Forcing iterations and Cichon's diagram

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Reals

The following are the typical spaces considered for the analysis of the real line

- *The Cantor space* $2^\omega = \prod_{n<\omega} 2 = \{f \mid f : \omega \to 2\}$ (recall that $2 = \{0, 1\}$).
- *The Baire space* $\omega^\omega = \prod_{n<\omega} \omega = \{f \mid f : \omega \to \omega\}$.
- $\mathbb{R}$ the set of real numbers.
- $[0, 1]$ the unit interval in $\mathbb{R}$.

These are perfect *Polish spaces* (second countable completely metrizable topological spaces).

- We refer as a *real* to an element of one of those spaces.
- Those spaces, as every perfect Polish space, have size $\mathfrak{c} := 2^{\aleph_0}$, the *size of the continuum*.

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Measure

On a perfect polish space X,

- $\mathcal{B}(X)$ is the σ-algebra generated by the open sets of X. A *borel subset of X* is a member of $\mathcal{B}(X)$.

On the cantor space 2^ω is defined a standard *measure* $\mu : \mathcal{B}(2^\omega) \to [0, 1]$.

- $N \subseteq 2^\omega$ is *null* if $N \subseteq B$ for some borel set such that $\mu(B) = 0$.
- $\mathcal{L}(2^\omega)$ is the σ-algebra generated by the open sets and the null sets of 2^ω. An object in that family is called *(Lebesgue) measurable*.
- The measure μ can be extended to a *complete measure* on $\mathcal{L}(2^\omega)$, which we still denote by μ. Here, $N \subseteq 2^\omega$ is null iff $\mu(N) = 0$.
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Some cardinal invariants for measure

We define the following cardinal numbers.

$\mathrm{add}(\mathcal{N})$ The *additivity of the null ideal* is the least size of a family $\mathcal{F} \subseteq \mathcal{N}(2^\omega)$ which union is not null.

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$\mathrm{non}(\mathcal{N})$ The *uniformity of the null ideal* is the least size of a non-null set of reals.

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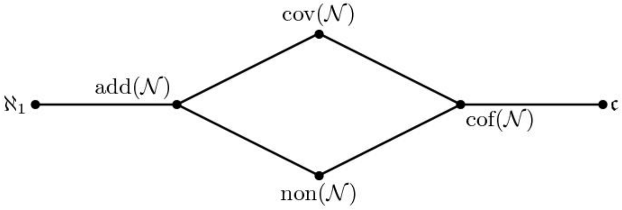
Likewise, a Lebesgue measure can be defined for the spaces ω^ω, $\mathbb{R}$ and $[0,1]$. The value of each of these cardinals doesn't change if we use one of those spaces.

The following diagram represents the order relation between these cardinals.



Category

Let X be a topological space.

- $A \subseteq X$ is *nowhere dense (n.w.d.)* if, for every G non-empty open in X, there exists a non-empty open $H \subseteq G$ that doesn't intersect A.
- $M \subseteq X$ is *meager* if it is the countable union of n.w.d. sets.
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The Baire Category Theorem

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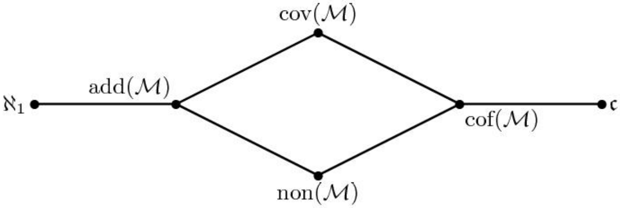

Two more cardinal invariants

In ω^ω, define

- For $f, g \in \omega^\omega$, $f \leq^* g$ means that $f(n) \leq g(n)$ for all but finitely many $n < \omega$. Here we say that f *is dominated by* g
- $\mathcal{F} \subseteq \omega^\omega$ is a *dominating family* if every real is dominated by some member of $\mathcal{F}$.

  - $\mathfrak{b}$ The *unbounding number* is the least size of a $\leq^*$-unbounded family of reals.
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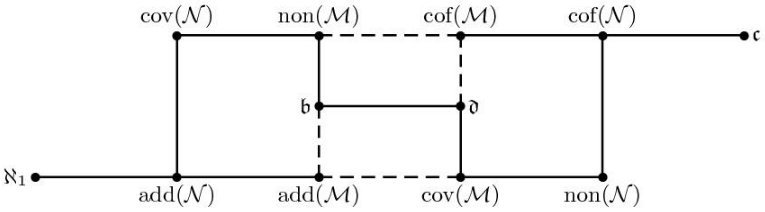

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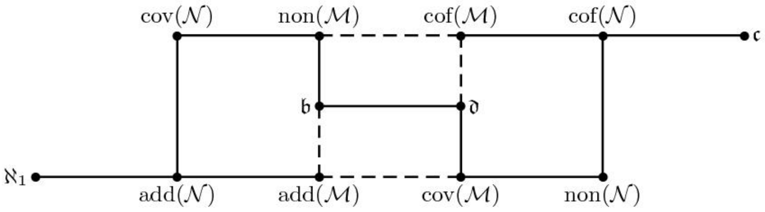

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- A *forcing notion* or *p.o. set* is a system $\langle \mathbb{P}, \leq, \mathbb{1} \rangle$ where $\leq$ is a reflexive and transitive relation in $\mathbb{P}$ and $\mathbb{1} \in \mathbb{P}$ is a maximum element. Elements in $\mathbb{P}$ are called *conditions* and $\mathbb{1}$ is the *trivial condition*.

From a model M of ZFC and G a $\mathbb{P}$-*generic set over* M, $M[G]$ is defined as a model of ZFC that extends M and contains G. In this context, we call M the *ground model* and $M[G]$ the *generic extension*. Every element in $M[G]$ has a $\mathbb{P}$-*name* in the ground model that "codes" it.

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$$V \overset{\mathbb{Q}_0\text{-ext.}}{\subseteq} V_1 \overset{\mathbb{Q}_1\text{-ext.}}{\subseteq} \cdots \subseteq V_\alpha \overset{\mathbb{Q}_\alpha\text{-ext.}}{\subseteq} V_{\alpha+1} \overset{\mathbb{Q}_{\alpha+1}\text{-ext.}}{\subseteq} \cdots \subseteq V_\delta$$

Fact

Assume δ is a limit ordinal with uncountable cofinality. In the context of a c.c.c. forcing iteration of length δ, any real in V_δ is in V_α for some $\alpha < \delta$.

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$$V \overset{\mathbb{Q}_0\text{-ext.}}{\subseteq} V_1 \overset{\mathbb{Q}_1\text{-ext.}}{\subseteq} \cdots \subseteq V_\alpha \overset{\mathbb{Q}_\alpha\text{-ext.}}{\subseteq} V_{\alpha+1} \overset{\mathbb{Q}_{\alpha+1}\text{-ext.}}{\subseteq} \cdots \subseteq V_\delta$$

Fact

Assume δ is a limit ordinal with uncountable cofinality. In the context of a c.c.c. forcing iteration of length δ, any real in V_δ is in V_α for some $\alpha < \delta$.

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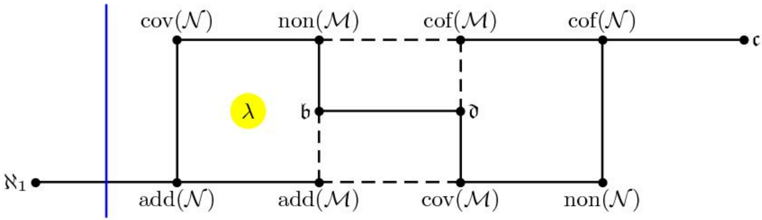
Examples on the Cichon's diagram

Assume that λ is an uncountable regular cardinal and that V is a model of $\mathrm{ZFC} + \mathrm{GCH}$.

Iteration of length λ using amoeba forcing $\mathbb{A}$



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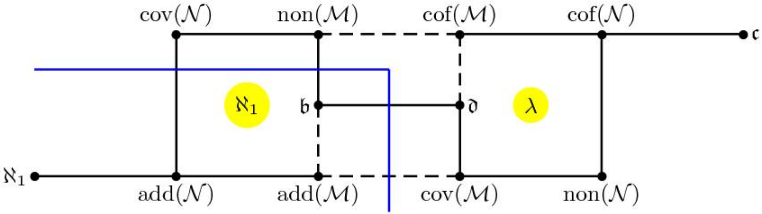

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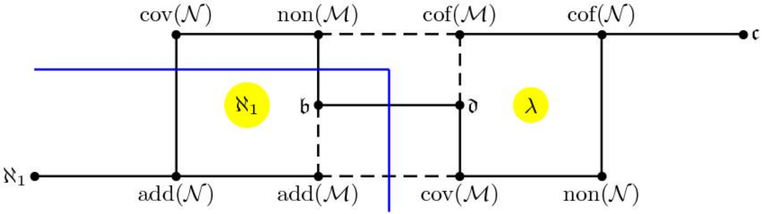

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Examples on the Cichon's diagram

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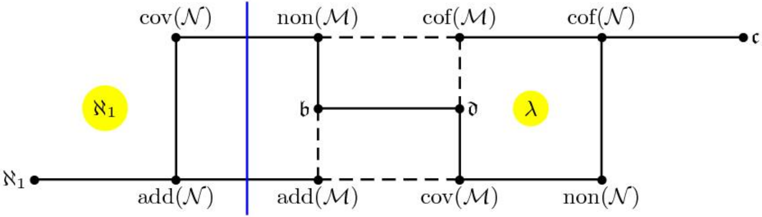
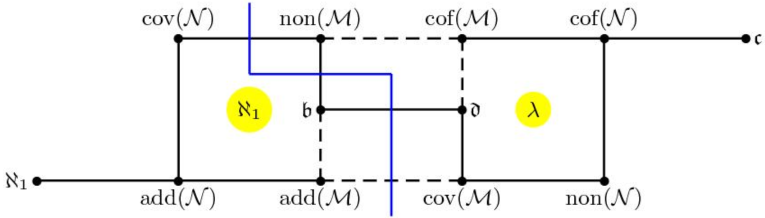


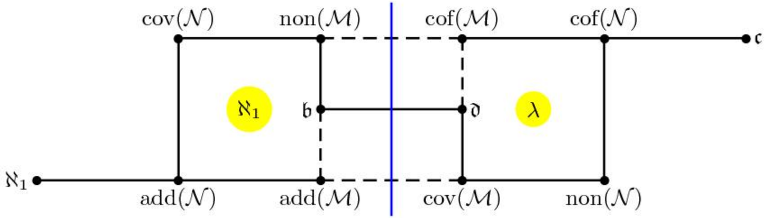
Examples on the Cichon's diagram

Iteration of length λ using e.d. forcing $\mathbb{E}$



Examples on the Cichon's diagram

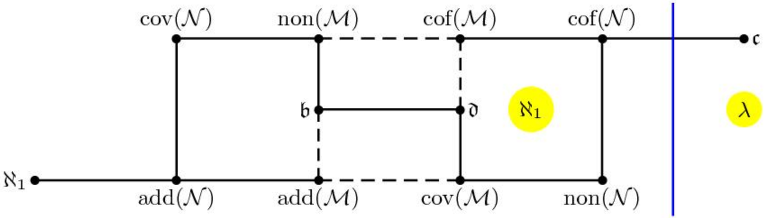
Iteration of length λ using cohen forcing $\mathbb{C}$



Dual case of the previous examples

Assume that λ is an uncountable regular cardinal and that V is a model of $\mathrm{ZFC} + \mathrm{add}(\mathcal{N}) = \mathfrak{c} = \lambda$.

Iteration of length ω_1 using amoeba forcing $\mathbb{A}$



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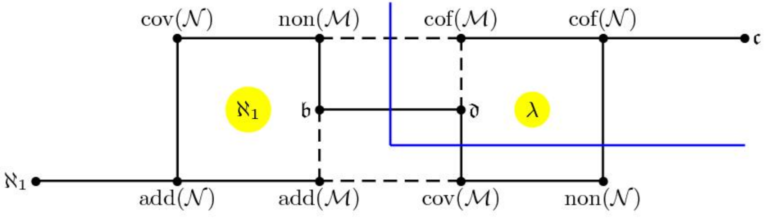

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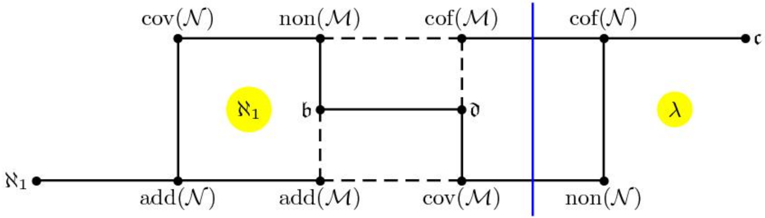

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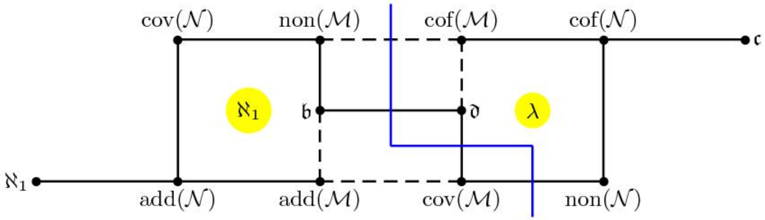
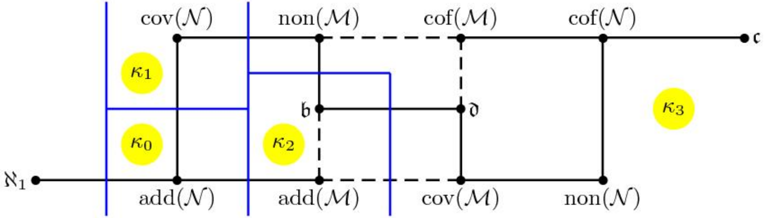
Iteration of length ω_1 using e.d. forcing $\mathbb{E}$



One example for the left side of the Cichon's diagram

Theorem

Assume $\kappa_0 \leq \kappa_1 \leq \kappa_2 \leq \kappa_3$ *are uncountable regular cardinals. It is consistent with* ZFC *that* $\aleph_1 \leq \kappa_0 = \mathrm{add}(\mathcal{N}) \leq \kappa_1 = \mathrm{cov}(\mathcal{N}) \leq \kappa_2 = \mathfrak{b} = \mathrm{add}(\mathcal{M}) \leq \kappa_3 = \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{N}) = \mathfrak{c}$.



How about the dual case?

- The c.c.c. forcing iteration technique works to get similar consistency results on the left side of the Cichon's diagram.
- The usual c.c.c. iteration technique doesn't seem to work on the dual case, that is, when we look at the right hand side of the Cichon's diagram.
- To get consistency results on the case when we look at the right hand side of the Cichon's diagram and we use two uncountable regular cardinals κ, λ (greater than $\aleph_1$), a new approach on the iteration technique is needed.

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Matrix of iterations of c.c.c. forcing notions (Shelah-Blass,1984)

For δ, γ ordinals, we consider a matrix iteration $\langle\langle\mathbb{P}_{\alpha,\xi}, \dot{\mathbb{Q}}_{\alpha,\xi}\rangle_{\xi<\gamma}\rangle_{\alpha\leq\delta}$ defined by the following conditions.

(1) $\mathbb{P}_{\delta,0} = \langle\mathbb{P}_{\alpha,0}, \dot{\mathbb{C}}\rangle_{\alpha<\delta}$ (f.s.i. of cohen forcing).

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(3) For $\alpha < \beta \leq \delta, \xi < \gamma$, $\dot{\mathbb{Q}}_{\alpha,\xi} \subseteq \dot{\mathbb{Q}}_{\beta,\xi}$ in the (β, ξ) extension.

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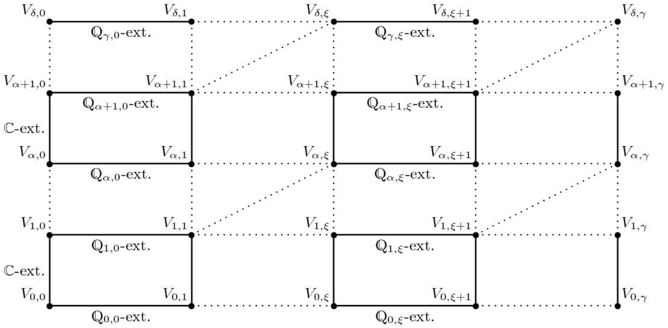
Matrix of iterations of c.c.c. forcing notions

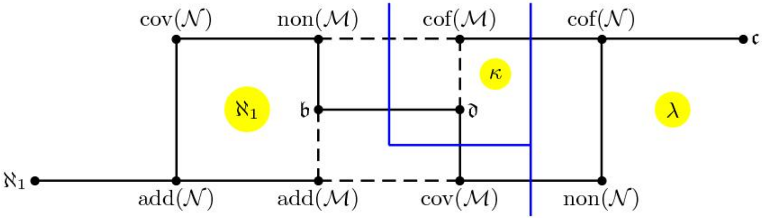
Like in the case of "linear" iterations, if $G_{\delta,\gamma}$ is $\mathbb{P}_{\delta,\gamma}$-generic over V, we consider the model $V_{\alpha,\xi}$ for $\alpha \leq \delta, \xi \leq \gamma$ as a $\mathbb{P}_{\alpha,\xi}$-extension. $V_{0,0} = V$ and the generic extensions can be seen as in the figure.



One application

Theorem

For $\kappa \leq \lambda$ *uncountable regular cardinals, it is consistent with* ZFC *that* $\aleph_1 = \mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{M}) \leq \kappa = \mathfrak{d} = \mathrm{cof}(\mathcal{M}) \leq \lambda = \mathrm{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \mathfrak{c}$.



Sketched proof

Start with V model of $\mathrm{ZFC} + \mathrm{add}(\mathcal{N}) = \mathfrak{c} = \lambda$. Let $f : \kappa \cdot \omega_1 \to \kappa$ such that $f(\xi) = \delta$, where $\xi = \kappa \cdot \epsilon + 2\delta + k$ for (unique) $\epsilon < \omega_1$, $\delta < \kappa$ and $k < 2$. Perform a matrix iteration $\langle\langle \mathbb{P}_{\alpha,\xi}, \dot{\mathbb{Q}}_{\alpha,\xi} \rangle_{\xi<\kappa\cdot\omega_1}\rangle_{\alpha\leq\kappa}$ as explained, such that, for $\xi < \kappa \cdot \omega_1$,

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In the extension $V_{\kappa,\kappa\cdot\omega_1}$, $\mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{M}) = \aleph_1$ because of the cofinally many e.d. and cohen reals added in the iteration. $\mathrm{non}(\mathcal{N}) = \mathfrak{c} = \lambda$ by preservation properties. $\mathfrak{d} = \mathrm{cof}(\mathcal{M}) = \kappa$ because of the dominating reals added by each $\dot{\mathbb{D}}_\xi$ ($\xi < \kappa \cdot \omega_1$) and by the preservation of the unbounding (cohen) reals added at each $(\alpha + 1, 0)$ step.

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Sketched proof

Start with V model of $\mathrm{ZFC} + \mathrm{add}(\mathcal{N}) = \mathfrak{c} = \lambda$. Let $f : \kappa \cdot \omega_1 \to \kappa$ such that $f(\xi) = \delta$, where $\xi = \kappa \cdot \epsilon + 2\delta + k$ for (unique) $\epsilon < \omega_1$, $\delta < \kappa$ and $k < 2$. Perform a matrix iteration $\langle\langle \mathbb{P}_{\alpha,\xi}, \dot{\mathbb{Q}}_{\alpha,\xi}\rangle_{\xi<\kappa\cdot\omega_1}\rangle_{\alpha\leq\kappa}$ as explained, such that, for $\xi < \kappa \cdot \omega_1$,

(I) If $\xi \equiv 0 \mathrm{mod} 2$, make $\dot{\mathbb{Q}}_{\alpha,\xi} = \dot{\mathbb{E}}$ ($\mathbb{P}_{\alpha,\xi}$-name).

(II) If $\xi \equiv 1 \mathrm{mod} 2$, make

$$\dot{\mathbb{Q}}_{\alpha,\xi} = \begin{cases} \mathbb{1}, & \text{if } \alpha \leq f(\xi), \\ \dot{\mathbb{D}}_\xi, & \text{if } \alpha > f(\xi), \end{cases}$$

where $\dot{\mathbb{D}}_\xi$ is the $\mathbb{P}_{f(\xi),\xi}$-name for $\mathbb{D}$.

In the extension $V_{\kappa,\kappa\cdot\omega_1}$, $\mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{M}) = \aleph_1$ because of the cofinally many e.d. and cohen reals added in the iteration. $\mathrm{non}(\mathcal{N}) = \mathfrak{c} = \lambda$ by preservation properties. $\mathfrak{d} = \mathrm{cof}(\mathcal{M}) = \kappa$ because of the dominating reals added by each $\dot{\mathbb{D}}_\xi$ ($\xi < \kappa \cdot \omega_1$) and by the preservation of the unbounding (cohen) reals added at each $(\alpha + 1, 0)$ step.

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(I) If $\xi \equiv 0 \bmod 2$, make $\dot{\mathbb{Q}}_{\alpha,\xi} = \dot{\mathbb{E}}$ ($\mathbb{P}_{\alpha,\xi}$-name).

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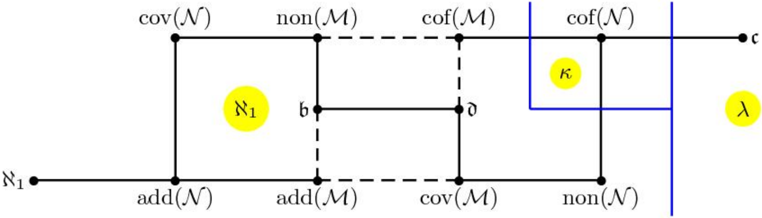
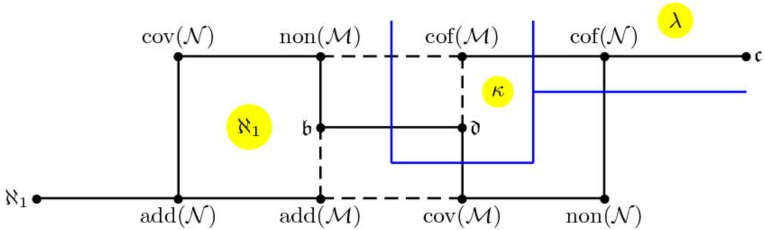
More applications

Similarly, we can obtain the consistency with ZFC of



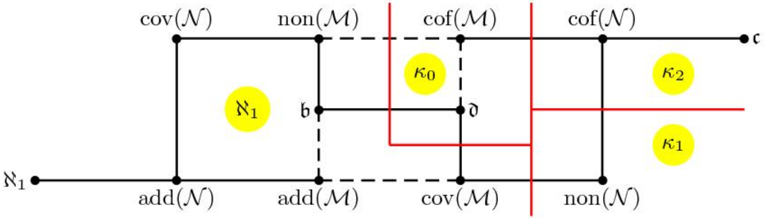
More applications



Questions

Question 1

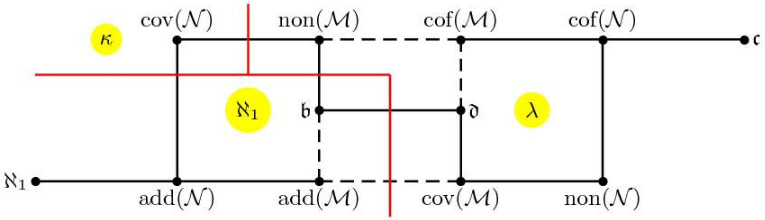
If $\aleph_1 < \kappa_0 < \kappa_1 < \kappa_2$ for $\kappa_0, \kappa_1, \kappa_2$ regular cardinals, is it consistent with ZFC that $\aleph_1 = \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) < \kappa_0 = \mathfrak{d} = \mathrm{cof}(\mathcal{M}) < \kappa_1 = \mathrm{non}(\mathcal{N}) < \kappa_2 = \mathrm{cof}(\mathcal{N}) = \mathfrak{c}$?



Questions

Question 2

If $\aleph_1 < \kappa < \lambda$ for κ, λ regular cardinals, is it consistent with ZFC that $\aleph_1 = \mathfrak{b} < \kappa = \mathrm{cov}(\mathcal{N}) < \lambda = \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mathfrak{c}$.



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有り難う御座います！

Thank you!