Finitely generated torsion-free groups and

non-commutatively slender groups

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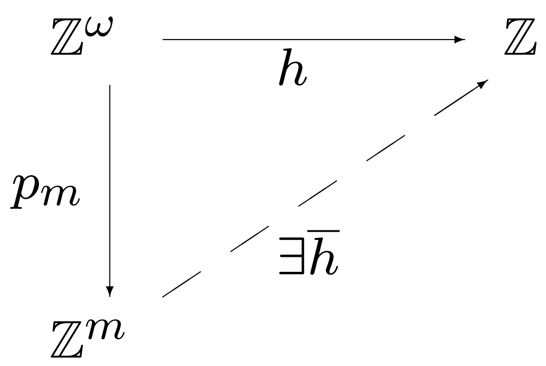
1.Specker'theorem and slender groups

E.Specker(1950)

$h : \mathbb{Z}^\omega \to \mathbb{Z}$ a homomorphism.

$\exists m$



$h = \overline{h} \circ p_m$ p_m:projection. $\overline{h}(x) = \sum_{i=0}^{m-1} x(i)h(e_i)$

e_i:i-th component is 1 , other components are all zero.

$$x = \sum_{i<\omega} x(i)e_i = \sum_{i=0}^{m-1} x(i)e_i + \sum_{m\leq i<\omega} x(i)e_i$$

$$\begin{aligned} h(x) &= h(\sum_{i=0}^{m-1} x(i)e_i) + h(\sum_{m\leq i<\omega} x(i)e_i) \\ &= \sum_{i=0}^{m-1} x(i)h(e_i) + h(\sum_{m\leq i<\omega} x(i)e_i) \\ &= \sum_{i=0}^{m-1} x(i)h(e_i) \end{aligned}$$

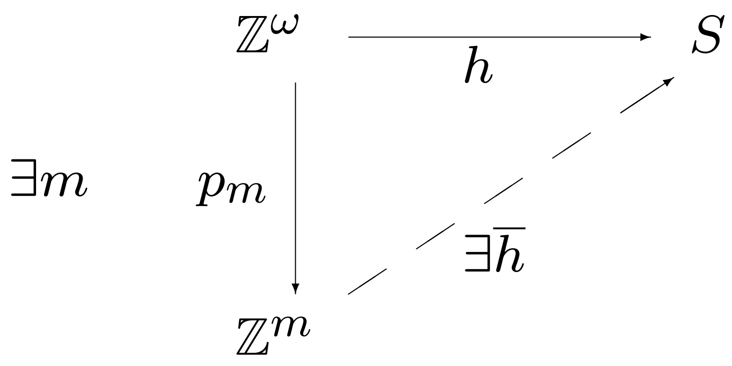
$h(x)$ is determined by only finite components of x.

h factors through a finitely generated free abelian group $\mathbb{Z}^m$.

Slenderness was introduced by J.Łoś.

An abelian group S is slender, if S satisfies the following diagram.

$h : \mathbb{Z}^\omega \to S$ a homomorphism.



$$h = \overline{h} \circ p_m$$

A slender group S satisfies Specker'theorem.

$\mathbb{Z}$ is a typical example of slender groups.

Theorem (L.Fuchs)

Direct sums of slender groups are slender.

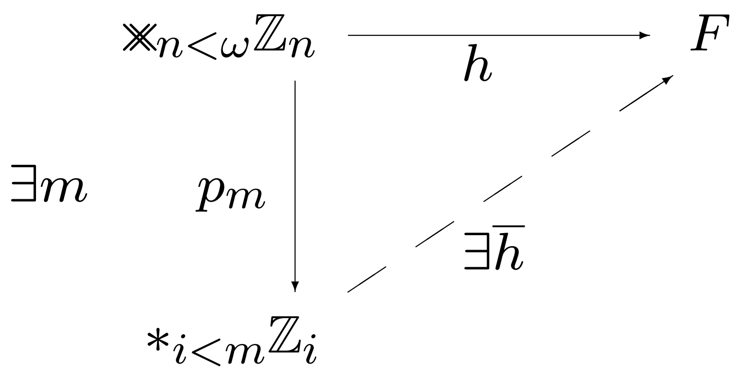
Theorem (R.J.Nunke) the characterization of slender groups.

An abelian group is slender if and only if, it is torsion-free and contains no copy of $\mathbb{Q}, \mathbb{Z}^\omega$, *or* p*-adic integer group* $\mathbb{J}_p$ *for any prime* p.

2.Non-commutative Specker'theorem and n-slender groups

G.Higman (1952)

Let F be a free group and $h : ⋇_{n<\omega}\mathbb{Z}_n \to F$ a homomorphism.



$h = \overline{h} \circ p_m$ p_m: canonical projection

$⋇_{n<\omega}\mathbb{Z}_n$ is the free complete product of copies of $\mathbb{Z}$.

It is isomorphic to the fundamental group of the Hawaiian earring.

n-slenderness was introduced by K.Eda in 1992.

A group S is n-slender if, S satisfies the following diagram.

$$\exists m \qquad \begin{array}{ccc} \divideontimes_{n<\omega}\mathbb{Z}_n & \xrightarrow{\quad h \quad} & S \\ \Big\downarrow p_m & \nearrow \exists\overline{h} & \\ *_{i<m}\mathbb{Z}_i & & \end{array} \qquad h = \overline{h} \circ p_m$$

A n-slender group satisfies non-commutative Specker'theorem.

$\mathbb{Z}$ is also a good example of n-slender groups.

Theorem(K.Eda)

Let A *be an abelian group.*

A *is slender if and only if,* A *is n-slender.*

Theorem(K.Eda)

Let $G_i (i \in I)$ *be n-slender. Then, the free product* $*_{i \in I} G_i$

and the restricted direct product $\prod_{i \in I}^{r} G_i = \{x \in \prod_{i \in I} G_i | \{i \in I | x(i) \neq e\}$ *is finite* $\}$ *are n-slender.*

There is a characterization of n-slender groups using fundamental groups.

Theorem(K.Eda)

$\pi_1(X, x)$ *is n-slender if and only if,*

for any homomorphism $h : \pi_1(\mathbb{H}, o) \to \pi_1(X, x)$*,*

there exists a continuous map $f : (\mathbb{H}, o) \to (X, x)$

such that $h = f_*$ *where* f_* *is the induced homomorphism.*

We can rephrase Higman's theorem in topological terms as follows:

Let h be a homomorphism from $\pi_1(\mathbb{H}, o)$ to $\pi_1(\mathbb{S}^1)$.

Then, there exists a continuous map $f : \mathbb{H} \to \mathbb{S}^1$ such that $h = f_*$.

Many things about wild algebraic topology can be reduced to the Hawaiian earring and

how the homomorphic image of the fundamental group of the Hawaiian earring can detect a point in the space in question.

It is due to the non-commutative Specker phenomenon.

Theorem(K.Eda)

Let X and Y be a one-dimensional Peano continua which are not semi-locally simply connected at any point.

Then, X and Y are homeomorphic if and only if, the fundamental groups of X and Y are isomorphic.

Theorem(K.Eda)

Let X and Y be one-dimensional Peano continua.

If the fundamental groups of X and Y are isomorphic, then X and Y are homotopy equivalent.

3.Examples of finitely generated groups which is n-slender

Definition

P, Q: properties for groups.

A group G is P by Q iff,

there exists a normal subgroup N such that N satisfies P and G/N satisfies Q.

Lemma

If G is n-slender by n-slender, then G is n-slender.

proof

Let N be a n-slender normal subgroup of G such that G/N is also n-slender,

$h : \circledast_{n<\omega}\mathbb{Z}_n \to G$ be a homomorphism,

and σ be the canonical projection from G to G/N.

Since G/N is n-slender, $\sigma \circ h[\circledast_{m_0 \leq n < \omega} \mathbb{Z}_n] = \{e\}$ for some $m_0 < \omega$.

It implies $h[\circledast_{m_0 \leq n < \omega} \mathbb{Z}_n] \leq N$.

$h[\circledast_{m_1 \leq n < \omega} \mathbb{Z}_n] = \{e\}$ for some $m_1 > m_0$ because N is n-slender.

It means G is n-slender.

Cor.1

$\pi_1(M_g)$ is n-slender for any g and $\pi_1(N_g)$ is n-slender for $g \geq 2$.

i.e; any torsion-free surface group is n-slender.

$\pi_1(M_g)$: the fundamental group of the orientable compact surface with genus g.

$\pi_1(N_g)$: the fundamental group of the non-orientable compact surface with genus g.

proof of Cor.1

$\pi_1(M_g) = < a_1, \cdots, a_g, b_1, \cdots, b_g \mid [a_1, b_1] \cdots [a_g, b_g] >$

$\pi_1(N_g) = < a_1, \cdots, a_g \mid a_1^2 \cdots a_g^2 >$

It is well known that any subgroup of surface groups with infinite index is free.

We can easily find homomorphisms from $\pi_1(M_g)$,$\pi_1(N_g)$ to some free abelian group respectively.

Since a kernel of such a homomorphism is free, we conclude any torsion-free surface group is n-slender.

Cor.2

$BS(1,m) = < a, b \mid aba^{-1} = b^m >$ is n-slender for any $1 \leq m < \omega$.

proof of Cor.2

Let be a homomorphism h from $BS(1,m)$ to $\mathbb{Z}$ such that $h(a) = 1$ and $h(b) = 0$.

It is well known that ker(h) is the additive group of m-adic rationals, which is a proper subgroup of $\mathbb{Q}$.

It implies ker(h) is n-slender, we conclude $BS(1,m)$ is n-slender.

Cor.3

$G(p,q) =< x, y \mid x^p = y^q >$ is n-slender where gcd$(p,q) = 1$ and $p, q \geq 2$.

proof of Cor.3

The abelianization of $G(p,q)$ is equal to $\mathbb{Z}$.

It is well know that the commutator subgroup of $G(p,q)$ is a free group of rank $(p-1)(q-1)$.

We conclude the torus knot group $G(p,q)$ is n-slender by lemmna.

4.Examples of f.g. torsion-free groups which are not n-slender

Firstly we conjectured that any torsion-free f.g. group is n-slender because it is true for abelian groups.

But,using famous results of G.Higman,

we can obtain a f.g. torsion-free group containing $\mathbb{Q}$, which is a counter example of our conjecture.

Theorem.1 (Higman,Neuman, and Neuman. 1949)

Every countable group C can be embedded in a group G generated by two elements.

The group G has a torsion element if and only if C does.

proof of Theorem.1

Assume that $C = < c_1, c_2, \cdots | R >$ and let $F = C * < a, b >$.

$\{b^{-n}ab^n \mid n < \omega\}$ freely generates a free subgroup of $< a, b >$.

$\{b, c_n a^{-n} b a^n \mid 1 \leq n < \omega\}$ freely generates in F.

$G = < F, t \mid t^{-1}at = b, t^{-1}b^{-n}ab^n t = c_n a^{-n} b a^n, n \geq 1 >$

G is generated by t and a. C is embedded in G.

By Theorem.1, we construct a torsion-free group 2-generated which contains $\mathbb{Q}$.

Such a group is not n-slender because $\mathbb{Q}$ is not slender.

Now, we introduce a more typical counter example using a remarkable theorem of G.Higman.

Theorem.2 (The Higman Embedding Theorem)

Every recursively presented group can be embedded in some finitely presented group.

By this theorem,

we get a finitely presented torsion-free group which contains $\mathbb{Q}$.

Questions

Q.1 : Is any torsion-free one-relator group n-slender?

Q.2 : What is the characterization of the n-slenderness?

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