

# Semistationary and stationary reflection

Hiroshi Sakai

Graduate School of Information Science,  
Nagoya University

## Abstract

We study the relationship between the semistationary reflection principle and stationary reflection principles. We show that for all regular cardinals  $\lambda \geq \omega_2$  the semistationary reflection principle in the space  $[\lambda]^\omega$  implies that every stationary subset of  $E_\omega^\lambda := \{\alpha \in \lambda \mid \text{cf}(\alpha) = \omega\}$  reflects. We also show that for all cardinals  $\lambda \geq \omega_3$  the semistationary reflection principle in  $[\lambda]^\omega$  does not imply the stationary reflection principle in  $[\lambda]^\omega$ .

## 1 Introduction

In this paper we compare the semistationary reflection principle with stationary reflection principles. The notion of semistationary sets and the semistationary reflection principle were introduced by Shelah [10](Ch.XIII §1). They are closely related to the semiproperness of posets. We review this:

**Notation 1.1.** *For countable sets  $x$  and  $y$ , we write  $x \sqsubseteq y$  if  $x \subseteq y$  and  $x \cap \omega_1 = y \cap \omega_1$ .*

**Definition 1.2** (Shelah [10] Ch.XIII, §1, 1.1.Def.). *Let  $W$  be a set with  $W \supseteq \omega_1$ . A subset  $S \subseteq [W]^\omega$  is called semistationary if the set  $\{y \in [W]^\omega \mid (\exists x \in S) x \sqsubseteq y\}$  is stationary in  $[W]^\omega$ .*

**Definition 1.3** (Shelah [10] Ch.XIII, §1, 1.5.Def.). *For a cardinal  $\lambda \geq \omega_2$ ,  $\text{SSR}([\lambda]^\omega)$ , the semistationary reflection principle in  $[\lambda]^\omega$ , is the following:*

$\text{SSR}([\lambda]^\omega) \equiv$  *For every semistationary  $S \subseteq [\lambda]^\omega$ , there exists  $W \subseteq \lambda$  such that  $|W| = \omega_1 \subseteq W$  and  $S \cap [W]^\omega$  is semistationary in  $[W]^\omega$ .*

In [10](Ch.XIII, §1, 1.4.Claim) Shelah shows that a poset  $\mathbb{P}$  is semiproper if and only if  $\mathbb{P}$  preserves  $\omega_1$  and preserves semistationary subsets of  $[W]^\omega$  for every  $W$ . He also shows that  $(\dagger)$  holds if and only if  $\text{SSR}([\lambda]^\omega)$  holds for every  $\lambda \geq \omega_2$ . Here  $(\dagger)$  is the principle, introduced in Foreman-Magidor-Shelah [3], that every poset preserving stationary subset of  $\omega_1$  is semiproper. This is known

---

MSC (2000) : Primary 03E55, Secondary 03E35

Key words : Semistationary reflection, Stationary reflection.

to have interesting consequences. Shelah [10](Ch.XII, §2) shows that if Namba forcing is semiproper then (a strong form of) Chang's conjecture holds. Hence  $(\dagger)$  implies Chang's conjecture. Also, Foreman-Magidor-Shelah [3] shows that  $(\dagger)$  implies precipitousness of the nonstationary ideal over  $\omega_1$ .

In this paper we compare the semistationary reflection principle with the stationary reflection principles defined below. For a regular cardinal  $\lambda$ ,  $E_\omega^\lambda$  denotes the set  $\{\alpha \in \lambda \mid \text{cf}(\alpha) = \omega\}$ .

**Definition 1.4.** *For a regular cardinal  $\lambda \geq \omega_2$ , let  $\text{SR}(\lambda)$  denote the following stationary reflection principle:*

$\text{SR}(\lambda) \equiv$  *For every stationary  $B \subseteq E_\omega^\lambda$ , there exists  $\gamma < \lambda$  such that  $B \cap \gamma$  is stationary.*

*For a cardinal  $\lambda \geq \omega_2$ , let  $\text{SR}([\lambda]^\omega)$  denote the stationary reflection principle in  $[\lambda]^\omega$ :*

$\text{SR}([\lambda]^\omega) \equiv$  *For every stationary  $S \subseteq [\lambda]^\omega$ , there exists  $W \subseteq \lambda$  such that  $|W| = \omega_1 \subseteq W$  and  $S \cap [W]^\omega$  is stationary in  $[W]^\omega$ .*

It is easy to see that  $\text{SR}([\lambda]^\omega)$  implies  $\text{SSR}([\lambda]^\omega)$ . (See Section 2.2.)

Our main results are as follows:

**Theorem 1.5.** *Let  $\lambda$  be a regular cardinal  $\geq \omega_2$ . Then  $\text{SSR}([\lambda]^\omega)$  implies  $\text{SR}(\lambda)$ .*

**Theorem 1.6.** *If  $\kappa$  is a supercompact cardinal, then there exists a generic extension in which  $\text{SSR}([\lambda]^\omega)$  holds for every  $\lambda \geq \omega_2$  but  $\text{SR}([\lambda]^\omega)$  does not hold for any  $\lambda \geq \omega_3$ .*

Foreman-Magidor-Shelah [3] shows that if  $\text{SR}([\lambda]^\omega)$  holds for every  $\lambda \geq \omega_2$  then  $(\dagger)$  holds. Theorem 1.6 claims that the converse is not true. Also, as we prove in Section 5,  $\text{SSR}([\omega_2]^\omega)$  implies  $\text{SR}([\omega_2]^\omega)$ . Theorem 1.6 is optimal in this sense.

This paper is organized as follows: In Section 2 we discuss some preliminaries for this paper. In Section 3 we present a certain type of stationary subset of  $[\lambda]^\omega$  which was first introduced by Shelah. This type of stationary set plays a central role in the proofs of both Theorems 1.5 and 1.6. In Section 4 we prove Theorem 1.5. In Section 5 we compare  $\text{SSR}([\lambda]^\omega)$  and  $\text{SR}([\lambda]^\omega)$ . Among other things, we prove Theorem 1.6.

## 2 Preliminaries

### 2.1 Notations

We follow the notations of Jech [4]. Here we present those which may not be general.

For a regular cardinal  $\gamma$  and an inaccessible cardinal  $\kappa$ , let  $\text{Col}(\gamma, < \kappa)$  denote the Lévy collapse which forces  $\kappa$  to be  $\gamma^+$ .

For a regular cardinal  $\gamma$  and a limit ordinal  $\delta > \gamma$ , let  $E_\gamma^\delta$  denote the set  $\{\alpha \in \delta \mid \text{cf}(\alpha) = \gamma\}$ . Note that if  $\text{cf}(\delta) > \gamma$  then  $E_\gamma^\delta$  is stationary in  $\delta$ .

For a set  $x$  of ordinals let

$$\text{s}\bar{\text{u}}\text{p } x := \sup\{\alpha + 1 \mid \alpha \in x\}.$$

In this paper we use  $\text{s}\bar{\text{u}}\text{p}$  rather than  $\sup$ . We are mainly interested in sets of ordinals which do not have a greatest element. For such  $x$ ,  $\text{s}\bar{\text{u}}\text{p } x = \sup x$ . The merit of using  $\text{s}\bar{\text{u}}\text{p}$  is that  $\text{s}\bar{\text{u}}\text{p } x$  is a limit ordinal if and only if  $x$  does not have a greatest element. This makes our definitions and arguments slightly simpler.

## 2.2 Basics on stationary sets and semistationary sets

For basics on the notion of club or stationary subsets of  $\mathcal{P}_\kappa W$  consult Jech [4]. When  $\kappa = \omega_1$ , we prefer to use  $[W]^\omega$  rather than  $\mathcal{P}_{\omega_1} W$ . A subset of  $[W]^\omega$  is said to be club (stationary) if it is club (stationary) in  $\mathcal{P}_{\omega_1} W$ . This paper uses the following two facts without any reference:

**Fact 2.1** (Kueker [6]). *Let  $\kappa$  be a regular uncountable cardinal,  $W$  be a set with  $\kappa \subseteq W$ , and let  $C \subseteq \mathcal{P}_\kappa W$  be a club. Then there exists a function  $f : [W]^{<\omega} \rightarrow W$  such that  $\{x \in \mathcal{P}_\kappa W \mid f''[x]^{<\omega} \subseteq x \wedge x \cap \kappa \in \kappa\} \subseteq C$ . If  $\kappa = \omega_1$  then there exists a function  $f : [W]^{<\omega} \rightarrow W$  such that  $\{x \in \mathcal{P}_\kappa W \mid f''[x]^{<\omega} \subseteq x\} \subseteq C$ .*

**Fact 2.2** (Menas [9]). *Let  $\kappa$  be a regular uncountable cardinal, and let  $W$  and  $\bar{W}$  be sets with  $\kappa \subseteq W \subseteq \bar{W}$ .*

- (1) *If  $C \subseteq \mathcal{P}_\kappa W$  is a club then the set  $\{\bar{x} \in \mathcal{P}_\kappa \bar{W} \mid \bar{x} \cap W \in C\}$  is a club in  $\mathcal{P}_\kappa \bar{W}$ . Hence if  $\bar{S} \subseteq \mathcal{P}_\kappa \bar{W}$  is stationary then the set  $\{\bar{x} \cap W \mid \bar{x} \in \bar{S}\}$  is stationary in  $\mathcal{P}_\kappa W$ .*
- (2) *If  $\bar{C} \subseteq \mathcal{P}_\kappa \bar{W}$  is a club then the set  $\{\bar{x} \cap W \mid \bar{x} \in \bar{C}\}$  contains a club in  $\mathcal{P}_\kappa W$ . Hence if  $S \subseteq \mathcal{P}_\kappa W$  is stationary then the set  $\{\bar{x} \in \mathcal{P}_\kappa \bar{W} \mid \bar{x} \cap W \in S\}$  is stationary in  $\mathcal{P}_\kappa \bar{W}$ .*

Basics on semistationary subsets of  $[W]^\omega$  was studied in Shelah [10] (Ch.XIII, §1). The following lemma is an analogy of Fact 2.2 for semistationary sets. In the case of (2), a stronger result holds. Part (2) of the following lemma illustrates a unique property of semistationary sets:

**Lemma 2.3.** *Let  $W$  and  $\bar{W}$  be sets with  $\omega_1 \subseteq W \subseteq \bar{W}$ .*

- (1) *If  $\bar{S} \subseteq [\bar{W}]^\omega$  is semistationary then the set  $\{\bar{x} \cap W \mid \bar{x} \in \bar{S}\}$  is semistationary.*
- (2) *If  $S \subseteq [W]^\omega$  is semistationary then  $S$  is also semistationary in  $[\bar{W}]^\omega$ .*

*Proof.* (1) is clear from Fact 2.2 (1). We prove (2).

Suppose that  $S \subseteq [W]^\omega$  is semistationary. Let  $T := \{y \in [W]^\omega \mid (\exists x \in S) x \sqsubseteq y\}$  and  $\bar{T} := \{\bar{y} \in [\bar{W}]^\omega \mid (\exists x \in S) x \sqsubseteq \bar{y}\}$ . Then  $T$  is stationary in  $[W]^\omega$ , and  $\bar{T} = \{\bar{y} \in [\bar{W}]^\omega \mid \bar{y} \cap W \in T\}$ . Hence  $\bar{T}$  is stationary in  $[\bar{W}]^\omega$ . Therefore  $S$  is semistationary in  $[\bar{W}]^\omega$ .  $\square$

### 2.3 Basics on reflection principles

In this paper we use the following reflection principles which are generalizations of  $\text{SR}([\lambda]^\omega)$  and  $\text{SSR}([\lambda]^\omega)$ :

**Definition 2.4.** For a cardinal  $\lambda \geq \omega_2$  and a regular cardinal  $\kappa$  with  $\omega_2 \leq \kappa \leq \lambda$ , let  $\text{SSR}([\lambda]^\omega, < \kappa)$  and  $\text{SR}([\lambda]^\omega, < \kappa)$  be the following reflection principles:

$\text{SR}([\lambda]^\omega, < \kappa) \equiv$  For every stationary  $S \subseteq [\lambda]^\omega$ , there exists  $W \in \mathcal{P}_\kappa \lambda$  such that  $\omega_1 \subseteq W \cap \kappa \in \kappa$  and  $S \cap [W]^\omega$  is stationary.

$\text{SSR}([\lambda]^\omega, < \kappa) \equiv$  For every semistationary  $S \subseteq [\lambda]^\omega$ , there exists  $W \in \mathcal{P}_\kappa \lambda$  such that  $\omega_1 \subseteq W \cap \kappa \in \kappa$  and  $S \cap [W]^\omega$  is semistationary.

Here we review basics about the above reflection principles. First we observe that these are generalizations of  $\text{SR}([\lambda]^\omega)$  and  $\text{SSR}([\lambda]^\omega)$ :

**Lemma 2.5.** Let  $\lambda$  be a cardinal  $\geq \omega_2$ .

- (1)  $\text{SR}([\lambda]^\omega)$  is equivalent to  $\text{SR}([\lambda]^\omega, < \omega_2)$ .
- (2)  $\text{SSR}([\lambda]^\omega)$  is equivalent to  $\text{SSR}([\lambda]^\omega, < \omega_2)$ .

*Proof.* (1). It suffices to show that  $\text{SR}([\lambda]^\omega)$  implies  $\text{SR}([\lambda]^\omega, < \omega_2)$ . Before starting, take a surjection  $\pi_\alpha : \omega_1 \rightarrow \alpha$  for each  $\alpha < \omega_2$  and let  $f : \omega_2 \times \omega_1 \rightarrow \omega_2$  be the function defined by  $f(\alpha, \xi) = \pi_\alpha(\xi)$  for each  $\langle \alpha, \xi \rangle \in \omega_2 \times \omega_1$ .

Assume that  $\text{SR}([\lambda]^\omega)$  holds. To show that  $\text{SR}([\lambda]^\omega, < \omega_2)$  holds, take an arbitrary stationary  $S \subseteq [\lambda]^\omega$ . We may assume that every element of  $S$  is closed under  $f$ . By  $\text{SR}([\lambda]^\omega)$ , we may choose a  $W \subseteq \lambda$  such that  $|W| = \omega_1 \subseteq W$  and  $S \cap [W]^\omega$  is stationary. Note that  $W$  is closed under  $f$  because stationary many elements of  $[W]^\omega$  are closed under  $f$ . Because  $\omega_1 \subseteq W$ , if  $\alpha \in W \cap \omega_2$  then  $\alpha \subseteq W$ . Hence  $W \cap \omega_2 \in \omega_2$ . Therefore  $W$  witnesses that  $\text{SR}([\lambda]^\omega, < \omega_2)$  holds for  $S$ .

(2). It suffices to show that  $\text{SSR}([\lambda]^\omega)$  implies  $\text{SSR}([\lambda]^\omega, < \omega_2)$ . Assume that  $\text{SSR}([\lambda]^\omega)$  holds. Take an arbitrary semistationary  $S \subseteq [\lambda]^\omega$ . Let  $W \subseteq \lambda$  be a witness of  $\text{SSR}([\lambda]^\omega)$  for  $S$  and let  $W' := W \cup \text{süp}(W \cap \omega_2)$ . Then  $\omega_1 \subseteq W' \cap \omega_2 \in \omega_2$ . Moreover  $S \cap [W]^\omega$  is semistationary in  $[W']^\omega$  by Lemma 2.3. Hence  $S \cap [W']^\omega$  is semistationary. Therefore  $W'$  witnesses  $\text{SSR}([\lambda]^\omega, < \omega_2)$  for  $S$ .  $\square$

Next we observe that  $\text{SR}([\lambda]^\omega, < \kappa)$  implies  $\text{SSR}([\lambda]^\omega, < \kappa)$ .

**Lemma 2.6.** Let  $\lambda$  be a cardinal  $\geq \omega_2$  and  $\kappa$  be a regular cardinal with  $\omega_2 \leq \kappa \leq \lambda$ . Then  $\text{SSR}([\lambda]^\omega, < \kappa)$  is equivalent to the following principle:

$\text{SSR}'([\lambda]^\omega, < \kappa) \equiv$  For every stationary  $S \subseteq [\lambda]^\omega$ , there exists  $W \in \mathcal{P}_\kappa \lambda$  such that  $\omega_1 \subseteq W \cap \kappa \in \kappa$  and  $S \cap [W]^\omega$  is semistationary.

Therefore  $\text{SR}([\lambda]^\omega, < \kappa)$  implies  $\text{SSR}([\lambda]^\omega, < \kappa)$ .

*Proof.* It suffices to show that  $\text{SSR}'$  implies  $\text{SSR}$ . Assume that  $\text{SSR}'([\lambda]^\omega, < \kappa)$  holds. Take an arbitrary semistationary  $S \subseteq [\lambda]^\omega$ . Then  $T := \{y \in [\lambda]^\omega \mid (\exists x \in S) x \sqsubseteq y\}$  is stationary. Let  $W \subseteq \lambda$  be a witness of  $\text{SSR}'([\lambda]^\omega, < \kappa)$  for  $T$ . Here note that

$$\{y \in [W]^\omega \mid (\exists x \in T \cap [W]^\omega) x \sqsubseteq y\} = \{y \in [W]^\omega \mid (\exists x \in S \cap [W]^\omega) x \sqsubseteq y\}.$$

Hence  $\{y \in [W]^\omega \mid (\exists x \in S \cap [W]^\omega) x \sqsubseteq y\}$  is stationary and thus  $S \cap [W]^\omega$  is semistationary. Therefore  $W$  witnesses  $\text{SSR}([\lambda]^\omega, < \kappa)$  for  $S$ .  $\square$

The following is very easy:

**Lemma 2.7.** *Let  $\lambda$  and  $\lambda'$  be cardinals and let  $\kappa$  and  $\kappa'$  be regular cardinals such that  $\omega_2 \leq \kappa \leq \kappa' \leq \lambda' \leq \lambda$ .*

- (1)  $\text{SR}([\lambda]^\omega, < \kappa)$  implies  $\text{SR}([\lambda']^\omega, < \kappa)$ .
- (2)  $\text{SSR}([\lambda]^\omega, < \kappa)$  implies  $\text{SSR}([\lambda']^\omega, < \kappa')$ .

*Proof.* (1) Assume that  $\text{SR}([\lambda]^\omega, < \kappa)$  holds. Take an arbitrary stationary  $S' \subseteq [\lambda']^\omega$ . Then  $S := \{x \in [\lambda]^\omega \mid x \cap \lambda' \in S'\}$  is stationary. Hence there exists  $W \in \mathcal{P}_\kappa \lambda$  such that  $\omega_1 \subseteq W \cap \kappa \in \kappa$  and  $S \cap [W]^\omega$  is stationary. Let  $W' := W \cap \lambda'$ . Then  $W' \in \mathcal{P}_{\kappa'} \lambda'$  and  $\omega_1 \subseteq W' \cap \kappa \in \kappa$ . Moreover  $S' \cap [W']^\omega = \{x \cap W' \mid x \in S \cap [W]^\omega\}$ . Thus  $S' \cap [W']^\omega$  is stationary. Therefore  $W'$  witnesses  $\text{SR}([\lambda']^\omega, < \kappa)$  for  $S'$ .

(2) Assume that  $\text{SSR}([\lambda]^\omega, < \kappa)$  holds. Take an arbitrary semistationary  $S' \subseteq [\lambda']^\omega$ . Let  $S, W$  and  $W'$  be as in (1). Then, using Lemma 2.3, the same argument as (1) shows that  $S' \cap [W']^\omega$  is semistationary. Let  $W'' := W' \cup \sup(W' \cap \kappa')$ . Then  $W'' \in \mathcal{P}_{\kappa'} \lambda'$  and  $\omega_1 \subseteq W'' \cap \kappa' \in \kappa'$ . Moreover  $S' \cap [W']^\omega$  is semistationary in  $[W'']^\omega$  by Lemma 2.3. Hence  $S' \cap [W'']^\omega$  is semistationary. Therefore  $W''$  witnesses  $\text{SSR}([\lambda']^\omega, < \kappa')$  for  $S'$ .  $\square$

We end this section with the following:

**Lemma 2.8.** *Let  $\lambda$  be a cardinal  $\geq \omega_2$  and  $\kappa$  be a regular cardinal with  $\omega_2 \leq \kappa \leq \lambda$ .*

- (1) (Feng-Jech [2]) *Assume that  $\text{SR}([\lambda]^\omega, < \kappa)$  holds. If  $S \subseteq [\lambda]^\omega$  is stationary then the set  $\{W \in \mathcal{P}_\kappa \lambda \mid S \cap [W]^\omega \text{ is stationary}\}$  is stationary in  $\mathcal{P}_\kappa \lambda$ .*
- (2) *Assume that  $\text{SSR}([\lambda]^\omega, < \kappa)$  holds. If  $S \subseteq [\lambda]^\omega$  is semistationary then the set  $\{W \in \mathcal{P}_\kappa \lambda \mid S \cap [W]^\omega \text{ is semistationary}\}$  is co-bounded, that is, there exists  $W^* \in \mathcal{P}_\kappa \lambda$  such that  $S \cap [W]^\omega$  is semistationary for every  $W \in \mathcal{P}_\kappa \lambda$  with  $W^* \supseteq W$ .*

*Proof.* (2) is clear from Lemma 2.3 (2). We prove (1).

Take an arbitrary stationary  $S \subseteq [\lambda]^\omega$  and an arbitrary function  $f : [\lambda]^{<\omega} \rightarrow \lambda$ . It suffices to find a  $W \in \mathcal{P}_\kappa \lambda$  such that  $W \cap \kappa \in \kappa$  and  $W$  is closed under  $f$ . Let  $S'$  be the set of all  $x \in S$  closed under  $f$ . Then  $S'$  is stationary. Hence there exists  $W \in \mathcal{P}_\kappa \lambda$  such that  $\omega_1 \subseteq W \cap \kappa \in \kappa$  and  $S' \cap [W]^\omega$  is stationary. Note that  $W$  is closed under  $f$  because stationary many elements of  $[W]^\omega$  are closed under  $f$ . Moreover  $S \cap [W]^\omega$  is stationary because  $S' \subseteq S$ . Therefore  $W$  is a desired one.  $\square$

### 3 Sup depending stationary sets

Here we present a type of stationary set which plays a central role in the proofs of both Theorems 1.5 and 1.6:

**Lemma 3.1** (The case when  $n = 1$  is due to Shelah). *Suppose that  $0 < n < \omega$  and that  $\mu_0 < \mu_1 < \dots < \mu_n$  are regular uncountable cardinals. Moreover, suppose that  $A \subseteq E_{\omega}^{\mu_0}$  is stationary and that, for each  $m$  with  $1 \leq m \leq n$ ,  $\langle A_{\alpha}^m \mid \alpha \in \mu_{m-1} \rangle$  is a sequence of stationary subsets of  $E_{\omega}^{\mu_m}$ . Let  $S$  be the set of all  $x \in \mathcal{P}_{\mu_0 \mu_n}$  such that*

- (1)  $x \cap \mu_0 \in A$ ,
- (2)  $\text{süp}(x \cap \mu_m) \in A_{\text{süp}(x \cap \mu_{m-1})}^m$  for each  $m$  with  $1 \leq m \leq n$ .

*Then  $S$  is stationary in  $\mathcal{P}_{\mu_0 \mu_n}$ .*

This type of stationary set was considered by Shelah, and in Shelah-Shioya [12] and Shelah [11], such sets are used to obtain consequences of the stationary reflection principle. The proof of the above lemma for the case when  $n = 1$  can be found in Shelah-Shioya [12]. Although there are no difficulties in generalization, we give the complete proof of Lemma 3.1.

We use the following game  $\mathfrak{D}$ :

**Definition 3.2.** *Suppose that  $0 < n < \omega$  and that  $\mu_0 < \mu_1 < \dots < \mu_n$  are regular uncountable cardinals. For an  $\alpha \in \mu_0$  and a function  $f : [\mu_n]^{<\omega} \rightarrow \mu_n$  let  $\mathfrak{D}(\langle \mu_0, \mu_1, \dots, \mu_n \rangle, \alpha, f)$  be the following two players game of length  $\omega$ :*

*In the  $k$ -th stage, first Player I plays a  $\langle \beta_k^m \mid 1 \leq m \leq n \rangle$  and then Player II plays a  $\langle \gamma_k^m \mid 1 \leq m \leq n \rangle$  so that  $\beta_k^m \leq \gamma_k^m < \mu_m$  for each  $m$ .*

|    |                                 |                                 |       |                                 |       |
|----|---------------------------------|---------------------------------|-------|---------------------------------|-------|
| I  | $\beta_0^1, \dots, \beta_0^n$   | $\beta_1^1, \dots, \beta_1^n$   | ..... | $\beta_k^1, \dots, \beta_k^n$   | ..... |
| II | $\gamma_0^1, \dots, \gamma_0^n$ | $\gamma_1^1, \dots, \gamma_1^n$ | ..... | $\gamma_k^1, \dots, \gamma_k^n$ | ..... |

*Player II wins if  $\text{cl}_f(\alpha \cup \{\gamma_k^m \mid 1 \leq m \leq n \wedge k \in \omega\}) \cap \mu_0 = \alpha$ , where  $\text{cl}_f(x)$  denotes the closure of  $x$  under  $f$ . Otherwise Player I wins.*

Note that  $\mathfrak{D}(\langle \mu_0, \mu_1, \dots, \mu_n \rangle, \alpha, f)$  is an open game for Player I. Hence it is determined. The following is a key lemma:

**Lemma 3.3.** *Suppose that  $0 < n < \omega$  and that  $\mu_0 < \mu_1 < \dots < \mu_n$  are regular uncountable cardinals. Then for every function  $f : [\mu_n]^{<\omega} \rightarrow \mu_n$ , there are club many  $\alpha \in \mu_0$  such that Player II has a winning strategy in  $\mathfrak{D}(\langle \mu_0, \mu_1, \dots, \mu_n \rangle, \alpha, f)$ .*

*Proof.* Take an arbitrary function  $f : [\mu_n]^{<\omega} \rightarrow \mu_n$  and let  $A$  be the set of all  $\alpha \in \mu_0$  such that Player I has a winning strategy in  $\mathfrak{D}(\langle \mu_0, \mu_1, \dots, \mu_n \rangle, \alpha, f)$ . It suffices to show that  $A$  is nonstationary.

Assume that  $A$  is stationary. For each  $\alpha \in A$ , take a winning strategy  $\sigma_{\alpha}$  for Player I in  $\mathfrak{D}(\langle \mu_0, \mu_1, \dots, \mu_n \rangle, \alpha, f)$ . Let  $\theta$  be a sufficiently large regular cardinal.

Then we can take an elementary submodel  $M$  of  $\langle \mathcal{H}_\theta, \in, \langle \sigma_\alpha \mid \alpha \in A \rangle \rangle$  such that  $\alpha^* := M \cap \mu_0 \in A$ .

By induction on  $k$ , we construct a sequence of moves  $\langle \beta_k^1, \dots, \beta_k^n, \gamma_k^1, \dots, \gamma_k^n \mid k \in \omega \rangle$  in  $\mathcal{D}(\langle \mu_0, \mu_1, \dots, \mu_n \rangle, \alpha^*, f)$  so that  $\gamma_k^1, \dots, \gamma_k^n \in M$  for each  $k \in \omega$ . Suppose that  $k \in \omega$  and that  $\langle \beta_l^1, \dots, \beta_l^n \rangle$  and  $\langle \gamma_l^1, \dots, \gamma_l^n \rangle$  have been chosen for each  $l < k$ . Let

$$\langle \beta_k^1, \dots, \beta_k^n \rangle := \sigma_{\alpha^*}(\langle \gamma_l^1, \dots, \gamma_l^n \mid l < k \rangle),$$

and for each  $m$  with  $1 \leq m \leq n$ , let

$$\gamma_k^m := \sup\{\pi_m \circ \sigma_\alpha(\langle \gamma_l^1, \dots, \gamma_l^n \mid l < k \rangle) \mid \alpha \in A\},$$

where  $\pi_m : \mu_1 \times \dots \times \mu_n \rightarrow \mu_m$  is the  $m$ -th projection. Clearly  $\beta_k^m \leq \gamma_k^m$  for each  $m$ . Note that  $\gamma_k^m < \mu_m$  since  $\mu_m$  is regular and  $A \subseteq \mu_0 < \mu_m$ . Note also that  $\gamma_k^m \in M$  because  $\langle \gamma_l^1, \dots, \gamma_l^n \mid l < k \rangle \in M$ . This completes the induction.

First note that  $\langle \beta_k^1, \dots, \beta_k^n, \gamma_k^1, \dots, \gamma_k^n \mid k \in \omega \rangle$  is a sequence of moves in  $\mathcal{D}(\langle \mu_0, \mu_1, \dots, \mu_n \rangle, \alpha^*, f)$  in which Player I has played according to winning strategy  $\sigma_{\alpha^*}$ . Hence Player I wins. On the other hand,  $\alpha^* \cup \{\gamma_k^m \mid 1 \leq m \leq n \wedge k \in \omega\} \subseteq M$ , and  $M$  is closed under  $f$ . Thus  $\text{cl}_f(\alpha^* \cup \{\gamma_k^m \mid 1 \leq m \leq n \wedge k \in \omega\}) \cap \mu_0 \subseteq M \cap \mu_0 = \alpha^*$ . Therefore  $\text{cl}_f(\alpha^* \cup \{\gamma_k^m \mid 1 \leq m \leq n \wedge k \in \omega\}) = \alpha^*$ , so that Player II wins with this sequence of moves. This is a contradiction.  $\square$

Now we prove Lemma 3.1:

*Proof of Lemma 3.1.* We prove Lemma 3.1 by induction on  $n$ . Suppose that  $n = 1$  or that  $n > 1$  and that the lemma holds for  $n - 1$ . We prove the lemma for  $n$ . Take an arbitrary function  $f : [\mu_n]^{<\omega} \rightarrow \mu_n$ . It suffices to find  $x^* \in S$  such that  $x^*$  is closed under  $f$  and  $x^* \cap \mu_0 \in \mu_0$ .

By Lemma 3.3, there exists  $\alpha^* \in A$  such that Player II has a winning strategy  $\sigma^*$  in  $\mathcal{D}(\langle \mu_0, \mu_1, \dots, \mu_n \rangle, \alpha^*, f)$ . Let  $S'$  be the set of all  $y \in \mathcal{P}_{\mu_1 \mu_n}$  such that

- (1)  $y \cap \mu_1 \in A_{\alpha^*}^1$ ,
- (2)  $\sup(y \cap \mu_m) \in A_{\sup(y \cap \mu_{m-1})}^m$  for each  $m$  with  $2 \leq m \leq n$ .

Then  $S'$  is stationary in  $\mathcal{P}_{\mu_1 \mu_n}$ . If  $n = 1$  then this is clear. ((2) claims nothing.) If  $n > 1$  then this follows from the lemma for  $n - 1$ .

Choose  $y^* \in S'$  such that  $\alpha^* \subseteq y^*$  and such that  $y^*$  is closed under  $\sigma^*$  and  $f$ . For each  $m$  with  $1 \leq m \leq n$ , take a cofinal sequence  $\langle \beta_k^m \mid k \in \omega \rangle$  in  $y^* \cap \mu_m$ . Moreover let

$$\langle \gamma_k^1, \dots, \gamma_k^n \rangle := \sigma^*(\langle \beta_l^1, \dots, \beta_l^n \mid l < k \rangle)$$

for each  $k \in \omega$ . Note that  $\langle \gamma_k^m \mid k \in \omega \rangle$  is a cofinal sequence in  $y^* \cap \mu_m$  for each  $m$  with  $1 \leq m \leq n$ . This is because  $y^*$  is closed under  $\sigma^*$ . Finally let  $x^* := \text{cl}_f(\alpha^* \cup \{\gamma_k^m \mid 1 \leq m \leq n \wedge k \in \omega\})$ . We show that  $x^* \in S$  and  $x^* \cap \mu_0 \in \mu_0$ .

First note that  $x^* \subseteq y^*$  because  $\alpha^* \cup \{\gamma_k^m \mid 1 \leq m \leq n \wedge k \in \omega\} \subseteq y^*$  and  $y^*$  is closed under  $f$ . Then, because  $\langle \gamma_k^m \mid k \in \omega \rangle$  is cofinal in  $y^* \cap \mu_m$ ,  $\sup(x^* \cap \mu_m) = \sup(y^* \cap \mu_m)$  for each  $m$  with  $1 \leq m \leq n$ . Hence, by (2) above,

- (i)  $\text{s}\bar{\text{u}}\text{p}(x^* \cap \mu_m) \in A_{\text{s}\bar{\text{u}}\text{p}(x^* \cap \mu_{m-1})}^m$  for each  $m$  with  $2 \leq m \leq n$ .

Moreover  $x^* \cap \mu_0 = \alpha^*$ . This is because  $\sigma^*$  is a winning strategy for Player II in  $\mathcal{D}(\langle \mu_0, \mu_1, \dots, \mu_n \rangle, \alpha^*, f)$ . Also recall that  $\text{s}\bar{\text{u}}\text{p}(x^* \cap \mu_1) = \text{s}\bar{\text{u}}\text{p}(y^* \cap \mu_1) \in A_{\alpha^*}^1$ . Thus

- (ii)  $\text{s}\bar{\text{u}}\text{p}(x^* \cap \mu_0) \in A$ ,  
 (iii)  $\text{s}\bar{\text{u}}\text{p}(x^* \cap \mu_1) \in A_{\text{s}\bar{\text{u}}\text{p}(x^* \cap \mu_1)}^1$ .

Now it follows from (i), (ii) and (iii) that  $x^* \in S$  and  $x^* \cap \mu_0 \in \mu_0$ .

This completes the proof.  $\square$

## 4 $\text{SSR}([\lambda]^\omega)$ and $\text{SR}(\lambda)$

In this section, we prove Theorem 1.5. In fact, we prove the following more general theorem:

**Theorem 4.1.** *Let  $\lambda$  and  $\kappa$  be regular cardinals such that  $\omega_2 \leq \kappa \leq \lambda$ . Then  $\text{SSR}([\lambda]^\omega, < \kappa)$  implies  $\text{SR}(\lambda)$ .*

By Lemma 2.5 and 2.7, Theorem 1.5 follows from Theorem 4.1. Theorem 4.1 can be easily obtained from Lemma 3.1 and the following lemma:

**Lemma 4.2.** *Let  $\lambda$  be a cardinal and  $\kappa$  be a regular cardinal such that  $\omega_2 \leq \kappa \leq \lambda$ . Assume that  $S \subseteq [\lambda]^\omega$  and that there exists  $W \in \mathcal{P}_\kappa \lambda$  such that  $\omega_1 \subseteq W \cap \kappa \in \kappa$  and  $S \cap [W]^\omega$  is semistationary. Let  $W^* \in \mathcal{P}_\kappa \lambda$  be such that*

- (1)  $\omega_1 \subseteq W^* \cap \kappa \in \kappa$  and  $S \cap [W^*]^\omega$  is semistationary,
- (2) for every  $W \in \mathcal{P}_\kappa \lambda$ , if  $\omega_1 \subseteq W \cap \kappa \in \kappa$  and  $S \cap [W]^\omega$  is semistationary then  $\text{s}\bar{\text{u}}\text{p} W^* \leq \text{s}\bar{\text{u}}\text{p} W$ .

Then

$$S_0 := \{y \in [W^*]^\omega \mid (\exists x \in S \cap [W^*]^\omega) x \sqsubseteq y \wedge \text{s}\bar{\text{u}}\text{p} x = \text{s}\bar{\text{u}}\text{p} y\}$$

is stationary in  $[W^*]^\omega$ .

*Proof.* Assume that  $S_0$  is not stationary. Then  $S_1 := \{y \in [W^*]^\omega \mid (\exists x \in S \cap [W^*]^\omega) x \sqsubseteq y \wedge \text{s}\bar{\text{u}}\text{p} x < \text{s}\bar{\text{u}}\text{p} y\}$  is stationary. For each  $y \in S_1$ , choose  $x_y \in S$  with  $x_y \sqsubseteq y$  and  $\text{s}\bar{\text{u}}\text{p} x_y < \text{s}\bar{\text{u}}\text{p} y$  and choose  $\alpha_y \in y$  with  $\text{s}\bar{\text{u}}\text{p} x_y \leq \alpha_y$ . Then there exists  $\alpha' \in W^*$  such that  $S' := \{y \in S_1 \mid \alpha_y = \alpha'\}$  is stationary. Let  $W' := W^* \cap \alpha'$ . Clearly  $W' \in \mathcal{P}_\kappa \lambda$  and  $\omega_1 \subseteq W' \cap \kappa \in \kappa$ . Moreover  $\text{s}\bar{\text{u}}\text{p} W' < \text{s}\bar{\text{u}}\text{p} W^*$ . Hence if we show that  $S \cap [W']^\omega$  is semistationary then this contradicts property (2) of  $W^*$ .

First note that if  $y \in S'$  then  $x_y \in [W']^\omega$  and  $x_y \sqsubseteq y \cap W'$ . Thus

$$\{y \cap W' \mid y \in S'\} \subseteq \{z \in [W']^\omega \mid (\exists x \in S \cap [W']^\omega) x \sqsubseteq z\}.$$

Now the left side is stationary in  $[W']^\omega$  because  $S'$  is stationary in  $[W^*]^\omega$ . Therefore the right side is stationary, that is,  $S \cap [W']^\omega$  is semistationary.

This completes the proof.  $\square$



*Proof of Theorem 4.1.* Assume that  $\text{SSR}([\lambda]^\omega, < \kappa)$  holds. Take an arbitrary stationary  $B \subseteq E_\omega^\lambda$ . We show that  $B$  reflects.

Take a pairwise disjoint sequence  $\langle B_\alpha \mid \alpha \in \omega_1 \rangle$  of stationary subsets of  $B$ . Let  $S$  be the set of all  $x \in [\lambda]^\omega$  such that  $x \cap \omega_1 \in \omega_1$  and  $\text{süp } x \in B_{x \cap \omega_1}$ . Then  $S$  is stationary by Lemma 3.1. By Lemma 4.2, there exists  $W \in \mathcal{P}_\kappa \lambda$  such that  $\omega_1 \subseteq W \cap \kappa \in \kappa$  and  $S_0 := \{y \in [W]^\omega \mid (\exists x \in S \cap [W]^\omega) x \sqsubseteq y \wedge \text{süp } x = \text{süp } y\}$  is stationary in  $[W]^\omega$ . Here note that  $S_0 \subseteq S$ . Hence  $S \cap [W]^\omega$  is stationary. We claim that  $\text{cf}(\text{süp } W) > \omega$ .

Clearly  $\text{süp } W$  is a limit ordinal. Assume that  $\text{cf}(\text{süp } W) = \omega$ . Then  $C := \{y \in [W]^\omega \mid \text{süp } y = \text{süp } W\}$  is club in  $[W]^\omega$ . But if  $y_1, y_2 \in S \cap C$  then  $\text{süp}(y_1 \cap \omega_1) = \text{süp}(y_2 \cap \omega_1)$  by the construction of  $S$ . Hence  $|\{y \cap \omega_1 \mid y \in S_0 \cap C\}| \leq 1$ . This contradicts  $\omega_1 \subseteq W$  and  $S \cap [W]^\omega$  is stationary.

Now  $\text{cf}(\text{süp } W) > \omega$  and  $S \cap [W]^\omega$  is stationary. Hence  $\{\text{süp } y \mid y \in S \cap [W]^\omega\}$  is stationary in  $\text{süp } W$ . Recall that  $\text{süp } y \in B$  for every  $y \in S$ . Therefore  $B \cap \text{süp } W$  is stationary. This completes the proof.  $\square$

## 5 $\text{SSR}([\lambda]^\omega)$ and $\text{SR}([\lambda]^\omega)$

As we mentioned in Section 1, first we prove that  $\text{SSR}([\omega_2]^\omega)$  implies  $\text{SR}([\omega_2]^\omega)$  and thus that they are equivalent. This is essentially proved in Todorćević [13]. After that, we prove Theorem 1.6.

**Theorem 5.1.**  $\text{SSR}([\omega_2]^\omega)$  implies  $\text{SR}([\omega_2]^\omega)$ .

*Proof.* Assume that  $\text{SSR}([\omega_2]^\omega)$  holds. To show that  $\text{SR}([\omega_2]^\omega)$  holds, take an arbitrary stationary  $S \subseteq [\omega_2]^\omega$ . Fix a bijection  $\pi_\alpha : \omega_1 \rightarrow \alpha$  for each  $\alpha \in [\omega_1, \omega_2)$ . We may assume that if  $x \in S$  then  $\omega_1 < \text{süp } x$  and  $x$  is closed under  $\pi_\alpha, \pi_\alpha^{-1}$  for each  $\alpha \in x \setminus \omega_1$ .

By Lemma 2.5 let  $\alpha^*$  be the least  $\alpha \in \omega_2$  such that  $S \cap [\alpha]^\omega$  is semistationary. Then let  $S_0$  be the set of all  $y \in [\alpha^*]^\omega$  such that

- (1) for some  $x \in S \cap [\alpha^*]^\omega$ ,  $x \sqsubseteq y$  and  $\text{süp } x = \text{süp } y$ ,
- (2)  $y$  is closed under  $\pi_\alpha, \pi_\alpha^{-1}$  for each  $\alpha \in y \setminus \omega_1$ .

By Lemma 4.2,  $S_0$  is stationary in  $[\alpha^*]^\omega$ . For each  $y \in S_0$ , choose an  $x_y \in S \cap [\alpha^*]^\omega$  witnessing (1).

Note that if  $y \in S_0$  then

$$y \cap \alpha = \pi_\alpha^{-1}(y \cap \omega_1) = \pi_\alpha^{-1}(x_y \cap \omega_1) = x_y \cap \alpha$$

for each  $\alpha \in x_y \setminus \omega_1$ . Then, because  $\text{süp } y = \text{süp } x_y$ ,  $y = x_y$  for every  $y \in S_0$ . Hence  $S_0 \subseteq S \cap [\alpha^*]^\omega$ . Therefore  $S \cap [\alpha^*]^\omega$  is stationary. This completes the proof.  $\square$

Now we turn our attention to Theorem 1.6. It is well-known that if a  $\lambda$ -supercompact cardinal is Lévy collapsed to  $\omega_2$  then  $\text{SR}([\lambda]^\omega)$  holds. It was shown by Shelah [10] that collapsing a  $\lambda$ -strongly compact cardinal suffices to obtain a model of  $\text{SSR}([\lambda]^\omega)$ . First we review this:

**Lemma 5.2** (Shelah [10] Ch.XIII, §1, 1.6.Claim, 1.10.Claim). *Suppose that  $\kappa$  is a  $\lambda$ -strongly compact cardinal, where  $\lambda$  is a cardinal  $\geq \kappa$ . Then  $\text{SSR}([\lambda]^\omega, < \kappa)$  holds. Moreover if  $\gamma$  is a regular uncountable cardinal  $< \kappa$  then  $\Vdash_{\text{Col}(\gamma, < \kappa)} \text{"SR}([\lambda]^\omega, < \kappa) \text{"}$ .*

*Proof.* Both statements can be proved by similar arguments, but the latter is slightly harder than the former. We will prove only the latter.

We discuss some preliminaries in  $V$ . Take a fine ultrafilter  $U$  over  $\mathcal{P}_\kappa \lambda$ . Let  $M$  be the transitive collapse of  $\text{Ult}(V, U)$ , and let  $j : V \rightarrow M$  be the ultrapower map. Moreover, let  $f : \mathcal{P}_\kappa \lambda \rightarrow \mathcal{P}_\kappa \lambda$  be a function such that  $f(W) = W \cup \text{süp}(W \cap \kappa)$  for each  $W \in \mathcal{P}_\kappa \lambda$  and let  $W^* := [f]_U \in M$ . Then  $j \text{"} \lambda \subseteq W^*$ , and in  $M$ ,  $W^* \in \mathcal{P}_{j(\kappa)} j(\lambda)$  and  $\omega_1 \subseteq W^* \cap j(\kappa) \in j(\kappa)$ .

Suppose that  $\gamma$  is a regular uncountable cardinal  $< \kappa$  and that  $G$  is a  $\text{Col}(\gamma, < \kappa)$ -generic filter over  $V$ . In  $V[G]$ , take an arbitrary stationary  $S \subseteq [\lambda]^\omega$ . We must show that, in  $V[G]$ , there exists  $W \in \mathcal{P}_\kappa \lambda$  such that  $\omega_1 \subseteq W \cap \kappa \in \kappa$  and  $S \cap [W]^\omega$  is semistationary.

Let  $\bar{G}$  be a  $\text{Col}(\gamma, < j(\kappa))$ -generic filter over  $V$  with  $\bar{G} \cap \text{Col}(\gamma, < \kappa) = G$ . We work in  $V[\bar{G}]$ . Define a map  $\bar{j} : V[\bar{G}] \rightarrow M[\bar{G}]$  by  $\bar{j}(\dot{a}_{\bar{G}}) = j(\dot{a})_{\bar{G}}$  for each  $\text{Col}(\gamma, < \kappa)$ -name  $\dot{a} \in V$ . Then  $\bar{j}$  is well-defined, and  $\bar{j} : V[\bar{G}] \rightarrow M[\bar{G}]$  is an elementary embedding which extends  $j$ . For simplicity of notation, we let  $j$  denote  $\bar{j}$ .

Note that  $S$  remains stationary because  $V[\bar{G}]$  is a  $\gamma$ -closed forcing extension of  $V[G]$ . Hence  $\{j \text{"} x \mid x \in S\}$  is stationary in  $[j \text{"} \lambda]^\omega$ . Moreover, for each  $x \in S$ ,  $j \text{"} x = j(x)$  because  $x$  is countable in  $V[G]$ . Thus  $\{j \text{"} x \mid x \in S\} \subseteq j(S)$ , and therefore  $j(S) \cap [j \text{"} \lambda]^\omega$  is stationary. Then  $j(S) \cap [W^*]^\omega$  is semistationary by Lemma 2.3 (2). This is also true in  $M[\bar{G}]$ . Hence  $W^*$  witnesses that the following holds in  $M[\bar{G}]$ :

There exists  $W \in \mathcal{P}_{j(\kappa)} j(\lambda)$  such that  $\omega_1 \subseteq W \cap j(\kappa) \in j(\kappa)$  and  $j(S) \cap [W]^\omega$  is semistationary.

Therefore, by the elementarity of  $j$ , it holds in  $V[G]$  that there exists  $W \in \mathcal{P}_\kappa \lambda$  such that  $\omega_1 \subseteq W \cap \kappa \in \kappa$  and  $S \cap [W]^\omega$  is semistationary. This completes the proof.  $\square$

We will prove that collapsing a  $\lambda$ -strongly compact cardinal does not suffice to obtain a model of  $\text{SR}([\lambda]^\omega)$ . The core of Theorem 1.6 is the following theorem. As we see later, Theorem 1.6 will be obtained by further Lévy collapsing  $\kappa$  to  $\omega_2$ .

**Theorem 5.3.** *If  $\kappa$  is a supercompact cardinal, then there exists a generic extension in which  $\kappa$  is a strongly compact cardinal and  $\text{SR}([\kappa^+]^\omega, < \kappa)$  does not hold.*

First we prove Theorem 5.3. Krueger [7] constructed a model in which  $\kappa$  is strongly compact but  $S(\kappa, \kappa^+) := \{x \in \mathcal{P}_\kappa \kappa^+ \mid \text{o.t.}(x) = (x \cap \kappa)^+\}$  is not stationary. (Note that  $S(\kappa, \kappa^+)$  is stationary if  $\kappa$  is  $\kappa^+$ -supercompact.) We show that  $\text{SR}([\kappa^+]^\omega, < \kappa)$  does not hold in this model.

We start with a review of Krueger's model. Krueger's model was obtained from a model with  $\kappa$  supercompact by two step forcing extension. The first step forces a partial square principle at  $\kappa$  with preserving supercompactness of  $\kappa$ . This type of partial square principle was first introduced by Baumgartner in his unpublished note, and Apter-Cummings [1] showed that it can hold at a supercompact cardinal. In fact, the first step of Krueger's construction is due to Apter-Cummings [1]. The second step is the iteration of Prikry forcing below  $\kappa$  which was developed by Magidor [8]. We summarize basic properties of these forcings below.

**Definition 5.4.** *For an uncountable cardinal  $\kappa$  and an  $E \subseteq \text{Lim}(\kappa^+)$ , let  $\square_\kappa^E$  be the following principle:*

$\square_\kappa^E \equiv$  *There exists a sequence  $\langle c_\beta \mid \beta \in E \rangle$  such that*

- (1)  *$c_\beta$  is a club in  $\beta$ .*
- (2) *if  $\text{cf}(\beta) < \kappa$  then  $\text{o.t.}(c_\beta) < \kappa$ ,*
- (3) *if  $\beta' \in \text{Lim}(c_\beta)$  then  $c_{\beta'} = c_\beta \cap \beta'$ ,*

*for each  $\beta, \beta' \in E$ .*

*We call a sequence  $\langle c_\beta \mid \beta \in E \rangle$  satisfying (1)-(3) above a  $\square_\kappa^E$ -sequence.*

The proof of the following fact can be also found in Krueger [7]:

**Fact 5.5** (Apter-Cummings [1]). *Assume that  $\kappa$  is a supercompact cardinal. Then there exists a poset  $\mathbb{P}$  with the following properties:*

- (1)  $\mathbb{P}$  *preserves supercompactness of  $\kappa$ .*
- (2)  $\Vdash_{\mathbb{P}}$  " $\square_\kappa^E$  *holds for*

$$E = \text{Lim}(\kappa^+) \setminus \bigcup \{E_{\alpha^+}^\kappa \mid \alpha \text{ is a measurable cardinal } < \kappa\}."$$

**Fact 5.6** (Magidor [8]). *Assume that  $\kappa$  is a supercompact cardinal. Then there exists a poset  $\mathbb{Q}$  with the following properties:*

- (1)  $\mathbb{Q}$  *has the  $\kappa^+$ -c.c.*
- (2)  $\Vdash_{\mathbb{Q}}$  " $\kappa$  *is strongly compact*".
- (3) *For every measurable cardinal  $\alpha < \kappa$ ,  $\Vdash_{\mathbb{Q}}$  " $\text{cf}(\alpha) = \omega$ ".*
- (4) *For every measurable cardinal  $\alpha < \kappa$ ,  $\mathbb{Q}$  can be factored as  $\mathbb{Q}_{\leq \alpha} * \dot{\mathbb{Q}}_{> \alpha}$  so that  $\mathbb{Q}_{\leq \alpha}$  has the  $\alpha^+$ -c.c. and  $\Vdash_{\mathbb{Q}_{\leq \alpha}}$  " $\dot{\mathbb{Q}}_{> \alpha}$  does not add subsets of  $\alpha^+$ ".*

Before starting the proof of Theorem 5.3, we give a technical lemma:

**Lemma 5.7.** *Suppose that  $\kappa$  and  $\theta$  are regular cardinals with  $\omega_2 \leq \kappa < \theta$ . Let  $M$  be an elementary submodel of  $\langle \mathcal{H}_\theta, \in \rangle$  such that  $M \cap \kappa \in \kappa$  and such that both  $\text{cf}(M \cap \kappa)$  and  $\text{cf}(\text{süp}(M \cap \kappa^+))$  are uncountable. Then  $M \cap \kappa^+$  is  $\omega$ -closed, that is,  $\text{süp } b \in M$  for every countable  $b \subseteq M \cap \kappa^+$ .*

*Proof.* We prove this by contradiction. Assume that  $b \subseteq M \cap \kappa^+$  is countable and that  $\sup b \notin M \cap \kappa^+$ . Then  $b$  does not have a greatest element. Also,  $b$  is bounded in  $M \cap \kappa^+$  because  $\text{cf}(\sup(M \cap \kappa^+))$  is uncountable. Let  $\beta^*$  be the least element of  $(M \cap \kappa^+) \setminus \sup b$ . Note that  $\beta^*$  is a limit ordinal and  $\sup(M \cap \beta^*) = \sup b < \beta^*$ . Take an increasing continuous cofinal map  $\sigma : \text{cf}(\beta^*) \rightarrow \beta^*$ .

First assume that  $\text{cf}(\beta^*) < \kappa$ . Then  $\text{cf}(\beta^*) \subseteq M$  because  $\text{cf}(\beta^*) \in M \cap \kappa \in \kappa$ . Hence  $\text{ran } \sigma \subseteq M$  and thus  $M \cap \beta^*$  is cofinal in  $\beta^*$ . This contradicts  $\sup(M \cap \beta^*) < \beta^*$ .

Next assume that  $\text{cf}(\beta^*) = \kappa$ . Then it is easy to see that  $\sup(M \cap \beta^*) = \sup(\sigma''(M \cap \kappa))$ . Hence  $\text{cf}(\sup(M \cap \beta^*))$  is uncountable because  $\text{cf}(M \cap \kappa)$  is uncountable. But this contradicts  $\sup(M \cap \beta^*) = \sup b$  and  $b$  is countable.

This completes the proof.  $\square$

Now we prove Theorem 5.3:

*Proof of Theorem 5.3.* Assume that  $\kappa$  is supercompact in  $V$ . Let  $V_0$  be a forcing extension of  $V$  by the poset  $\mathbb{P}$  of Fact 5.5, and let  $V_1$  be a forcing extension of  $V_0$  by the poset  $\mathbb{Q}$  of Fact 5.6. It suffices to show that  $\text{SR}([\kappa^+]^\omega, < \kappa)$  does not hold in  $V_1$ . Before starting, we summarize properties of  $V_0$  and  $V_1$ .

In  $V_0$ , let  $E := \text{Lim}(\kappa^+) \setminus \bigcup \{E_{\alpha^+}^\kappa \mid \alpha \text{ is a measurable cardinal } < \kappa\}$ . Then  $\square_\kappa^E$  holds in  $V_0$ . Let  $\langle c_\gamma \mid \gamma \in E \rangle$  be a  $\square_\kappa^E$ -sequence. Note that, in  $V_0$ , there are unboundedly many measurable cardinals below  $\kappa$ .

$V_1$  is a  $\kappa^+$ -c.c. forcing extension of  $V_0$ . Moreover, in  $V_1$ , the following hold:

(\*1) If  $\alpha < \kappa$  is a measurable cardinal in  $V_0$  the  $\text{cf}(\alpha) = \omega$ .

(\*2) Suppose that  $\alpha < \kappa$  is measurable in  $V_0$ . Let  $\gamma := (\alpha^+)^{V_0}$ . Then  $(E_\alpha^\gamma)^{V_0}$  is a stationary subset of  $E_\omega^\gamma$ .

(\*3)  $E_\omega^{\kappa^+} \subseteq E$ .

(\*2) and (\*3) hold by Fact 5.6 (4).

Now we show that  $\text{SR}([\kappa^+]^\omega, < \kappa)$  does not hold in  $V_1$ . We work in  $V_1$ . We show that there exists a stationary  $S \subseteq [\kappa^+]^\omega$  such that  $\{W \in \mathcal{P}_\kappa \kappa^+ \mid S \cap [W]^\omega \text{ is stationary}\}$  is nonstationary in  $\mathcal{P}_\kappa \kappa^+$ . By Lemma 2.8 this suffices.  $S$  will be constructed using Lemma 3.1.

First take a pairwise disjoint partition  $\langle A_\xi \mid \xi < \omega_1 \rangle$  of  $E_\omega^\kappa$  into stationary sets. Next take an injection  $\sigma : E_\omega^\kappa \rightarrow \kappa$  such that, for every  $\alpha \in \kappa$ ,  $\sigma(\alpha) > \alpha$  and  $\sigma(\alpha)$  is measurable in  $V_0$ . Then let  $B_\alpha := (E_{\sigma(\alpha)}^\kappa)^{V_0}$  for each  $\alpha \in E_\omega^\kappa$ . Note the following:

- $\langle B_\alpha \mid \alpha \in E_\omega^\kappa \rangle$  is a pairwise disjoint sequence of stationary subsets of  $E_\omega^{\kappa^+}$ .
- For each  $\alpha \in E_\omega^\kappa$  and each  $\beta \in B_\alpha$ ,  $\text{cf}^{V_0}(\beta) > \alpha$ .

Now let  $S$  be the set of all  $x \in [\kappa^+]^\omega$  such that

- (1)  $\sup(x \cap \kappa) \in A_{\sup(x \cap \omega_1)}$ ,
- (2)  $\sup(x) \in B_{\sup(x \cap \kappa)}$ .

Then  $S$  is stationary by Lemma 3.1. We show that  $\{W \in \mathcal{P}_{\kappa\kappa^+} \mid S \cap [W]^\omega \text{ is stationary}\}$  is nonstationary. Take a sufficiently large regular cardinal  $\theta$  and let  $\Omega$  be the set of all  $M \in \mathcal{P}_{\kappa}\mathcal{H}_\theta$  such that  $M \cap \kappa \in \kappa$  and  $M \prec \langle \mathcal{H}_\theta, \in, \mathcal{H}_\theta^{V_0}, \kappa, \langle c_\gamma \mid \gamma \in E \rangle \rangle$ . It suffices to show that if  $M \in \Omega$  then  $S \cap [M \cap \kappa^+]^\omega$  is nonstationary. The proof of this splits into two cases:

**Case 1:**  $\text{cf}(M \cap \kappa) = \omega$  or  $\text{cf}(\text{süp}(M \cap \kappa^+)) = \omega$ .

First suppose that  $\text{cf}(M \cap \kappa) = \omega$ . Then  $C := \{x \in [M \cap \kappa^+]^\omega \mid \text{süp}(x \cap \kappa) = M \cap \kappa\}$  is club in  $[M \cap \kappa^+]^\omega$ . Note that if  $x, y \in S \cap C$  then  $\text{süp}(x \cap \omega_1) = \text{süp}(y \cap \omega_1)$  by (1) of the construction of  $S$ . Thus  $|\{\text{süp}(x \cap \omega_1) \mid x \in S \cap C\}| \leq 1$ . But  $\omega_1 \subseteq M \cap \kappa^+$ . Hence  $S \cap C$  is nonstationary in  $[M \cap \kappa^+]^\omega$ . Therefore  $S \cap [M \cap \kappa^+]^\omega$  is nonstationary.

The case when  $\text{cf}(\text{süp}(M \cap \kappa^+)) = \omega$  is similar. Suppose that  $\text{cf}(\text{süp}(M \cap \kappa^+)) = \omega$ . Then  $C := \{x \in [M \cap \kappa^+]^\omega \mid \text{süp}(x) = \text{süp}(M \cap \kappa^+)\}$  is club in  $[M \cap \kappa^+]^\omega$ . If  $x, y \in S \cap C$  then  $\text{süp}(x \cap \kappa) = \text{süp}(y \cap \kappa)$  by (2) of the construction of  $S$  and thus  $\text{süp}(x \cap \omega_1) = \text{süp}(y \cap \omega_1)$  by (1). Hence  $|\{x \cap \omega_1 \mid x \in S \cap C\}| \leq 1$ . Therefore  $S \cap [M \cap \kappa^+]^\omega$  is nonstationary.  $\blacksquare$ (Case 1)

**Case 2:** Both  $\text{cf}(M \cap \kappa)$  and  $\text{cf}(\text{süp}(M \cap \kappa^+))$  are uncountable.

First we claim the following:

**Claim .**  $\text{cf}^{V_0}(\text{süp}(M \cap \kappa^+)) \leq M \cap \kappa$ .

$\vdash$  Let  $\delta := \text{süp}(M \cap \kappa^+)$ .

First suppose that  $\delta \in E$ . Note that  $M \cap \kappa^+$  is  $\omega$ -closed by Lemma 5.7. Hence, by (\*3),  $\text{Lim}(c_\delta) \cap M \cap E$  is unbounded in  $\delta$ . Moreover if  $\beta \in \text{Lim}(c_\delta) \cap M \cap E$  then  $\text{o.t.}(c_\delta \cap \beta) = \text{o.t.}(c_\beta) \in M \cap \kappa$ . Thus  $\text{o.t.}(c_\delta) \leq M \cap \kappa$ . But  $\text{cf}^{V_0}(\delta) \leq \text{o.t.}(c_\delta)$  because  $c_\delta \in V_0$ . Therefore  $\text{cf}^{V_0}(\delta) \leq M \cap \kappa$ .

Next suppose that  $\delta \notin E$ . Then there exists  $\alpha < \kappa$  such that, in  $V_0$ ,  $\alpha$  is a measurable cardinal and  $\text{cf}(\delta) = \alpha^+$ . Then, by (\*2),  $(E_\alpha^\delta)^{V_0}$  is a stationary subset of  $E_\omega^\delta$ . On the other hand,  $M \cap \kappa^+$  is  $\omega$ -closed unbounded in  $\delta$ . Thus there exists  $\beta \in M \cap \kappa^+$  such that  $\text{cf}^{V_0}(\beta) = \alpha$ . Then, by the elementarity of  $M$ ,  $(\alpha^+)^{V_0} \in M \cap \kappa$ . Therefore  $\text{cf}^{V_0}(\delta) = (\alpha^+)^{V_0} < M \cap \kappa$ .

This completes the proof of the claim.  $\dashv$

Take an increasing continuous sequence  $\langle \beta_\gamma \mid \gamma < \text{cf}^{V_0}(\text{süp}(M \cap \kappa^+)) \rangle \in V_0$  which is cofinal in  $\text{süp}(M \cap \kappa^+)$ . Let  $C$  be the set of all  $x \in [M \cap \kappa^+]^\omega$  such that, for some limit  $\gamma < \text{cf}^{V_0}(\text{süp}(M \cap \kappa^+))$ ,  $\text{süp} x = \beta_\gamma$  and  $\gamma \leq \text{süp}(x \cap \kappa)$ . Then  $C$  is club in  $[M \cap \kappa^+]^\omega$  by the claim above. Moreover if  $x \in C$  and  $\text{süp} x = \beta_\gamma$  then

$$\text{cf}^{V_0}(\text{süp} x) = \text{cf}^{V_0}(\beta_\gamma) \leq \gamma \leq \text{süp}(x \cap \kappa)$$

That is,  $\text{cf}^{V_0}(\text{süp} x) \leq \text{süp}(x \cap \kappa)$  for every  $x \in C$ . On the other hand, if  $x \in S$  then  $\text{cf}^{V_0}(\text{süp} x) > \text{süp}(x \cap \kappa)$  by (2) of the construction of  $S$ . Thus  $S \cap C = \emptyset$ . Therefore  $S \cap [M \cap \kappa^+]^\omega$  is nonstationary.  $\blacksquare$ (Case 2)

This completes the proof of Theorem 5.3.  $\square$

Now we turn our attention to Theorem 1.6. As we mentioned before, the forcing of Theorem 1.6 followed by Lévy collapsing  $\kappa$  to  $\omega_2$  gives Theorem 1.6.

Let  $V_0$ ,  $V_1$  and  $E$  be as in the proof of Theorem 1.6. Let  $V_2$  be an extension of  $V_1$  by  $\text{Col}(\omega_1, < \kappa)$ . Then, by Lemma 5.2,  $\text{SSR}([\lambda]^\omega)$  holds in  $V_2$  for every  $\lambda \geq \omega_2$ . So, by Lemma 2.5 (1) and 2.7 (1), it suffices to show that  $\text{SR}([\omega_3]^\omega, < \omega_2)$  does not hold in  $V_2$ .

Here note that  $V_2$  is a  $\kappa^+$ -c.c. forcing extension of  $V_0$  and that (\*1), (\*2) and (\*3) all hold in  $V_2$ . (\*2) holds because  $\text{Col}(\omega_1, < \kappa)$  preserves stationary subsets of  $E_\omega^\gamma$ . (\*3) holds because  $\text{Col}(\omega_1, < \kappa)$  preserves ordinals having uncountable cofinalities. Hence the same argument shows that  $\text{SR}([\kappa^+]^\omega, < \kappa)$  does not hold in  $V_2$ . But  $\kappa = \omega_2$  in  $V_2$ . Therefore  $\text{SR}([\omega_3]^\omega, < \omega_2)$  does not hold in  $V_2$ .

## References

- [1] A. W. Apter and J. Cummings, *A global version of a theorem of Ben-David and Magidor*, Ann. Pure. Appl. Logic **102** (2000), no.3, 199-222.
- [2] Q. Feng and T. Jech, *Local clubs, reflection, and preserving stationary sets*, Proc. London Math. Soc. (3) **58** (1989), no. 2, 237-257.
- [3] M. Foreman, M. Magidor and S. Shelah, *Martin's maximum, saturated ideals and nonregular ultrafilters I*, Ann. of Math. (2) **127** (1988), no.1, 1-47.
- [4] T. Jech, *Set Theory*, 3rd edition, Springer-Verlag, Berlin, 2002.
- [5] A. Kanamori, *The Higher Infinite*, Springer-Verlag, 1994.
- [6] D. W. Kueker, *Countable approximations and Löwenheim-Skolem theorems*, Ann. Math. Logic **11** (1977), no. 1, 57-103.
- [7] J. Krueger, *Strong compactness and stationary sets*, J. Symbolic Logic **70** (2005), no. 3, 767-777.
- [8] M. Magidor, *How large is the first strongly compact cardinal?*, Ann. Math. Logic **10** (1976), no. 1, 33-57.
- [9] T.K. Menas, *On strong compactness and supercompactness*, Ann. Math. Logic **7** (1974/75), 327-359.
- [10] S. Shelah, *Proper and Improper Forcing*, Perspectives in Mathematical Logic **29**, Springer-Verlag, Berlin, 1998.
- [11] S. Shelah, *Stationary reflection implies SCH*, preprint.
- [12] S. Shelah and M. Shioya, *Nonreflecting stationary sets in  $\mathcal{P}_\kappa\lambda$* , Adv. Math. **199** (2006), 185-191.

- [13] S. Todorćević, *Conjectures of Rado and Chang and cardinal arithmetic*, in *Finite and infinite combinatorics in sets and logic*, edited by N. Sauer, R. Woodrow, B. Sands, Kluwer Academic Publishers, 1993, 385-398.