Continuum hypothesis and diamond principle

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Theorem (Cantor's theorem)

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$$|X| < |\mathcal{P}(X)|.$$

For a cardinal λ, we have

$$\lambda < 2^{\lambda}.$$

Continuum hypothesis

Theorem (Cantor's theorem)

For every set X, we have

$$|X| < |\mathcal{P}(X)|.$$

For a cardinal λ, we have

$$\lambda < 2^{\lambda}.$$

$\Longrightarrow$ For an infinite cardinal λ, does there exist a cardinal μ such that $\lambda < \mu < 2^{\lambda}$?

Continuum hypothesis

Definition

The Continuum Hypothesis(CH) is the statement

$$2^{\omega} = \omega_1.$$

For an infinite cardinal λ, CH_{λ} is the statement

$$2^{\lambda} = \lambda^{+}.$$

The statement

$$\forall\lambda\ \mathrm{CH}_{\lambda}$$

is called the Generalized Continuum Hypothesis(GCH).

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Theorem (Gödel, Cohen)

GCH is independent of the axioms of ZFC.

Diamond principle

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- $C \subset \kappa$ is a club set iff
  - $0 < \forall\delta < \kappa\big(\sup(C \cap \delta) = \delta \rightarrow \delta \in C\big)$ (closed)
  - $\sup C = \kappa$ (unbounded)

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club set : measure 1

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Definition

Assume κ is a regular cardinal and $S \subset \kappa$ is a stationary set. Then, $\Diamond_S$ is the following statement: there exists $\langle S_\delta : \delta \in S\rangle$ such that for every $A \subset \kappa$,

$$\{\delta \in S : A \cap \delta = S_\delta\}$$

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Remark

If $S \subset T$ then $\Diamond_S \to \Diamond_T$.

Diamond principle

Proposition

Assume λ is an infinite cardinal and $S \subset \lambda^+$ is a stationary set. Then,

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Assume λ is an infinite cardinal and $S \subset \lambda^+$ is a stationary set. Then,

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Proof.

Assume $\langle S_\delta : \delta \in S \rangle$ is a $\diamondsuit_S$-sequence.
Since $C := \{\delta < \lambda^+ : \delta > \lambda\} = (\lambda, \lambda^+)$ is club, for every $A \subset \lambda$, there exists $\delta \in S \cap C$ such that

$$A = A \cap \delta = S_\delta.$$

Then

$$2^\lambda = |\mathcal{P}(\lambda)| \leq |S \cap C| = \lambda^+.$$

□

Diamond principle

Fact (Jensen)

$(2^\omega = \omega_1) + \neg\Diamond_{\omega_1}$ *is consistent.*

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Question 1

Assume λ *is an uncountable cardinal. What kind of* $S \subset \lambda^+$ *entail*

$$2^\lambda = \lambda^+ \to \Diamond_S \quad ?$$

Diamond principle

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Assume λ *is an uncountable cardinal. What kind of* $S \subset \lambda^+$ *entail*

$$2^{\lambda} = \lambda^+ \to \Diamond_S \quad ?$$

Question 2

For every uncountable cardinal λ*,*

$$2^{\lambda} = \lambda^+ \to \Diamond_{\lambda^+}$$

holds?

Shelah's theorem

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Assume $\mu < \kappa$ are regular cardinals.

- $S^\kappa_\mu := \{\alpha < \kappa : \mathrm{cf}(\alpha) = \mu\}$,
- $S^\kappa_{\neq\mu} := \{\alpha < \kappa : \mathrm{cf}(\alpha) \neq \mu\}$.

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Lemma

Assume $\mu < \kappa$ are regular cardinals. Then S^κ_μ is stationary.

For uncountable cardinal λ, $S^{\lambda^+}_{\neq\mathrm{cf}(\lambda)}$ is stationary.

Shelah's theorem

Theorem (Shelah)

Assume λ is an uncountable cardinal and $S \subset S^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$ is a stationary set. Then,

$$2^\lambda = \lambda^+ \to \Diamond_S$$

holds.

Shelah's theorem

Theorem (Shelah)

Assume λ is an uncountable cardinal and $S \subset S^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$ is a stationary set. Then,

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Corollary

For every uncountable cardinal λ,

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For every uncountable cardinal λ,

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Question 2 is solved !!

A problem about Shelah's theorem

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Fact

If λ is a regular cardinal, then $(2^\lambda = \lambda^+) + \neg\Diamond_{S^{\lambda^+}_{\mathrm{cf}(\lambda)}}$ is consistent.

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Question1-2

Assume λ is a singular cardinal. Then, For every stationary $S \subset S^{\lambda^+}_{\mathrm{cf}(\lambda)}$,

$$2^\lambda = \lambda^+ \to \diamondsuit_S$$

holds?

A problem about Shelah's theorem

Definition

Assume λ is a singular cardinal and $S \subset \lambda^+$ is a stationary set. $I[S;\lambda]$ is a set such that

$$\begin{aligned} T \in I[S;\lambda] \leftrightarrow T \subset \mathrm{Tr(S)} \text{ and } & \\ & \exists d : [\lambda^+]^2 \to \mathrm{cf}(\lambda) \text{ normal, subadditive} \\ & \exists C \subset \lambda^+ \text{ club} \\ & \forall \gamma \in T \cap C \cap S^{\lambda^+}_{>\mathrm{cf}(\lambda)} \; \exists S_\gamma \subset \gamma \cap S \text{ stationary} \\ & \left(\sup d\left[[S_\gamma]^2\right] < \mathrm{cf}(\lambda)\right). \end{aligned}$$

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& \left(\sup d\left[[S_\gamma]^2\right] < \mathrm{cf}(\lambda) \right).
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- $\mathrm{Tr}(S) = \{\alpha < \lambda^+ : \mathrm{cf}(\alpha) > \omega, S \cap \alpha \text{ is stationary in } \alpha\}$,

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$$\forall \gamma \in T \cap C \cap S^{\lambda^+}_{>\mathrm{cf}(\lambda)}\ \exists S_\gamma \subset \gamma \cap S \text{ stationary}$$
$$\left(\sup d\left[[S_\gamma]^2\right] < \mathrm{cf}(\lambda)\right).$$

- $\mathrm{Tr}(S) = \{\alpha < \lambda^+ : \mathrm{cf}(\alpha) > \omega, S \cap \alpha \text{ is stationary in } \alpha\}$,
- For $d : [\lambda^+]^2 \to \mathrm{cf}(\lambda)$,
  - d is subadditive $\leftrightarrow \forall \alpha \leq \forall \beta \leq \forall \gamma < \lambda^+ \big(d(\alpha,\gamma) \leq \max\{d(\alpha,\beta), d(\beta,\gamma)\}\big)$,
  - d is normal $\leftrightarrow \forall \beta < \lambda^+ \forall i < \mathrm{cf}(\lambda)\big(|\{\alpha < \beta : d(\alpha,\beta) \leq i\}| < \lambda\big)$.

A problem about Shelah's theorem

Theorem (Rinot)

Assume λ is a singular cardinal and $S \subset \lambda^+$ is a stationary set. Then, if $I[S;\lambda]$ contains a stationary set,

$$2^\lambda = \lambda^+ \to \Diamond_S$$

holds.

A problem about Shelah's theorem

Theorem (Rinot)

Assume λ is a singular cardinal and $S \subset \lambda^+$ is a stationary set. Then, if $I[S;\lambda]$ contains a stationary set,

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holds.

Question1-3

Assume λ is a singular cardinal. Then for every stationary $S \subset S^{\lambda^+}_{\mathrm{cf}(\lambda)}$, does $I[S;\lambda]$ contain stationary sets ?
Does $I[S^{\lambda^+}_{\mathrm{cf}(\lambda)};\lambda]$ contain stationary sets ?

Sketch of the proof of Shelah's theorem

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Theorem (Shelah)

Assume λ is an uncountable cardinal and $S \subset S^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$ is a stationary set. Then,

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Assume λ is an uncountable cardinal and $S \subset S^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$ is a stationary set. Then,

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Sketch of proof (Komjáth).

Sketch of the proof of Shelah's theorem

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Assume λ is an uncountable cardinal and $S \subset S^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$ is a stationary set. Then,

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Sketch of proof (Komjáth).

Claim

There exist a sequence $\langle A^\delta \in [\delta]^{<\lambda} : \delta \in S \rangle$ and an enumeration of $[\lambda \times \lambda^+]^{\leq \lambda}$, $\langle X_\beta : \beta < \lambda^+ \rangle$, such that for every $Z \subset \lambda \times \lambda^+$,

$$S_Z := \left\{ \delta \in S : \sup \left\{ \alpha \in A^\delta : \exists \beta \in A^\delta (Z \cap (\lambda \times \alpha) = X_\beta) \right\} = \delta \right\}$$

is stationary.

Sketch of the proof of Shelah's theorem

Now, we define a sequence of subsets of λ^+, $\langle Y_\gamma : \gamma < \lambda \rangle$, and a decreasing sequence of clubs, $\langle C_\gamma : \gamma < \lambda \rangle$.

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Put $Y_0 := C_0 := \lambda^+$.

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Now, we define a sequence of subsets of λ^+, $\langle Y_\gamma : \gamma < \lambda \rangle$, and a decreasing sequence of clubs, $\langle C_\gamma : \gamma < \lambda \rangle$.
Put $Y_0 := C_0 := \lambda^+$.
Assume $\langle Y_\tau : \tau < \gamma \rangle$ and $\langle C_\tau : \tau < \gamma \rangle$ are defined for some $\gamma < \lambda$.

- If there exist $Y \subset \lambda^+$ and club $C \subset \bigcap_{\tau<\gamma} C_\tau$ such that for every $\delta \in S \cap C$, we have

$$\delta = \bigcup \left\{ \alpha \in A^\delta : \exists \beta \in A^\delta \forall \tau < \gamma \big(Y_\tau \cap \alpha = (X_\beta)_\tau \big) \right\} \text{ then}$$

$$\exists \langle \alpha, \beta \rangle \in A^\delta \times A^\delta \left[\forall \tau < \gamma \big(Y_\tau \cap \alpha = (X_\beta)_\tau \big) \wedge Y \cap \alpha \neq (X_\beta)_\gamma \right],$$

  then, put $Y_\gamma = Y, C_\gamma = C$.
- Otherwise, terminate the recursion.

$(X_\beta)_\gamma := \{\xi : \langle \gamma, \xi \rangle \in X_\beta\} \subset \lambda^+$.

Sketch of the proof of Shelah's theorem

Claim

There exists $\gamma^* < \lambda$ such that the recursion terminates in γ^*.
Then, for every $Y \subset \lambda^+$ and club $C \subset \bigcap_{\gamma < \gamma^*} C_\gamma$, there exists $\delta \in S \cap C$, such that

$$\delta = \bigcup \left\{ \alpha \in A^\delta : \exists \beta \in A^\delta \forall \gamma < \gamma^* \left(Y_\gamma \cap \alpha = (X_\beta)_\gamma \right) \right\} \text{ and}$$

$$\forall \langle \alpha, \beta \rangle \in A^\delta \times A^\delta \left[\forall \gamma < \gamma^* \left(Y_\gamma \cap \alpha = (X_\beta)_\gamma \right) \rightarrow Y \cap \alpha = (X_\beta)_{\gamma^*} \right].$$

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Put $C^* := \bigcap_{\gamma<\gamma^*} C_\gamma$.

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Put $C^* := \bigcap_{\gamma<\gamma^*} C_\gamma$.
For $\delta \in S \cap C^*$, define

$$S_\delta := \bigcup_{\langle\alpha,\beta\rangle \in A^\delta \times A^\delta} \left\{(X_\beta)_{\gamma^*} : \forall\gamma < \gamma^* \left(Y_\gamma \cap \alpha = (X_\beta)_\gamma\right)\right\}.$$

Sketch of the proof of Shelah's theorem

Then, for every $Y \subset \lambda^+$ and club $C \subset \lambda^+$, there exists $\delta \in S \cap C^* \cap C$, we have

$$\begin{aligned}
Y \cap \delta &= Y \cap \bigcup \left\{ \alpha \in A^\delta : \exists \beta \in A^\delta \forall \gamma < \gamma^* \left(Y_\gamma \cap \alpha = (X_\beta)_\gamma \right) \right\} \\
&= \bigcup_{\langle \alpha, \beta \rangle \in A^\delta \times A^\delta} \left\{ Y \cap \alpha : \forall \gamma < \gamma^* \left(Y_\gamma \cap \alpha = (X_\beta)_\gamma \right) \right\} \\
&= \bigcup_{\langle \alpha, \beta \rangle \in A^\delta \times A^\delta} \left\{ (X_\beta)_{\gamma^*} : \forall \gamma < \gamma^* \left(Y_\gamma \cap \alpha = (X_\beta)_\gamma \right) \right\} \\
&= S_\delta .
\end{aligned}$$

Sketch of the proof of Shelah's theorem

Then, for every $Y \subset \lambda^+$ and club $C \subset \lambda^+$, there exists $\delta \in S \cap C^* \cap C$, we have

$$\begin{aligned} Y \cap \delta &= Y \cap \bigcup \left\{ \alpha \in A^\delta : \exists \beta \in A^\delta \forall \gamma < \gamma^* \left(Y_\gamma \cap \alpha = (X_\beta)_\gamma \right) \right\} \\ &= \bigcup_{\langle \alpha, \beta \rangle \in A^\delta \times A^\delta} \left\{ Y \cap \alpha : \forall \gamma < \gamma^* \left(Y_\gamma \cap \alpha = (X_\beta)_\gamma \right) \right\} \\ &= \bigcup_{\langle \alpha, \beta \rangle \in A^\delta \times A^\delta} \left\{ (X_\beta)_{\gamma^*} : \forall \gamma < \gamma^* \left(Y_\gamma \cap \alpha = (X_\beta)_\gamma \right) \right\} \\ &= S_\delta . \end{aligned}$$

$\langle S_\delta : \delta \in S \cap C^* \rangle$ is $\Diamond_{S \cap C^*}$-sequence.
Then, $\Diamond_S$ holds. $\square$

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