The difficulty of marriage

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joint work with

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What is Marriage Problem?

Marriage problem

There are boys and girls and each boy is acquainted with finitely many girls. Under what condition is it possible that all boys marry one of his acquaintances?

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Let $(B, G; R)$ be a bipartite graph such that each vertex in B is adjacent to finitely many vertices in G. Under what condition is it possible that there exists a matching of B ?

- A graph is **bipartite** if the vertices can be devided so that every edges consist of two another kind of vertices.
- A **matching of** B is a set of independent edges such that every vertices in B is incident with an edge in it.

Theorem (P. Hall, 1935)

Let $\mathbb{G} = (B, G; R)$ be a finite bipartite graph.
Then there is a matching of B if $\mathbb{G}$ satisfies Hall condition.

- Hall condition (for B) : for all $X \subset_{fin} B$, $|N(X)| \geq |X|$.

Here $N(X)$ is the set of vertices in G which are adjacent to the vertecies in X. $|N(X)| \geq |X|$ means that n boys know at least n girls.

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This theorem can be extended to infinite graph.

Theorem (M. Hall, 1948)

Let $\mathbf{G} = (B, G; R)$ *be a bipartite graph and* B-*locally finite.*
Then $\mathbf{G}$ *has a solution if* $\mathbf{G}$ *satisfies Hall condition.*

- B-locally finite : for all $b \in B$, $|N(b)| < \infty$.
- Solution of $\mathbf{G}$: injection $M : B \to G$ s.t. $M \subset R$.

That is, B-locally finite means that each boy is acquainted with finitely many girls and the solution means the appropriate marriage.

Note that if "B-locally finite" is dropped, the above statement does not hold.

Marriage Theorems and Recursive Graph Theory

Let $\mathbf{G}$ be a recursive bipartite (countable) graph and $\mathbf{G}$ is B-locally finite. Then if $\mathbf{G}$ satisfies Hall condition, $\mathbf{G}$ has a solution by the previous theorem.

Now can we take the solution recursively? The answer is "no".

Theorem (A.Manaster and J.Rosenstein, 1972)

There exists a recursive bipartite graph which is B-locally finite and satisfies Hall condition, but has no recursive solution.

Can we modify this to have a recursive solution?

- B-strong locally finite : the function $p(b) = |N(b)|$ is recursive.
- G-strong locally finite : the function $p(g) = |N(g)|$ is recursive.

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Expanding Hall Condition

There is a function h_B s.t.
$$h_B(0) = 0 \ \& \ \forall n \forall X \subset_{fin} B(|X| \geq h_B(n) \rightarrow |N(X)| - |X| \geq n).$$

That is, expanding Hall condition requires that if the number of boys is greater than $h_B(n)$, they knows extra n girls.

Theorem (H.Kierstead, 1983)

If $\mathbb{G}$ is recursive bipartite graph which is B, G-strong locally finite and satisfies expanding Hall condition with a recursive h_B, then $\mathbb{G}$ has a recursive solution.

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Theorem (H.Kierstead, 1983)

There exists a recursive bipartite graph which is B, G-strong locally finite and satisfies expanding Hall condition, but has no recursive solution.

Kierstead also indicated that a bipartite graph which satisfies expanding Hall condition has a solution even if "*B*-locally finite" is dropped.

Fact.

If $\mathbf{G}$ is a bipartite graph which satisfies expanding Hall condition, then $\mathbf{G}$ has a solution.

What is Reverse Mathematics?

- The aim of reverse mathematics is classifying the mathematical theorems by the difficulty.

- For that reason, we look for the set existence axiom which is exactly needed to prove each theorem. That is, we check which set existence axiom is necessary and sufficient to the theorem over base system.

- In reverse mathematics, we prove not only a theorem form axioms but also an axiom from the theorem. That is the reason why this research program is called "reverse mathematics".

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- Although there are so many mathematical theorems, most of ordinary mathematical theorems are classified to the following five classes in this method.

strictly	$\Pi^1_1\text{-CA}_0$	RCA_0 + Π^1_1 set existence axiom
stronger	ATR_0	RCA_0 + arithmetical transfinite recursion
$\Uparrow$	ACA_0	RCA_0 + arithmetical set existence axiom
$\Uparrow$	WKL_0	RCA_0 + weak König's lemma
	RCA_0	discrete ordered semiring + Σ^0_1-induction
		+ recursive set existence axiom

RCA_0 is our base formal system. It just guarantees the existence of recursive sets.

Now we analyze how difficult each marriage problem is !

Marriage theorems provides an typical example of reverse mathematics. In my talk, only RCA_0 and WKL_0 and ACA_0 appear.

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Reverse Mathematics of Marriage Theorems

Hirst analyzed the strength of several marriage theorems with respect to reverse mathematics.

Theorem (J.Hirst, 1990)

The following is provable within RCA$_0$.
If $\mathbf{G} = (B, G; R)$ *is a bipartite graph which satisfies Hall condition and* B *is finite, then* $\mathbf{G}$ *has a solution.*

This theorem is used throughout as a basic tool to anaryze the strength of marriage theorems.

- $RCA_0 \vdash B'_H GM \leftrightarrow ACA_0$
- $RCA_0 \vdash B''_H GM \leftrightarrow WKL_0$

$B'_H GM$: If $\mathbf{G} = (B, G; R)$ is a bipartite graph which is B-locally finite and satisfies Hall condition, then $\mathbf{G}$ has a solution.

$B''_H GM$: If $\mathbf{G} = (B, G; R)$ is a bipartite graph which is B-strong locally finite and satisfies Hall condition, then $\mathbf{G}$ has a solution.

"B-strong locally finite" is written as follows within RCA_0.

$$\exists p : B \to \mathbb{N} \text{ s.t.} \forall b, g((b, g) \in R \to g < p(b))$$

Hirst also analyzed the strength of symmetric marriage theorems.

Theorem (J.Hirst, 1990)

- $RCA_0 \vdash B'_H G'_H M_s \leftrightarrow ACA_0$
- $RCA_0 \vdash B''_H G''_H M_s \leftrightarrow WKL_0$

$B'_H G'_H M_s$: If $\mathbf{G} = (B, G; R)$ is a bipartite graph which is B, G-locally finite and satisfies Hall condition for B and G, then $\mathbf{G}$ has a symmetric solution.

$B''_H G''_H M_s$: If $\mathbf{G} = (B, G; R)$ is a bipartite graph which is B, G-strong locally finite and satisfies Hall condition for B and G, then $\mathbf{G}$ has a symmetric solution.

- "G-locally finite" and "G-strong locally finite" and "Hall Condition for G" are defined in the same manner for B.
- A **symmetric solution** is a bijectve solution. That is, the symmetric solution requires any girl also can marry.

"Expanding Hall condition" and "expanding Hall condition with recursive h_B" are written as follows within RCA_0.

- H′ :
  $\forall n \exists m \forall X \subset_{fin} B(|N(X)| - |X| \geq 0 \land |X| \geq m \rightarrow |N(X)| - |X| \geq n)$
- H″ : $\exists h_B : B \rightarrow \mathbb{N}$ s.t.
  $\forall n \forall X \subset_{fin} B(|N(X)| - |X| \geq 0 \land |X| \geq h_B(n) \rightarrow |N(X)| - |X| \geq n)$

We analyze the strength of all the considerable marriage theorems and symmetric marriage theorems with respect to reverse mathematics.

	Hall condition	Expanding Hall condition	Strongly expanding Hall condition	
ACA_0	$*$	$B_{H'}GM$	$B_{H''}GM$	
	$*$	$B_{H'}G'M$	$B_{H''}G'M$	
	$*$	$B_{H'}G''M$	$B_{H''}G''M$	($\dashv RCA_0 + \Sigma_3^0\text{-IND}$)
	$B'_H GM$ (Hirst)	$B'_{H'}GM$	$B'_{H''}GM$	
	$B'_H G'M$	$B'_{H'}G'M$	$B'_{H''}G'M$	
	$B'_H G''M$	$B'_{H'}G''M$	$B'_{H''}G''M$	($\dashv RCA_0 + \Sigma_3^0\text{-IND}$)
WKL_0	$B''_H GM$ (Hirst)	$B''_{H'}GM$	$B''_{H''}GM$	
	$B''_H G'M$	$B''_{H'}G'M$	$B''_{H''}G'M$	
	$B''_H G''M$	$B''_{H'}G''M$	$B''_{H''}G''M$	(Kierstead)
RCA_0				

$*$: false X' : X-locally finite X'' : X-strong locally finite

Corollary.

If $\mathbf{G}$ is recursive bipartite graph which is G-strong locally finite and satisfies expanding Hall condition with a recursive h_B, then $\mathbf{G}$ has a recursive solution.

$\because$)$\mathsf{RCA}_0 + \Sigma^0_3\text{-IND} \vdash \mathsf{B}_{\mathsf{H}''}\mathsf{G}''\mathsf{M}.$

Corollary.

There exists a recursive bipartite graph which is B-strong locally finite and G-locally finite and satisfies expanding Hall codition with a recursive h_B, but does not have a recursive solution.

$\because$)$\mathsf{RCA}_0 \vdash \mathsf{B}''_{\mathsf{H}''}\mathsf{G}'\mathsf{M} \rightarrow \mathsf{WKL}_0.$

The reasonable question is whether Σ^0_3 induction is essential for proving $\mathsf{B}_{\mathsf{H}''}\mathsf{G}''\mathsf{M}$ and $\mathsf{B}'_{\mathsf{H}''}\mathsf{G}''\mathsf{M}$.

Conjecture

$\mathsf{RCA}_0 \nvdash \mathsf{B}_{\mathsf{H}''}\mathsf{G}''\mathsf{M}.$

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The strength of symmetric mariiage theorems

$B_H G_H M_s$	$B_H G_{H'} M_s$	$B_H G_{H''} M_s$	$B_{H'} G_{H'} M_s$	$B_{H'} G_{H''} M_s$	$B_{H''} G_{H''} M_s$
$B_H G'_H M_s$	$B_H G'_{H'} M_s$	$B_H G'_{H''} M_s$	$B_{H'} G'_{H'} M_s$	$B_{H'} G'_{H''} M_s$	$B_{H''} G'_{H''} M_s$
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$B'_H G_H M_s$	$B'_H G_{H'} M_s$	$B'_H G_{H''} M_s$	$B'_{H'} G_{H'} M_s$	$B'_{H'} G_{H''} M_s$	$B'_{H''} G_{H''} M_s$
$B'_H G'_H M_s$(Hirst)	$B'_H G'_{H'} M_s$	$B'_H G'_{H''} M_s$	$B'_{H'} G'_{H'} M_s$	$B'_{H'} G'_{H''} M_s$	$B'_{H''} G'_{H''} M_s$
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False   ACA$_0$ WKL$_0$ RCA$_0$

- It does not make sense for each assertion to consider the assertion which assumptions for B and G are interchanged, for those are pariwise equivalent clearly. Therefore the above table lists strength of all considerable symmetric marriage theorems.
- There is a counter example of the green assertions.

Corollary.

If **G** is recursive bipartite graph which is B, G-strong locally finite and satisfies expanding H.c. for B and G with a recursive h_B and h_G, then **G** has a recursive sym. solution.

$\because$)$\mathsf{RCA_0} \vdash \mathsf{B}''_{\mathsf{H}''}, \mathsf{G}''_{\mathsf{H}''}, \mathsf{M_s}$.

Corollary.

There exists a recursive bipartite graph which is B, G-strong locally finite and satisfies expanding H.c. for B and G with a recursive h_G, but does not have a recursive sym. solution.

$\because$)$\mathsf{RCA_0} \vdash \mathsf{B}''_{\mathsf{H}'}, \mathsf{G}''_{\mathsf{H}''}, \mathsf{M_s} \rightarrow \mathsf{WKL_0}$.

Corollary.

There exists a recursive bipartite graph which is B-strong locally finite and G-locally finite and satisfies expanding H.c. for B and G with a rec. h_B and h_G, but does not have a recursive sym. solution.

$\because$)$\mathsf{RCA_0} \vdash \mathsf{B}''_{\mathsf{H}''}, \mathsf{G}'_{\mathsf{H}''}, \mathsf{M_s} \rightarrow \mathsf{ACA_0}$.

Indexed Marriage Problem and Recursive Solution

Theorem (A.Manaster and J.Rosenstein, 1972)

There exists a recursive bipartite graph which is B, G-strong locally finite and satisfies Hall condition, but has no recursive solution.

As the method of modifying the above marriage ploblem to have a recursive solution, the following two ways are considerable.

1. Strengthening the codition about locally finite
2. Strengthening the Hall condition $\longrightarrow$ by Kierstead.

We go about the modification by the first method.

Indexed Marraige Theorem

Let $B = G = \omega$ and $R \subset \omega \times \omega$ and there is a function $q : \omega \to \omega$ s.t. $\forall (i, j) \in R (j \leq i + q(i))$.
Then $\mathbf{G} = (B, G; R)$ has a solution if $\mathbf{G}$ satisfies Hall condition.

- We regard the function q as the parameter and consider the indexed marriage theorems dependent on q.
- By the result of Manaster-Rosenstein, in general we can not take the solution of this indexed marriage problem recursively even if the function q is recursive.
- But if $q(i) \equiv 0$, clearly we can take the solution recursively.

$\Rightarrow$ Where is the boundary?

Let $B = G = \omega$ and $R \subset \omega \times \omega$ and there is a function $q : \omega \to \omega$ s.t. $\forall (i, j) \in R (j \leq i + q(i))$.
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⇒Where is the boundary?

Theorem

$\mathsf{RCA}_0 \vdash (q : \mathbb{N} \to \mathbb{N} \wedge \forall i(q(i) \leq \bar{k})) \to \mathsf{IM}^{(q)}$.

$\mathsf{IM}^{(q)}$: If $\mathbf{G} = (\mathbb{N}, \mathbb{N}; R)$ is a bipartite graph which satisfies Hall condition and $\forall (i, j) \in R(j \leq i + q(i))$, then $\mathbf{G}$ has a solution.

Corollary.

Suppose q is a function from ω to ω and bounded by some constant.
If $\mathbf{G} = (\omega, \omega; R)$ be a recursive bipartite countable graph which satisfies Hall condition and $\forall (i, j) \in R(j \leq i + q(i))$,
then $\mathbf{G}$ has a recursive solution.

Remark.

$\mathsf{RCA}_0 + \Pi^0_2\text{-IND} \ \vdash (q : \mathbb{N} \to \mathbb{N} \wedge \exists k \forall i(q(i) \leq k)) \to \mathsf{IM}^{(q)}$.

Question.

$\mathsf{RCA}_0 \vdash (q : \mathbb{N} \to \mathbb{N} \wedge \exists k \forall i(q(i) \leq k)) \to \mathsf{IM}^{(q)}$? or not?

Theorem

$RCA_0 \vdash (q : \mathbb{N} \to \mathbb{N} \land \forall i(q(i) \leq \bar{k})) \to IM^{(q)}$.

$IM^{(q)}$: If $\mathbf{G} = (\mathbb{N}, \mathbb{N}; R)$ is a bipartite graph which satisfies Hall condition and $\forall(i, j) \in R(j \leq i+q(i))$, then $\mathbf{G}$ has a solution.

Corollary.

Suppose q is a function from ω to ω and bounded by some constant.
If $\mathbf{G} = (\omega, \omega; R)$ be a recursive bipartite countable graph which satisfies Hall condition and $\forall(i, j) \in R(j \leq i + q(i))$,
then $\mathbf{G}$ has a recursive solution.

Remark.

$RCA_0 + \Pi_2^0\text{-IND} \vdash (q : \mathbb{N} \to \mathbb{N} \land \exists k \forall i(q(i) \leq k)) \to IM^{(q)}$.

Question.

$RCA_0 \vdash (q : \mathbb{N} \to \mathbb{N} \land \exists k \forall i(q(i) \leq k)) \to IM^{(q)}$? or not?

Theorem

$RCA_0 \vdash [U(q) \rightarrow IM^{(q)}] \leftrightarrow WKL_0$,
where $U(q)$ *means*
$q : \mathbb{N} \rightarrow \mathbb{N} \land \forall k \exists i(q(i) > i + k) \land \forall i, i'(i < i' \rightarrow q(i) \leq q(i'))$

Corollary.

Suppose q is a recursive function from ω to ω and unbounded and nondecreasing.
Then there exists a recursive bipartite graph $\mathbf{G} = (\omega, \omega; R)$ which satisfies Hall condition and $\forall(i, j) \in R(j \leq i + q(i))$,
but has no recursive solution.

We also analyze the symmetric indexed marriage problem.

Theorem

$\mathsf{RCA}_0 \vdash (q : \mathbb{N} \to \mathbb{N} \land \forall i(q(i) \le \bar{k})) \to \mathsf{IM}_s^{(q)}$.

$\mathsf{IM}_s^{(q)}$: If $\mathbf{G} = (\mathbb{N}, \mathbb{N}; R)$ is a bipartite graph which satisfies symmetric Hall condition and $\forall(i, j) \in R(j \le i + q(i) \land i \le j + q(j))$, then $\mathbf{G}$ has a symmetric solution.

Corollary.

Suppose q is a function from ω to ω and bounded by some constant.
If $\mathbf{G} = (\omega, \omega; R)$ be a recursive bipartite countable graph which satisfies symmetric Hall condition and
$\forall(i, j) \in R(j \le i + q(i) \land i \le j + q(j))$, then $\mathbf{G}$ has a recursive symmetric solution.

Remark.

$\mathsf{RCA}_0 + \Pi_2^0\text{-IND} \vdash (q : \mathbb{N} \to \mathbb{N} \land \exists k \forall i(q(i) \le k)) \to \mathsf{IM}_s^{(q)}$.

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Thank you for your attention.

...Who should I marry?