1984): "The evolution of the labor market for medical interns and residents: A case study in game theory," *Journal of Political Economy*, 92, 991–1016.

——— (1991): "A natural experiment in the organization of entry-level labor markets: Regional markets for new physicians and surgeons in the United Kingdom," *American Economic Review*, 81, 415–440.

Roth, A. E., and M. A. O. Sotomayor (1990): *Two-Sided Matching: A Study in Game-Theoretic Modelling and Analysis*. Cambridge University Press.

Roth, A. E., and J. H. Vande Vate (1990): "Random paths to stability in two-sided matching," *Econometrica*, 58, 1475–80.

Sasaki, H., and M. Toda (1992): "Consistency and Characterization of the Core of Two-Sided Matching Problems," *Journal of Economic Theory*, 56, 218–227.

Thomson, W. (2011): "Consistency and its converse: an introduction," *Review of Economic Design*, 15, 257–291.