Interpretability and the arithmetized completeness theorem

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- **The notion of interpretability was explicitly introduced by Tarski(1954) and systematically investigated by Feferman and Orey(1960).**
- **In this talk, we introduce a result about interpretability proved by using the arithmetized completeness theorem in Feferman(1960).**

Contents

Definition

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- $t(x = y) \equiv x = y$;
- $\forall c \in \mathcal{L}$: constant, $\exists \eta_c(x)$: $\mathcal{L}'$-formula s.t. $t(x = c) \equiv \eta_c(x)$;
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- $t(\neg\varphi) \equiv \neg t(\varphi)$ for any $\mathcal{L}$-formula φ;
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- $\cdots$;
- $\exists d(x)$: $\mathcal{L}'$-formula s.t.
  $t(\exists x \varphi(x)) \equiv \exists x(d(x) \wedge t(\varphi(x)))$ for any $\mathcal{L}$-formula $\varphi(x)$.

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S is **interpretable** in T ($S \leq T$)
$\overset{\text{def.}}{\Leftrightarrow}$ $\exists t$: interpretation of S in T.

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- **If $S \leq T$ and T is consistent, then S is consistent.**
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- $\mathcal{L}_A := \{0, S, +, \times, <\}$: language of arithmetic.
- PA: Peano arithmetic (Basic axioms of arithmetic with induction scheme for all $\mathcal{L}_A$-formulas).
- ZF: Zermelo-Fraenkel set theory (Inf: Axiom of infinity).

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- $\mathsf{ZF} - \mathsf{Inf} \leq \mathsf{PA}$.
- $\mathsf{ZF} \not\leq \mathsf{PA}$.
- $\mathsf{PA} + \mathsf{Con}_{\mathsf{PA}} \not\leq \mathsf{PA}$.
- $\mathsf{PA} + \neg\mathsf{Con}_{\mathsf{PA}} \leq \mathsf{PA}$.
- $\cdots$.

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- **Feferman's theorem is proved by using the arithmetized completeness theorem.**

Countable completeness theorem

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- $\{\varphi_n(x_n)\}_{n\in\omega}$: **primitive recursive(p.r.) enumeration of all $\mathcal{L}_C$-formulas with one free-variable.**
- $\exists Z := \{\exists x_n \varphi_n(x_n) \to \varphi_n(c_{i_n}) \mid n \in \omega\}$: **p.r. set s.t. $T + Z$ is a conservative extension of T.**

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- **Then $T + Z$ is consistent (Henkin extension).**

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S: **r.e.** $\mathcal{L}$**-theory,** T: $\mathcal{L}_A$**-theory. (**$\mathcal{L}$: **countable)**

$\exists \sigma(x)$: Σ_1 **formula s.t.** $\forall \varphi$: $\mathcal{L}$**-sentence,**

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For Σ_1 numeration $\sigma(x)$ of S in T,
we can construct a Σ_1 formula $\mathrm{Pr}_\sigma(x)$ s.t. $\forall\varphi$: $\mathcal{L}$-sentence,

$$S \vdash \varphi \Leftrightarrow T \vdash \mathrm{Pr}_\sigma(\ulcorner\varphi\urcorner).$$

$\mathrm{Pr}_\sigma(x)$ is called the **provability predicate** of $\sigma(x)$.

$\sigma(x), \sigma'(x)$: **numerations of S and S' in T respectively.**

Define

- $(\sigma|n)(x) := \sigma(x) \wedge x \leq \bar{n}$.
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- Define the p.r. set Z as above. Let $\zeta(x)$ be a suitable numeration of Z s.t.
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The arithmetized completeness theorem

$\forall \sigma(x)$: **numeration of T, $\exists \xi(x)$: $\mathcal{L}_A$-formula s.t.**

1. **PA** $\vdash$ **Con**$_\sigma \to$ **Hcm**$_\xi$ **and**
2. **PA** $\vdash \forall x(\mathrm{Pr}_\sigma(x) \to \xi(x))$.

Proof.

- $\{\theta_n\}_{n\in\omega}$: p.r. enumeration of all $\mathcal{L}_C$-sentences.
- $\xi(x)\cdots$ "x is contained in the leftmost consistent path".
- PA $\vdash \forall x(\mathrm{Pr}_\sigma(x) \to \xi(x))$.
- PA $\vdash$ Con$_{\sigma\vee\zeta} \to$ Hcm$_\xi$.

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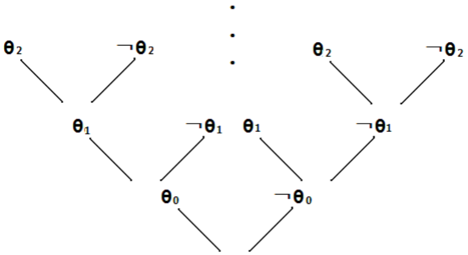

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- φ: $\mathcal{L}_S$-sentence s.t. $S \vdash \varphi$. $T \vdash \mathrm{Pr}_\sigma(\ulcorner \varphi \urcorner)$.

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- $\forall \varphi$, $T + \mathsf{Con}_\sigma \vdash t(\varphi) \leftrightarrow \xi(\ulcorner \varphi \urcorner)$.
- t is a translation of $\mathcal{L}_S$ into $\mathcal{L}_{T+\mathsf{Con}_\sigma}$.
- φ: $\mathcal{L}_S$-sentence s.t. $S \vdash \varphi$. $T \vdash \mathrm{Pr}_\sigma(\ulcorner \varphi \urcorner)$.
- $T + \mathsf{Con}_\sigma \vdash \xi(\ulcorner \varphi \urcorner)$. $T + \mathsf{Con}_\sigma \vdash t(\varphi)$.

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- **For** $c \in \mathcal{L}_S$: **constant,**
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- **Since** $S \leq T$, $T \vdash t(\varphi)$.

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- $\mathcal{N}$ **is a model of** S.

Definition

$\mathcal{M}, \mathcal{N}$: **models of arithmetic.**

$\mathcal{M}$ **is an initial segment of** $\mathcal{N}$ $(\mathcal{M} \subseteq_e \mathcal{N}) \overset{\text{def.}}{\Leftrightarrow}$

1. $|\mathcal{M}| \subseteq |\mathcal{N}|$ **and**
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Theorem(Orey(1961), Hájek (1971,1972))

For any consistent r.e. extensions S, T of PA, T.F.A.E.:

(i) **$S \leq T$.**

(ii) **$\forall \mathcal{M} \models T \, \exists \mathcal{N} \models S$ s.t. $\mathcal{M} \subseteq_e \mathcal{N}$.**

(iii) **$\forall \theta$: Π_1 sentence, $S \vdash \theta \Rightarrow T \vdash \theta$.**

(iv) **$\forall \sigma(x)$: Σ_1 numeration of S $\forall n \in \omega$, $T \vdash \mathrm{Con}_{\sigma|n}$.**

(i) $S \leq T$.

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(i) $\Rightarrow$ (ii)

- Let θ be an interpretation of S in T.
- Let $\mathcal{M}$ be any model of T.

(i) $S \leq T$.

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- Let θ be an interpretation of S in T.
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- Let θ be an interpretation of S in T.
- Let $\mathcal{M}$ be any model of T.
- Let $\mathcal{N}$ be a model of S defined by t and $\mathcal{M}$.
- Define a function f in $\mathcal{M}$ satisfying $f(0^{\mathcal{M}}) = 0^{\mathcal{N}}$ and $f(S^{\mathcal{M}}(a)) = S^{\mathcal{N}}(f(a))$.

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- Define a function f in $\mathcal{M}$ satisfying $f(0^{\mathcal{M}}) = 0^{\mathcal{N}}$ and $f(S^{\mathcal{M}}(a)) = S^{\mathcal{N}}(f(a))$.
- Then f is an isomorphism of an initial segment of $\mathcal{N}$.

(ii) $\forall \mathcal{M} \models T \exists \mathcal{N} \models S$ s.t. $\mathcal{M} \subseteq_e \mathcal{N}$.

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(ii) ⇒ (iii)

- **Let θ be any Π_1 sentence s.t. $S \vdash \theta$.**
- **Let $\mathcal{M}$ be any model of T.**
- **By (ii), $\exists \mathcal{N} \models S$ s.t. $\mathcal{M} \models \mathcal{N}$.**

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- $\mathcal{N} \models \theta$.
- Since θ is Π_1, $\mathcal{M} \models \theta$.

(ii) $\forall \mathcal{M} \models T \exists \mathcal{N} \models S$ s.t. $\mathcal{M} \subseteq_e \mathcal{N}$.

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- Since θ is Π_1, $\mathcal{M} \models \theta$.
- By completeness theorem, $T \vdash \theta$.

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- **By Mostowski's theorem,** $\forall n \in \omega$, $S \vdash \mathrm{Con}_{\sigma|n}$.

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- Let $\sigma^*(x) :\equiv \sigma(x) \wedge \mathrm{Con}_{\sigma|x}$.

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- Let $\sigma^*(x) :\equiv \sigma(x) \wedge \mathsf{Con}_{\sigma|x}$.
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- **Model theoretic proof of the second incompleteness theorem**
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Theorem

T, S: consistent r.e. extensions of PA. $\mathcal{M}$: models of T.
If $\mathcal{M} \models \mathrm{Con}_S$, then
$\exists \mathcal{N} \models S$ s.t. $\mathcal{M} \subseteq_e \mathcal{N}$ and
$\exists \xi(x)$: $\mathcal{L}_A$-formula s.t.

- $\forall \varphi$, $(\mathcal{M} \models \mathrm{Pr}_S(\ulcorner \varphi \urcorner) \Rightarrow \mathcal{N} \models \varphi)$,
- $\forall \varphi$, $(\mathcal{M} \models \xi(\ulcorner \varphi \urcorner) \Leftrightarrow \mathcal{N} \models \varphi)$.

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- $\forall \varphi$, $(\mathcal{M} \models \mathrm{Pr}_S(\ulcorner \varphi \urcorner) \Rightarrow \mathcal{N} \models \varphi)$,
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- **Suppose $\forall \mathcal{M} \models T$, $\mathcal{M} \models \mathrm{Con}_T$.**
- **By using Theorem, lead a contradiction.**

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Theorem(Lindström)

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1. S is faithful interpretable in T.
2. $S \leq T$ and $\forall\varphi, (T \vdash \mathrm{Pr}_{\phi}(\ulcorner\varphi\urcorner) \Rightarrow S \vdash \varphi)$.
3. $\forall\theta : \Pi_1$ sentence, $(S \vdash \theta \Rightarrow T \vdash \theta)$ and
   $\forall\sigma : \Sigma_1$ sentence, $(T \vdash \sigma \Rightarrow S \vdash \theta)$ and

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- $\mathcal{D}_T := (D_T, \leq)$.

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- $D_T :=$ the set of degrees of extensions of T.
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- $\mathcal{D}_T := (D_T, \leq)$.

Theorem(Lindström)

T is an r.e. consistent extension of PA, then $\mathcal{D}_T$ is a distributive lattice.

Open problem

S, T is Σ_1-sound r.e. extensions of PA. $\mathcal{D}_S \simeq \mathcal{D}_T$?

- **Interpretability logic (Visser, 1990)(Shavrukov, 1988)(Berarducci, 1990)**

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Propositional modal logic **ILM**

ILM $=$ **GL**$+$ **the following axioms:**

- $\Box(A \to B) \to A \rhd B$;
- $A \rhd B \land B \rhd C \to A \rhd C$;
- $A \rhd C \land B \rhd C \to (A \lor B) \rhd C$;
- $A \rhd B \to (\Diamond A \to \Diamond B)$;
- $\Diamond A \rhd A$;
- $A \rhd B \to ((A \land \Box C) \rhd (B \land \Box C))$.

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- $A \rhd B \to ((A \land \Box C) \rhd (B \land \Box C))$.

Theorem(Shavrukov,Berarducci)

$\forall A$, (**ILM** $\vdash A \Leftrightarrow A$ **is arithmetically valid**).

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