SEMIPROPER IDEALS

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Abstract. We say that an ideal on $\mathcal{P}_\kappa\lambda$ is semiproper if the corresponding poset $\mathbb{P}_I$ is semiproper. In this paper we investigate properties of semiproper ideals on $\mathcal{P}_\kappa\lambda$.

1. Introduction

Large cardinal properties of ideals on $\mathcal{P}_\kappa\lambda$ such as saturation, presaturation, and precipitousness have been studied extensively by many set theorists. These are properties extracted from the dual ideals of supercompact filters or strongly compact filters on $\mathcal{P}_\kappa\lambda$. These properties are large cardinal properties in the sense of consistency strength. In general the existence of an ideal on $\mathcal{P}_\kappa\lambda$ with such a property does not imply that either κ or λ is a large cardinal.

In many cases these properties of ideals can be characterized by the properties of posets corresponding to those ideals. For example, an ideal I on $\mathcal{P}_\kappa\lambda$ is γ-saturated if and only if the corresponding $\mathbb{P}_I$, the poset of all I-positive sets ordered by inclusion, has the γ-chain condition (γ-c.c.). I is also called precipitous if and only if for every $\mathbb{P}_I$-generic filter G, the ultrapower of V by G is well-founded. In the development of the theory of forcing, many properties of posets have been introduced, e.g. σ-closure, σ-Baireness, properness, semiproperness, e.t.c.. We can define the properties of ideals corresponding to these as follows: Let us say that an ideal I on $\mathcal{P}_\kappa\lambda$ is σ-strategically closed (σ-Baire, proper*, semiproper, etc.) if $\mathbb{P}_I$ is σ-strategically closed (σ-Baire, proper, semiproper, etc.). (Usually if $\mathcal{P}_\kappa\lambda$ does not belong to an ideal I over $\mathcal{P}_\kappa\lambda$, then we say I is proper. To distinguish from this properness, we add the superscript $*$.)

In Matsubara [9], [10], σ-strategically closed ideals, σ-Baire ideals and proper* ideals were researched. In this paper, we study semiproper ideals on $\mathcal{P}_\kappa\lambda$. Our main results are as follows:

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Theorem 1.1. *Assume that, for some cardinal $\lambda \geq \omega_2$, there is a κ-complete semiproper ideal on $\mathcal{P}_\kappa\lambda$ whose dual filter is fine. Then $2^{\aleph_0} \leq \aleph_2$ and Chang's Conjecture holds.*

Theorem 1.2. *Assume κ is a regular cardinal, λ is a cardinal $\geq \lambda$ and there is a κ-complete semiproper ideal on $\mathcal{P}_\kappa\lambda$ whose dual filter is fine. Then $\square_\gamma$ fails for every cardinal γ such that $\kappa \leq \gamma^+ \leq \lambda$.*

Theorem 1.3. *The following is consistent relative to some large cardinal axiom: There are regular cardinals $\kappa \leq \lambda$ and a stationary $S \subseteq E^{\kappa,\lambda}_\omega$ such that $NS_{\kappa,\lambda} \restriction S$ is semiproper, where $NS_{\kappa,\lambda}$ is the nonstationary ideal on $\mathcal{P}_\kappa\lambda$ and $E^{\kappa,\lambda}_\omega = \{x \in \mathcal{P}_\kappa\lambda \mid x \cap \kappa \in \kappa \wedge cf(x \cap \kappa) = \omega\}$.*

From Theorem 1.1 and Theorem 1.2, we can see that semiproperness is a large cardinal property.

Theorem 1.2 is a generalization of Solovay's theorem that if κ is λ-supercompact then $\square$-principles fail between κ and λ. Matsubara [10] generalized Solovay's theorem and showed that if there is a normal σ-strategically closed ideal on $\mathcal{P}_\kappa\lambda$ then $\square$-principles fail between κ and λ. Theorem 1.2 is a further generalization.

In Theorem 1.3, note that $NS_{\kappa,\lambda} \restriction S$ cannot be proper *. In fact if I is a normal ideal on $\mathcal{P}_\kappa\lambda$ such that $E^{\kappa,\lambda}_\omega$ is in the dual filter of I then an easy calculation shows that $cf(\kappa) = \omega$ in the generic ultrapower. Hence $\mathbb{P}_I$ makes the cofinality of κ countable and so $\mathbb{P}_I$ is not proper. Theorem 1.3 says that such an ideal I can be semiproper.

This paper is organized as follows. In section 2, we make basic analyses of semiproper ideals. Among other things, we show Theorem 1.1 and Theorem 1.2. In section 3, we prove Theorem 1.3. More precisely, we prove that, under some condition on the ground model, if a supercompact cardinal is Levy collapsed to λ^+ then there is a stationary $S \subseteq E^{\kappa,\lambda}_\omega$ such that $NS_{\kappa,\lambda} \restriction S$ is semiproper.

Notations and Definitions: We believe our notations are basic ones. Here we only present our notations and basic definitions related to $\mathcal{P}_\kappa W$ and ideals on $\mathcal{P}_\kappa W$. For those which are not presented here, see Jech [5] or Kanamori [6]. Let κ be a regular uncountable cardinal and let W be a set with $\kappa \subseteq W$.

For an ideal I on $\mathcal{P}_\kappa W$, I^+ denotes the set of all I-positive sets, $\mathcal{F}_I$ denotes the dual filter of I and $\mathbb{P}_I$ denotes the poset $\langle I^+, \subseteq\rangle$. For an $S \in I^+$, $I \restriction S$

denotes the restriction of I to S, that is, the ideal on $\mathcal{P}_\kappa W$ such that for every $X \subseteq \mathcal{P}_\kappa W$, $X \in I \restriction S$ if and only if $X \cap S \in I$.

An ideal I on $\mathcal{P}_\kappa W$ is called a κ-ideal if I is a proper κ-complete ideal on $\mathcal{P}_\kappa W$ whose dual filter $\mathcal{F}_I$ is fine, i.e. $\{x \in \mathcal{P}_\kappa W \mid w \in x\} \in \mathcal{F}_I$ for every $w \in W$. $NS_{\kappa,W}$, the nonstationary ideal on $\mathcal{P}_\kappa W$, is a normal κ-ideal on $\mathcal{P}_\kappa W$.

A κ-ideal I on $\mathcal{P}_\kappa W$ is called semiproper if the corresponding poset $\mathcal{P}_I$ is semiproper. In this paper, we adopt the following definition of semiproperness of posets. Let $\mathbb{P}$ be a poset. For a set M, an $(M, \mathbb{P})$-semimaster condition q is a condition $q \in \mathbb{P}$ such that $q \Vdash$ "$\tau \in \omega_1^V \to \tau \in M$" for every $\mathbb{P}$-name $\tau \in M$. $\mathbb{P}$ is called semiproper if for every sufficiently large regular cardinal θ, every countable elementary submodel M of $\langle \mathcal{H}_\theta, \in \rangle$ with $\mathbb{P} \in M$ and every $p \in \mathbb{P} \cap M$, there is an $(M, \mathbb{P})$-semimaster condition q below p.

Other properties of ideals such as σ-strategically closure, properness*, etc. are defined in the same fashion. A κ-ideal I on $\mathcal{P}_\kappa W$ is called σ-strategically closed if $\mathbb{P}_I$ is σ-strategically closed and I is called proper* if $\mathbb{P}_I$ is proper. (For the definition of σ-strategically closure and properness of posets see Jech [5].) Moreover I is called ω_1-stationary preserving or ω_1-preserving if $\mathbb{P}_I$ preserves stationary subsets of ω_1 or preserves ω_1, respectively. The order of the strength of these properties is as follows:

$$\begin{aligned} &\sigma\text{-strategically closed} \;>\; \text{proper}^* \;>\; \text{semiproper} \\ &>\; \omega_1\text{-stationary preserving} \;>\; \omega_1\text{-preserving} \end{aligned}$$

Next we give notations for subsets of $\mathcal{P}_\kappa W$. For a regular $\gamma < \kappa$, let $E^\kappa_\gamma := \{\alpha \in \kappa \mid cf(\alpha) = \gamma\}$ and $E^{\kappa,W}_\gamma := \{x \in \mathcal{P}_\kappa W \mid x \cap \kappa \in E^\kappa_\gamma\}$.

Assume $W' \subseteq W$ and $\langle X_w \mid w \in W' \rangle$ is a family of subsets of $\mathcal{P}_\kappa W$ indexed by elements of W'. Then $\nabla \langle X_w \mid w \in W' \rangle$ or $\nabla_{w \in W'} X_w$ denotes the diagonal union $\{x \in \mathcal{P}_\kappa W \mid \exists w \in W', w \in x \in X_w\}$. Note that if $X \subseteq \nabla_{w \in W'} X_w$ is stationary in $\mathcal{P}_\kappa W$ then there is a $w \in W'$ such that $X \cap X_w$ is stationary. That is, $\{X_w \mid w \in W' \wedge X_w \text{ is stationary}\}$ is predense below $\nabla_{w \in W'} X_w$ in $\mathbb{P}_{NS_{\kappa,W}}$.

Finally we present basic facts on $\mathcal{P}_\kappa W$ which are used in this paper. The proof of Fact 1.5 can be found also in Jech [5] (Theorem 8.27).

Fact 1.4. *Let κ and λ be regular uncountable cardinals such that $\kappa \leq \lambda$. Then for every club $C \subseteq \mathcal{P}_\kappa \mathcal{H}_\lambda$, there is an $A \subseteq \mathcal{H}_\lambda$ such that $\{M \in \mathcal{P}_\kappa \mathcal{H}_\lambda \mid M \prec \langle \mathcal{H}_\lambda, \in, A \rangle \wedge M \cap \kappa \in \kappa\} \subseteq C$.*

Proof. Let $C \subseteq \mathcal{P}_\kappa \mathcal{H}_\lambda$ be a club. Then there is a function $F : [\mathcal{H}_\lambda]^{<\omega} \to \mathcal{P}_\kappa \mathcal{H}_\lambda$ such that $C_F \subseteq C$, where $C_F := \{x \in \mathcal{P}_\kappa \mathcal{H}_\lambda \mid \forall s \in [x]^{<\omega},\ F(s) \subseteq x\}$. Note that $F \subseteq \mathcal{H}_\lambda$. We show that $\{M \in \mathcal{P}_\kappa \mathcal{H}_\lambda \mid M \prec \langle \mathcal{H}_\lambda, \in, F\rangle \wedge M \cap \kappa \in \kappa\} \subseteq C$.

Take an arbitrary M such that $M \prec \langle \mathcal{H}_\lambda, \in, F\rangle$ and $M \cap \kappa \in \kappa$. It suffices to show that $M \in C_F$. Let $s \in [M]^{<\omega}$. Then $F(s), |F(s)| \in M$ by the elementarity of M. Moreover there is a bijection $g \in M$ from $|F(s)|$ to $F(s)$. Note that $|F(s)| \subseteq M$ because $|F(s)| \in M \cap \kappa$ and $M \cap \kappa \in \kappa$. Hence $F(s) = g[\,|F(s)|\,] \subseteq M$ and so $M \in C_F$. □

Fact 1.5. (Menas [12]) Let κ be a regular uncountable cardinal and $W, \bar{W}$ be sets with $\kappa \subseteq W \subseteq \bar{W}$.

(1) If $X \subseteq \mathcal{P}_\kappa W$ is stationary then $\{\bar{x} \in \mathcal{P}_\kappa \bar{W} \mid \bar{x} \cap W \in X\}$ is stationary in $\mathcal{P}_\kappa \bar{W}$.
(2) If $\bar{X} \subseteq \mathcal{P}_\kappa \bar{W}$ is stationary then $\{\bar{x} \cap W \mid \bar{x} \in \bar{X}\}$ is stationary in $\mathcal{P}_\kappa W$.

Fact 1.6. *Let κ be a regular uncountable cardinal and $W, \bar{W}$ be sets with $\kappa \subseteq W \subseteq \bar{W}$ and $|W| = |\bar{W}|$. Let $\pi : \bar{W} \to W$ be a bijection and let $\bar{C} \subseteq \mathcal{P}_\kappa \bar{W}$ be a club consisting of all $\bar{x} \in \mathcal{P}_\kappa \bar{W}$ which are closed under π and π^{-1}.*

(1) *For every $\bar{X} \subseteq \bar{C}$, $\bar{X}$ is stationary in $\mathcal{P}_\kappa \bar{W}$ if and only if $\{\bar{x} \cap W \mid \bar{x} \in \bar{X}\}$ is stationary in $\mathcal{P}_\kappa W$.*
(2) *Assume $\bar{X} \subseteq \bar{C}$ is stationary and let $X := \{\bar{x} \cap W \mid \bar{x} \in \bar{X}\}$. Then $\mathbb{P}_{NS_{\kappa,\bar{W}} \restriction \bar{X}}$ and $\mathbb{P}_{NS_{\kappa,W} \restriction X}$ are isomorphic.*
(3) *Assume $X \subseteq \mathcal{P}_\kappa W$ is stationary, γ is an ordinal and $\{X_\xi \mid \xi < \gamma\} \subseteq \mathbb{P}_{NS_{\kappa,W}}$ is predense below X. Let $\bar{X} := \{\bar{x} \in \mathcal{P}_\kappa \bar{W} \mid \bar{x} \cap W \in X\}$ and $\bar{X}_\xi := \{\bar{x} \in \mathcal{P}_\kappa \bar{W} \mid \bar{x} \cap W \in X_\xi\}$ for each $\xi < \gamma$. Then $\{\bar{X}_\xi \mid \xi < \gamma\}$ is predense below $\bar{X}$ in $\mathbb{P}_{NS_{\kappa,\bar{W}}}$.*

Proof. (1). Let $\bar{X} \subseteq \bar{C}$. If $\bar{X}$ is stationary in $\mathcal{P}_\kappa \bar{W}$ then $\{\bar{x} \cap W \mid \bar{x} \in \bar{X}\}$ is stationary in $\mathcal{P}_\kappa W$ by Fact 1.5 (2). On the other hand, assume $\bar{X}$ is nonstationary in $\mathcal{P}_\kappa \bar{W}$. Let $\bar{B} \subseteq \mathcal{P}_\kappa \bar{W}$ be a club such that $\bar{B} \cap \bar{X} = \emptyset$. Then $\{\pi[\bar{x}] \mid \bar{x} \in \bar{B}\}$ is club in $\mathcal{P}_\kappa W$ and $\{\pi[\bar{x}] \mid \bar{x} \in \bar{B}\} \cap \{\pi[\bar{x}] \mid \bar{x} \in \bar{X}\} = \emptyset$. Moreover if $\bar{x} \in \bar{X}$ then $\pi[\bar{x}] = \bar{x} \cap W$ because $\bar{x}$ is closed under π and π^{-1}. Hence $\{\pi[\bar{x}] \mid \bar{x} \in \bar{X}\} = \{\bar{x} \cap W \mid \bar{x} \in \bar{X}\}$. Therefore $\{\bar{x} \cap W \mid \bar{x} \in \bar{X}\}$ is nonstationary.

(2). Let $\pi^* : \mathcal{P}(\mathcal{P}_\kappa \bar{W}) \to \mathcal{P}(\mathcal{P}_\kappa W)$ be such that $\pi^*(\bar{Y}) := \{\pi[\bar{x}] \mid \bar{x} \in \bar{Y}\}$. Note that $\pi^*(\bar{X}) = X$ by the argument above. Then it is easy to see that $\pi^* \restriction \mathbb{P}_{NS_{\kappa,\bar{W}} \restriction \bar{X}}$ is an isomorphism from $\mathbb{P}_{NS_{\kappa,\bar{W}} \restriction \bar{X}}$ to $\mathbb{P}_{NS_{\kappa,W} \restriction X}$.

(3). Take an arbitrary stationary $\bar{Y} \subseteq \bar{X}$. We must find $\xi < \gamma$ such that $\bar{Y} \cap \bar{X}_\xi$ is stationary. We may assume $\bar{Y} \subseteq \bar{C}$. First let $Y := \{\bar{x} \cap W \mid \bar{x} \in \bar{Y}\}$. Then Y is a stationary subset of X and so there is a $\xi < \gamma$ with $Y \cap X_\xi$ stationary. Here note that $Y \cap X_\xi = \{\bar{x} \cap W \mid \bar{x} \in \bar{Y} \cap \bar{X}_\xi\}$. Moreover $\bar{Y} \cap \bar{X}_\xi \subseteq \bar{C}$. Therefore $\bar{Y} \cap \bar{X}_\xi$ is stationary in $\mathcal{P}_\kappa \bar{W}$ by (1). □

2. Basic Analysis of Semiproper Ideals

In this section we investigate properties of semiproper ideals.

First we discuss precipitousness. In Matsubara [9], it is shown that every proper κ-ideal on $\mathcal{P}_\kappa\lambda$ is precipitous. We do not know whether semiproperness implies precipitousness. In Gitik-Shelah [3], it was shown that if $2^\kappa = \kappa^+$ then every κ-complete ideal I on κ such that $\mathbb{P}_I$ preserves ω_1 is precipitous. The following is the $\mathcal{P}_\kappa\lambda$ version of this.

Theorem 2.1. *Assume κ is a regular uncountable cardinal, λ is a cardinal $\geq \kappa$ and $2^{\lambda^{<\kappa}} = \lambda^+$. Assume that I is a normal ω_1-preserving κ-ideal on $\mathcal{P}_\kappa\lambda$. Then I is precipitous.*

Theorem 2.1 follows from the following two facts.

Fact 2.2. *Assume γ is a regular uncountable cardinal and $\mathbb{P}$ is a poset of cardinality $\leq \gamma$. Then $\Vdash_{\mathbb{P}}$ "$cf(\gamma) = |\gamma|$ ".*

Proof. Let γ be a regular uncountable cardinal and $\mathbb{P}$ be a poset with $|\mathbb{P}| \leq \gamma$. We show that if $\dot{a}$ is a $\mathbb{P}$-name of an unbounded subset of γ then there is a function $f \in V$ from γ to γ such that $\Vdash_{\mathbb{P}}$ "$f[\dot{a}] = \gamma$". Clearly this suffices.

Fix an enumeration $\langle p_\xi \mid \xi < \gamma \rangle$ of $\mathbb{P}$. For each $\langle \xi, \eta \rangle \in {}^2\kappa$, we define $\alpha_{\xi\eta} < \gamma$ and $q_{\xi\eta} \in \mathbb{P}$ by induction on the lexicographical order $<_{\text{lex}}$ of ${}^2\kappa$. Assume that $\langle \xi, \eta \rangle \in {}^2\kappa$ and that $\alpha_{\xi'\eta'}$ was defined for each $\langle \xi', \eta' \rangle <_{\text{lex}} \langle \xi, \eta \rangle$. Then let $\alpha_{\xi\eta}$ and $q_{\xi\eta}$ be such that

- $\alpha_{\xi\eta} > \alpha_{\xi'\eta'}$ for every $\langle \xi', \eta' \rangle <_{\text{lex}} \langle \xi, \eta \rangle$,
- $q_{\xi\eta} \leq p_\xi$ and $q_{\xi\eta} \Vdash_{\mathbb{P}}$ "$\alpha_{\xi\eta} \in \dot{a}$".

We can take such $\alpha_{\xi\eta}$ and $q_{\xi\eta}$ because γ is regular and $\dot{a}$ is a $\mathbb{P}$-name of an unbounded subset of γ. Now let $f : \gamma \to \gamma$ be a function such that $f(\alpha_{\xi\eta}) = \eta$ for each $\langle \xi, \eta \rangle \in {}^2\kappa$.

We show that $\Vdash_{\mathbb{P}}$ "$f[\dot{a}] = \gamma$". Take an arbitrary $\eta < \gamma$ and $p \in \mathbb{P}$, say $p = p_\xi$. Then $q_{\xi\eta} \leq p$. and $q_{\xi\eta} \Vdash$ "$\eta \in f[\dot{a}]$". The latter is because $q_{\xi\eta} \Vdash_{\mathbb{P}}$ "$\alpha_{\xi\eta} \in \dot{a}$" and $f(\alpha_{\xi\eta}) = \eta$. This implies that $\Vdash_{\mathbb{P}}$ "$f[\dot{a}] = \gamma$".

□

Fact 2.3. (Baumgartner-Taylor [1]) *Assume κ is a regular uncountable cardinal, λ is a cardinal $\geq \kappa$ and $2^{\lambda^{<\kappa}} = \lambda^+$. Moreover assume that I is a normal κ-ideal on $\mathcal{P}_\kappa\lambda$ such that $\Vdash_{\mathbb{P}_I}$ "$cf((\lambda^+)^V) > \omega$". Then I is precipitous.*

Proof of Theorem 2.1. Let κ, λ and I be as in the theorem. Because $2^{\lambda^{<\kappa}} = \lambda^+$, $|\mathbb{P}_I| \leq \lambda^+$. Hence, by Fact 2.2 applied for $\gamma = \lambda^+$, $\Vdash_{\mathbb{P}_I}$ "$cf((\lambda^+)^V) = |(\lambda^+)^V|$". Moreover $\Vdash_{\mathbb{P}_I}$ "$|(\lambda^+)^V| > \omega$" because I is ω_1-preserving. Therefore $\Vdash_{\mathbb{P}_I}$ "$cf((\lambda^+)^V) > \omega$". Then, by Fact 2.3, I is precipitous. □

It is easy to see $\mathcal{P}_{\omega_1}\lambda$ cannot carry semiproper ideals. Assume λ is an uncountable cardinal, I is an ω_1-ideal on $\mathcal{P}_{\omega_1}\lambda$ and G is a $\mathbb{P}_I$-generic filter over V. Then the critical point of the generic ultrapower map is ω_1^V. Therefore ω_1^V is countable in the ultrapower of V by G and thus is in $V[G]$. Note that this argument does not need precipitousness. Hence I is not ω_1-preserving.

Theorem 2.4. *Assume that λ is an uncountable cardinal and that I is an ω_1-ideal on $\mathcal{P}_{\omega_1}\lambda$. Then I is not ω_1-preserving. So I is not semiproper.*

Next we discuss consequences of the existence of semiproper ideals on $\mathcal{P}_\kappa\lambda$. The argument uses the following principle.

Definition 2.5. *For a regular uncountable cardinal $\gamma \geq \omega_2$, let Φ_γ be the following statement:*

> *For every sufficiently large regular cardinal θ, there is a club $C \subseteq [\mathcal{H}_\theta]^\omega$ which satisfies the following conditions.*
> (1) *For every $M \in C$, $M \prec \langle \mathcal{H}_\theta, \in \rangle$.*
> (2) *For every $M \in C$ and $\alpha < \gamma$ there is an $N \in C$ such that:*
> (a) *$M \subseteq N$,*
> (b) *$M \cap \omega_1 = N \cap \omega_1$,*
> (c) *$sup(N \cap \gamma) > \alpha$.*

Φ_{ω_2} is known as a strong form of Chang's Conjecture and it is easy to see that Φ_{ω_2} implies Chang's Conjecture. Shelah [13] showed that if Namba forcing is semiproper then Φ_{ω_2} holds and Todorčević [15] showed that Rado's Conjecture implies Φ_{ω_2}. Here we show that if there is a semiproper κ-ideal on $\mathcal{P}_\kappa\lambda$ then Φ_γ holds for every regular cardinal between κ and λ. For this we prepare a standard lemma on the Skolem hull.

Lemma 2.6. *Let θ be a regular uncountable cardinal, Δ be a well ordering of $\mathcal{H}_\theta$ and $A \subseteq \mathcal{H}_\theta$. Assume that $M \prec \langle \mathcal{H}_\theta, \in, \Delta, A\rangle$ and that $\emptyset \neq D \subseteq E \in M$. Let*

$$N := \{f(d) \mid \exists n \in \omega, f : {}^nE \to \mathcal{H}_\theta \wedge f \in M \wedge d \in {}^nD\}.$$

Then N is the Skolem hull of $M \cup D$ in $\langle \mathcal{H}_\theta, \in, \Delta, A\rangle$.

Proof. Let $\mathcal{A} := \langle \mathcal{H}_\theta, \in, \Delta, A\rangle$. Clearly N is included in the Skolem hull of $M \cup D$ in $\mathcal{A}$. By taking f as the identity function on E, we can see $D \subseteq N$. For each $a \in M$, by taking f as the constant function on E with value a, we can see $a \in N$. So $M \cup D \subseteq N$. Hence it suffices to show that $N \prec \mathcal{A}$.

We use Tarski-Vaught's criterion. Assume $\varphi(v, v_1, ..., v_n)$ is a formula, $a_1, ..., a_n \in N$ and $\mathcal{A} \models \exists v\varphi[v, a_1, ..., a_n]$. It suffices to show that there exists an $a \in N$ such that $\mathcal{A} \models \varphi[a, a_1, ..., a_n]$.

For each $k = 1, ..., n$, let $m_k \in \omega$, $d_k \in {}^{m_k}E$ and $f_k : {}^{m_k}E \to \mathcal{H}_\theta$ be a function such that $f_k \in M$ and $f_k(d_k) = a_k$. Let $m := \Sigma_{k=1}^n m_k$ and $d := d_1{}^\frown d_2{}^\frown ...{}^\frown d_n$. We define a function f on ${}^mE$ as follows. Assume $e \in {}^mE$. Let $\langle e_k \mid k = 1, ..., n\rangle$ be such that $e = e_1{}^\frown e_2{}^\frown ...{}^\frown e_n$ and $e_k \in {}^{m_k}E$. If there is a b such that $\mathcal{A} \models \varphi[b, f_1(e_1), ..., f_n(e_n)]$ then let $f(e)$ be the Δ-least such b. Otherwise let $f(e) := 0$. Because θ is regular and $E \in \mathcal{H}_\theta$, $f \in \mathcal{H}_\theta$. Moreover f is definable in $\mathcal{A}$ from the parameters $E, f_1, ..., f_n$ which are in M. So, by the elementarity, $f \in M$ and therefore $f(d) \in N$. On the other hand, the definition of f and the assumption on $a_1, ..., a_n$ implies that $\mathcal{A} \models \varphi[f(d), a_1, ..., a_n]$. Now we have shown there is an $a \in N$ such that $\mathcal{A} \models \varphi[a, a_1, ..., a_n]$. □

Lemma 2.7. *Assume κ is a regular uncountable cardinal, λ is a cardinal $\geq \kappa$ and there is a semiproper κ-ideal on $\mathcal{P}_\kappa\lambda$. Then Φ_γ holds for every regular cardinal γ such that $\kappa \leq \gamma \leq \lambda$.*

Proof. First note that $\kappa \geq \omega_2$ by Theorem 2.4. Let I be a semiproper κ-ideal on $\mathcal{P}_\kappa\lambda$ and γ be a regular cardinal with $\kappa \leq \gamma \leq \lambda$. To show Φ_γ, let θ be a sufficiently large cardinal. Fix a well ordering Δ of $\mathcal{H}_\theta$. Then let C be the set of all countable $M \prec \langle \mathcal{H}_\theta, \in, \Delta\rangle$ with $\kappa, \lambda, I, \gamma \in M$. We show that C witnesses Φ_γ for θ.

(1) of Φ_γ is clear. We show that (2) holds. Take an arbitrary $M \in C$ and $\alpha < \gamma$. Then because $\mathbb{P}_I$ is semiproper we can take an $(M, \mathbb{P}_I)$-semimaster condition $X \in I^+$. We may assume that

- for every function $f : \mathcal{P}_\kappa\lambda \to \omega_1$ which is in M and for every $\xi < \omega_1$, if $f^{-1}[\{\xi\}] \cap X \in I$ then $f^{-1}[\{\xi\}] \cap X = \emptyset$.

For this, note that $Z := \bigcup\{f^{-1}[\{\xi\}] \cap X \mid f : \mathcal{P}_\kappa\lambda \to \omega_1 \wedge f \in M \wedge f^{-1}[\{\xi\}] \cap X \in I\}$ is in I because M is countable and I is ω_2-complete. Simply replace X by $X \setminus Z$.

Now, because $\mathcal{F}_I$ is fine, there is an $x \in X$ with $\alpha \in x$. Let $N := \{f(x) \mid f : \mathcal{P}_\kappa\lambda \to \mathcal{H}_\theta \wedge f \in M\}$. Then by Lemma 2.6 applied for $D = \{x\}$ and $E = \mathcal{P}_\kappa\lambda$, $M \cup \{x\} \subseteq N \prec \langle \mathcal{H}_\theta, \in, \Delta\rangle$. Therefore $M \subseteq N \in C$. Moreover, because $\alpha < sup(x \cap \gamma) \in N$, $\alpha < sup(N \cap \gamma)$. So all we have to show is $M \cap \omega_1 = N \cap \omega_1$.

For this it suffices to show that if $f \in M$ is a function from $\mathcal{P}_\kappa\lambda$ to ω_1 then $f(x) \in M$. Take an arbitrary function $f \in M$ from $\mathcal{P}_\kappa\lambda$ to ω_1. First let $\tau \in M$ be the $\mathbb{P}_I$-name of an ordinal in ω_1 such that for each $\xi < \omega_1$, if $f^{-1}[\{\xi\}] \in I^+$ then $f^{-1}[\{\xi\}] \Vdash$ "$\tau = \xi$". Then because X is an $(M, \mathbb{P}_I)$-semimaster condition, $X \cap f^{-1}[\{\xi\}] \in I$ for each $\xi \in \omega_1 \setminus M$. So, by the assumption on X, $f^{-1}[\{\xi\}] \cap X = \emptyset$ for each $\xi \in \omega_1 \setminus M$. Because $x \in X$, this implies that $f(x) \in M$. This completes the proof. □

Corollary 2.8. *Assume λ is a cardinal $\geq \omega_2$ and there is a semiproper ω_2-ideal on $\mathcal{P}_{\omega_2}\lambda$. Then Chang's Conjecture holds.*

Todorčević [15] showed that if Φ_{ω_2} holds then for every stationary $S \subseteq [\omega_2]^\omega$ there exists an $\alpha < \omega_2$ such that $S \cap [\alpha]^\omega$ is stationary in $[\alpha]^\omega$. Moreover Todorčević [16] showed that this type of stationary reflection principle implies $2^{\aleph_0} \leq \aleph_2$. Hence:

Corollary 2.9. *Assume λ is a cardinal $\geq \omega_2$ and there is a semiproper ω_2-ideal on $\mathcal{P}_{\omega_2}\lambda$. Then for every stationary $S \subseteq [\omega_2]^\omega$, there exists an $\alpha < \omega_2$ such that $S \cap [\alpha]^\omega$ is stationary in $[\alpha]^\omega$. Therefore $2^{\aleph_0} \leq \aleph_2$.*

Next we discuss □-principles. It is known that Chang's Conjecture implies the failure of $\square_{\omega_1}$. Here we show that Φ_{γ^+} implies the failure of $\square_\gamma$. For an uncountable cardinal γ, recall that $\square_\gamma$ is the following statement:

There is a sequence $\langle C_\alpha \mid \alpha \in Lim(\gamma^+)\rangle$ such that

(1) C_α is closed unbounded in α for each $\alpha \in Lim(\gamma^+)$,

(2) if $\beta \in Lim(C_\alpha)$ then $C_\beta = C_\alpha \cap \beta$ for each $\alpha,\ \beta \in Lim(\gamma^+)$,

(3) $o.t.(C_\alpha) \leq \gamma$ for each $\alpha \in Lim(\gamma^+)$.

(For a set A of ordinals, $Lim(A)$ denotes the set of all $\alpha \in A$ such that $sup(A \cap \alpha) = \alpha$.) $\langle C_\alpha \mid \alpha \in Lim(\gamma^+)\rangle$ satisfying (1)-(3) is called $\square_\gamma$-sequence.

We use the following easy lemma.

Lemma 2.10. *Let γ be an uncountable cardinal and assume that $\square_\gamma$ holds. Then there is a $\square_\gamma$-sequence $\langle C_\alpha \mid \alpha \in Lim(\gamma^+)\rangle$ such that $\{\alpha \in Lim(\gamma^+) \mid o.t.(C_\alpha) = \omega_1\}$ is stationary in γ^+.*

Proof. Let $\langle C_\alpha \mid \alpha \in Lim(\gamma^+)\rangle$ be an arbitrary $\square_\gamma$-sequence. We modify $\langle C_\alpha \mid \alpha \in Lim(\gamma^+)\rangle$. Let $E := E^{\gamma^+}_{\omega_1}$.

First let $E_\zeta := \{\alpha \in E \mid o.t.(C_\alpha) = \zeta\}$ for each $\zeta \leq \gamma$ and let ρ be the least $\zeta \leq \gamma$ such that E_ζ is stationary in γ^+. Because $E = \bigcup_{\zeta \leq \gamma} E_\zeta$, there is such ρ. Then take a club $C \subseteq \gamma^+$ such that $C \cap E_\zeta = \emptyset$ for every $\zeta < \rho$ and let $\sigma : \gamma^+ \to C$ be the increasing enumeration of C. Here note that if $\alpha \in Lim(C) \cap E_\rho$ then $o.t.(C_\alpha \cap C) = \omega_1$. Because $C_\alpha \cap C$ is club in α, $o.t.(C_\alpha \cap C) \geq \omega_1$. Assume $o.t.(C_\alpha \cap C) \gneq \omega_1$. Let β be the ω_1-th element of $C \cap C_\alpha$. Then $\beta < \alpha$ and $C_\beta = C_\alpha \cap \beta$. Hence $\beta \in E_\zeta$ for some $\zeta < \rho$. Thus $\beta \in C \cap E_\zeta$ for some $\zeta < \rho$. This contradicts the construction of C. Therefore $o.t.(C_\alpha \cap C) \leq \omega_1$.

Now, for each $\alpha \in Lim(\gamma^+)$, define C^*_α as follows: If $C_{\sigma(\alpha)} \cap C$ is unbounded in $\sigma(\alpha)$ then let $C^*_\alpha := \sigma^{-1}[C_{\sigma(\alpha)} \cap C]$. Assume $C_{\sigma(\alpha)} \cap C$ is not unbounded in $\sigma(\alpha)$. Then note that $cf(\alpha) = \omega$. Let C^*_α be an arbitrary unbounded subset of α of order type ω.

It is easy to check that $\langle C^*_\alpha \mid \alpha \in Lim(\gamma^+)\rangle$ is a $\square_\gamma$-sequence. Moreover, by the observation before, if $\alpha \in Lim(\gamma^+) \cap \sigma^{-1}[E_\rho]$ then $o.t.(C^*_\alpha) = \omega_1$. On the other hand, note that $\sigma^{-1}[E_\rho]$ is stationary in γ^+ because E_ρ is stationary and σ is continuous. So $\{\alpha \in Lim(\gamma^+) \mid o.t.(C^*_\alpha) = \omega_1\}$ is stationary in γ^+. This completes the proof. $\square$

Lemma 2.11. *Let γ be an uncountable cardinal. Then $\Phi_{\gamma^+} \to \neg\square_\gamma$.*

Proof. Assume that both Φ_{γ^+} and $\square_\gamma$ hold. By Lemma 2.10 let $\langle C_\alpha \mid \alpha \in Lim(\gamma^+)\rangle$ be a $\square_\gamma$-sequence such that $\{\alpha \in Lim(\gamma^+) \mid o.t.(C_\alpha) = \omega_1\}$ is stationary in γ^+. Let θ be a sufficiently large regular cardinal and let $C \subseteq [\mathcal{H}_\theta]^\omega$ be a club witnessing Φ_{γ^+}.

Claim 2.11.1. *There is an* $N \prec \langle \mathcal{H}_\theta, \in \rangle$ *such that* $\gamma^+, \langle C_\alpha \mid \alpha \in Lim(\gamma^+) \rangle \in N$, $o.t.(C_{sup(N \cap \gamma^+)}) = \omega_1$ *and* $N \cap \omega_1 \in \omega_1$.

Proof of Claim. Let Δ be a well-ordering of $\mathcal{H}_\theta$. Then there is a $K \prec \langle \mathcal{H}_\theta, \in, \Delta, C \rangle$ such that $\gamma^+, \langle C_\alpha \mid \alpha \in Lim(\gamma^+) \rangle \in K$, $\alpha^* := K \cap \gamma^+ \in \gamma^+$ and $o.t.(C_{\alpha^*}) = \omega_1$. Note that every proper initial segment of C_{α^*} is in K because $C_\alpha \in K$ for each $\alpha < \alpha^*$ and $\langle C_\alpha \mid \alpha \in Lim(\gamma^+) \rangle$ is a $\square_\gamma$-sequence. Let $\langle \alpha_\xi \mid \xi < \omega_1 \rangle$ be the increasing enumeration of C_{α^*}.

Now, by induction on $\xi < \omega_1$, define $M_\xi \in C$ as follows. Let M_0 be an element of C with $\gamma^+, \langle C_\alpha \mid \alpha \in Lim(\gamma^+) \rangle \in M$. If ξ is limit then let $M_\xi := \bigcup_{\eta < \xi} M_\eta$. Finally assume M_ξ has been defined for some $\xi < \omega_1$. Then, recalling that C witnesses Φ_{γ^+}, let $M_{\xi+1}$ be the Δ-least $N \in C$ such that $M_\xi \subseteq N$, $M_\xi \cap \omega_1 = N \cap \omega_1$ and $\alpha_\xi < sup(N \cap \gamma^+)$. This completes the inductive definition. Note that, because every initial segment of C_{α^*} is in K, every initial segment of this induction can be carried out in K. So $M_\xi \in K$ for each $\xi < \omega_1$. Hence $M_\xi \subseteq K$ for each $\xi < \omega_1$ because M_ξ is countable.

Let $N := \bigcup_{\xi < \omega_1} M_\xi$. Then $N \subseteq K$ and so $sup(N \cap \gamma^+) \leq \alpha^*$. On the other hand, $sup(N \cap \gamma^+) \geq \alpha^*$ because $\alpha_\xi < sup(M_{\xi+1} \cap \gamma^+) < sup(N \cap \gamma^+)$ for each $\xi < \omega_1$. Hence $sup(N \cap \gamma^+) = \alpha^*$ and so $o.t.(C_{sup(N \cap \gamma^+)}) = \omega_1$. Moreover clearly $\gamma^+, \langle C_\alpha \mid \alpha \in Lim(\gamma^+) \rangle \in N \prec \langle \mathcal{H}_\theta, \in \rangle$. Therefore N is the one desired. $\square$.*Claim*

Now we return to the proof of the lemma. Let N be the one obtained from Claim 2.11.1. Let $\alpha^* := sup(N \cap \gamma^+)$ and $\langle \alpha_\xi \mid \xi < \omega_1 \rangle$ be the increasing enumeration of C_{α^*}. Then we can take a limit $\xi < \omega_1$ such that $N \cap \omega_1 < \xi$ and $N \cap \alpha_\xi$ is unbounded in α_ξ. Let $\beta := min(N \setminus \alpha_\xi)$. Note that β is a limit ordinal and $C_\beta \in N$.

Claim 2.11.2.

(1) $C_{\alpha^*} \cap \alpha_\xi = C_\beta \cap \alpha_\xi = C_{\alpha_\xi}$.
(2) $C_{\alpha_\xi} \cap N$ *is unbounded in* α_ξ.

Proof of Claim. Because ξ is limit $C_{\alpha^*} \cap \alpha_\xi = C_{\alpha_\xi}$. Moreover, by the elementarity of N, it is easy to see that $N \cap C_\beta$ is unbounded in α_ξ. This implies that $\alpha_\xi \in Lim(C_\beta) \cup \{\beta\}$ and therefore $C_\beta \cap \alpha_\xi = C_{\alpha_\xi}$. Hence $C_{\alpha_\xi} \cap N$ is unbounded in α_ξ.

$\square$.*Claim*

By the claim above, there is an η such that $N \cap \omega_1 < \eta < \xi$ and $\alpha_\eta \in N$. Then, again by the claim, α_η is the η-th element of C_β. Then, because C_β and α_η are in N, $\eta \in N$. This contradicts $N \cap \omega_1 < \eta < \omega_1$. $\square$

Corollary 2.12. *Assume κ is a regular uncountable cardinal, λ is a cardinal $\geq \kappa$ and there is a semiproper κ-ideal on $\mathcal{P}_\kappa\lambda$. Then $\square_\gamma$ fails for every cardinal γ with $\kappa \leq \gamma^+ \leq \lambda$.*

Here we turn our attention to (†), where (†) is the statement that for every poset $\mathbb{P}$, $\mathbb{P}$ preserves stationary subsets of ω_1 if and only if $\mathbb{P}$ is semiproper. In Foreman-Magidor-Shelah [2], (†) was introduced and shown to have interesting consequences. Here we see that if (†) holds then $\square_\gamma$ fails for every uncountable cardinal γ.

As mentioned before, Shelah [13] showed that if Namba forcing is semiproper then Φ_{ω_2} holds. This can be generalized for an arbitrary regular cardinal $\gamma \geq \omega_2$. For each regular cardinal $\gamma \geq \omega_2$, let $Nm(\gamma)$ be the poset of all trees $T \subseteq {}^{<\omega}\gamma$ such that $|\{s \in T \mid t \subseteq s\}| = \gamma$ for every $t \in T$. The order is defined by inclusion. Note that $Nm(\gamma)$ forces $cf(\gamma) = \omega$. It is known that $Nm(\gamma)$ preserves stationary subsets of ω_1. Therefore (†) implies $Nm(\gamma)$ is semiproper.

Lemma 2.13. *Let γ be a regular cardinal $\geq \omega_2$. If $Nm(\gamma)$ is semiproper then Φ_γ holds. Hence if (†) holds then Φ_γ holds for every regular cardinal $\gamma \geq \omega_2$.*

Proof. Assume $Nm(\gamma)$ is semiproper. To show Φ_γ, take a sufficiently large regular cardinal θ. Fix a well ordering Δ of $\mathcal{H}_\theta$. Let C be the set of all countable $M \prec \langle \mathcal{H}_\theta, \in, \Delta\rangle$ with $\gamma \in M$. Then C is club in $[\mathcal{H}_\theta]^\omega$ and satisfies (1) of the definition of Φ_γ. We show that C satisfies (2).

Take an arbitrary $M \in C$ and $\alpha \in \gamma$. For each $\beta \in \gamma$, let $N_\beta := \{f(\beta) \mid f : \gamma \to \mathcal{H}_\theta \wedge f \in M\}$. Then by Lemma 2.6, N_β is the Skolem hull of $M \cup \{\beta\}$ in $\langle \mathcal{H}_\theta, \in, \Delta\rangle$. It suffices to show that there exists $\beta > \alpha$ with $N_\beta \cap \omega_1 = M \cap \omega_1$.

Let G be an $Nm(\gamma)$-generic filter containing an $(M, Nm(\gamma))$-semimaster condition. Working in $V[G]$, we show that there exists such a β. Let $M[G] := \{\tau_G \mid \tau \in M \wedge \tau \text{ is an } Nm(\gamma)\text{-name}\}$. Then $M[G] \cap \omega_1 = M \cap \omega_1$ because G contains an $(M, Nm(\gamma))$-semimaster condition. Moreover $M[G] \cap \gamma$ is cofinal in γ. This is because $Nm(\gamma)$ forces $cf(\gamma) = \omega$ and $Nm(\gamma) \in M \prec \langle \mathcal{H}_\theta, \in, \Delta\rangle$. Let $\beta > \alpha$ be an element of $M[G] \cap \gamma$. Then it is easy to see that $N_\beta \subseteq M[G]$. Hence $M \cap \omega_1 \subseteq N_\beta \cap \omega_1 \subseteq M[G] \cap \omega_1 = M \cap \omega_1$. Thus $N_\beta \cap \omega_1 = M \cap \omega_1$. This completes the proof. $\square$

Corollary 2.14. *If* (†) *holds then* $\square_\gamma$ *fails for every uncountable cardinal* γ.

Solovay [14] showed that if κ is a λ-strongly compact cardinal then the Singular Cardinal Hypothesis holds between κ and λ. Matsubara [10] showed that if there is a normal σ-strategically closed κ-ideal on $\mathcal{P}_\kappa\lambda$ then the Singular Cardinal Hypothesis holds between κ and λ. We end this section with the following question:

Question . *Does the existence of a semiproper* κ*-ideal on* $\mathcal{P}_\kappa\lambda$ *imply the Singular Cardinal Hypothesis between* κ *and* λ *?*

3. Consistency

In this section we discuss the consistency of the existence of semiproper ideals. In Matsubara [10] the following was shown.

Theorem 3.1. *Suppose* γ *is a regular uncountable cardinal and* κ *and* λ *are cardinals such that* $\gamma < \kappa \leq \lambda$ *and* κ *is a* λ*-supercompact. Let* I *be a normal maximal* κ*-ideal on* $\mathcal{P}_\kappa\lambda$*. Then, in* $V^{Col(\gamma,<\kappa)}$, $\bar{I}$ *is a normal* σ*-strategically closed* κ*-ideal on* $\mathcal{P}_\kappa\lambda^{V[G]}$*, where* $Col(\gamma,<\kappa)$ *is the Levy collapse making* κ *become* γ^+ *and* $\bar{I}$ *is the ideal on* $\mathcal{P}_\kappa\lambda^{V[G]}$ *generated by* I.

Therefore even if κ is a successor cardinal, $\mathcal{P}_\kappa\lambda$ can carry a normal σ-strategically closed κ-ideal.

On the other hand, there is an obvious limitation. If I is a normal κ-ideal on $\mathcal{P}_\kappa\lambda$ with $E^{\kappa,\lambda}_\omega \in I^+$ then I is not proper*. (If G is a $\mathbb{P}_I$-generic filter with $E^{\kappa,\lambda}_\omega \in G$ then $cf(\kappa) = \omega$ in the generic ultrapower of V by G and thus is in $V[G]$. Hence $\mathbb{P}_I$ is not proper.) In fact, it is known that if $\gamma^+ < \kappa$ then there is no normal proper* κ-ideal on $\mathcal{P}_\kappa\lambda$ with $E^{\kappa,\lambda}_\gamma \in I^+$. See Matsubara-Shelah [11].

In this section we show that it is possible to have a normal semiproper κ-ideal I on $\mathcal{P}_\kappa\lambda$ with $E^{\kappa,\lambda}_\omega \in I^+$. More precisely, using the method developed in Foreman-Magidor-Shelah [2], we show that under some assumption on the ground model, if a supercompact cardinal is Levy collapsed to λ^+ then there is a stationary $S \subseteq E^{\kappa,\lambda}_\omega$ such that $NS_{\kappa,\lambda} \restriction S$ is semiproper.

We begin with a review of the notion of internal approachability which is introduced in Foreman-Magidor-Shelah [2]. We only use the notion of internal approachability of length ω.

Definition 3.2. *A set x is said to be internally approachable of length ω (or in brief I.A. of length ω) if there is a $\subseteq$-increasing sequence $\langle x_n \mid n \in \omega \rangle \subseteq x$ such that $x = \bigcup_{n\in\omega} x_n$. For a regular uncountable cardinal κ and a regular cardinal $\lambda \geq \kappa$, let*

$$IA_\omega^{\kappa,\lambda} := \{ M \in \mathcal{P}_\kappa \mathcal{H}_\lambda \mid M \prec \langle \mathcal{H}_\lambda, \in \rangle \wedge M \textit{ is I.A. of length } \omega \} .$$

Note that $IA_\omega^{\kappa,\lambda}$ is stationary in $\mathcal{P}_\kappa \mathcal{H}_\lambda$. The following lemma is basic:

Lemma 3.3. *Let λ be a regular uncountable cardinal and Δ be a well ordering of $\mathcal{H}_\lambda$. Then the following holds:*

(1) *Suppose $\langle M_n \mid n \in \omega \rangle$ is a sequence such that for each $n \in \omega$, $M_n \prec \langle \mathcal{H}_\lambda, \in \rangle$ and M_n is I.A. of length ω. Then $\bigcup_{n\in\omega} M_n$ is also I.A. of length ω.*
(2) *Let $A \subseteq \mathcal{H}_\lambda$. Assume $x \subseteq \mathcal{H}_\lambda$ is I.A. of length ω and $a \subseteq \mathcal{H}_\lambda$ is a countable set. Then the Skolem hull of $x \cup a$ in $\langle \mathcal{H}_\lambda, \in, \Delta, A \rangle$ is I.A. of length ω.*

Proof. (1) is easy. We show (2). Let A, x and a be as in (2). Let $\langle x_n \mid n \in \omega \rangle \subseteq x$ be such that $\bigcup_{n\in\omega} x_n = x$ and let $\langle a_n \mid n \in \omega \rangle$ be an enumeration of a. Next let $\langle \varphi_n \mid n \in \omega \rangle$ be an enumeration of all formulae and assume that each φ_n has k_n free variables. For the convenience, by adding dummy variables if necessary, assume that $k_n \geq 2$ for each $n \in \omega$. For each $n \in \omega$, let h_n be the Skolem function for φ_n defined as follows. For each $b_1, ..., b_{k_n-1} \in \mathcal{H}_\lambda$, let $h_n(b_1, ..., b_{k_n-1})$ be the Δ-least b such that $\langle \mathcal{H}_\lambda, \in, \Delta, A \rangle \models \varphi_n(b, b_1, ..., b_{k_n-1})$ if such b exists and let $h_n(b_1, ..., b_{k_n-1}) := 0$ otherwise.

Now let N be the Skolem hull of $x \cup a$ in $\langle \mathcal{H}_\lambda, \in, \Delta, A \rangle$. Then

$$N = \bigcup_{n\in\omega} h_n[{}^{k_n-1}(x \cup a)] = \bigcup_{n\in\omega} \bigcup_{m\in\omega} h_n[{}^{k_n-1}(x_m \cup \{a_0, ..., a_{m-1}\})] .$$

For each $n, m \in \omega$, let $y_{n,m} := h_n[{}^{k_n-1}(x_m \cup \{a_0, ..., a_{m-1}\})]$. Then $y_{n,m} \in \mathcal{H}_\lambda$ and $y_{n,m}$ is definable in $\langle \mathcal{H}_\lambda, \in, \Delta, A \rangle$ from parameters $x_m, a_0, ..., a_{m-1}$. Hence $y_{n,m} \in N$ for each $n, m \in \omega$. Therefore if we let $y_l := \bigcup_{n,m<l} y_{n,m}$ for each $l \in \omega$ then $y_l \in N$ and $\bigcup_{l\in\omega} y_l = N$. This implies N is I.A. of length ω. □

Now we show that some stationary reflection principle, which holds if a supercompact cardinal is Levy collapsed to λ^+, implies $NS_{\kappa,\mathcal{H}_\lambda} \restriction IA_\omega^{\kappa,\lambda}$ is ω_1-stationary preserving. In the proof, we use Facts 1.4, 1.5 and 1.6.

Theorem 3.4. *Let κ, λ and θ be regular cardinals such that $\omega_2 \leq \kappa \leq \lambda < 2^{2^{<\lambda}} < \theta$. Assume that for every $\bar{X} \subseteq IA_\omega^{\kappa,\theta}$ which is stationary in $\mathcal{P}_\kappa\mathcal{H}_\theta$, there is a W such that $\mathcal{H}_\lambda \subseteq W$, $|W| = |\mathcal{H}_\lambda|$ and $\bar{X} \cap \mathcal{P}_\kappa W$ is stationary in $\mathcal{P}_\kappa W$.*

Then $NS_{\kappa,\mathcal{H}_\lambda} \restriction IA_\omega^{\kappa,\lambda}$ is ω_1-stationary preserving.

We use the following lemma:

Lemma 3.5. *Let κ, λ and θ be regular cardinals such that $\omega_2 \leq \kappa \leq \lambda < 2^{2^{<\lambda}} < \theta$. Assume that for every $\bar{X} \subseteq IA_\omega^{\kappa,\theta}$ which is stationary in $\mathcal{P}_\kappa\mathcal{H}_\theta$, there is a W such that $\mathcal{H}_\lambda \subseteq W$, $|W| = |\mathcal{H}_\lambda|$ and $\bar{X} \cap \mathcal{P}_\kappa W$ is stationary in $\mathcal{P}_\kappa W$.*

Let $X \subseteq IA_\omega^{\kappa,\lambda}$ be stationary in $\mathcal{P}_\kappa\mathcal{H}_\lambda$ and for each $\beta < \omega_1$, let $l(\beta) \in On$ and let $\langle X_\xi^\beta \mid \xi < l(\beta)\rangle$ be a maximal antichain below X in $\mathbb{P}_{NS_{\kappa,\mathcal{H}_\lambda}}$. (Note that $l(\beta) \leq |\mathcal{P}(\mathcal{P}_\kappa\mathcal{H}_\lambda)| = 2^{2^{<\lambda}}$.) Moreover let $\bar{X} := \{\bar{x} \in \mathcal{P}_\kappa\mathcal{H}_\theta \mid \bar{x} \cap \mathcal{H}_\lambda \in X\}$ and $\bar{X}_\xi^\beta := \{\bar{x} \in \mathcal{P}_\kappa\mathcal{H}_\theta \mid \bar{x} \cap \mathcal{H}_\lambda \in X_\xi^\beta\}$ for each $\xi < l(\beta)$.

Then $\bigcap_{\beta\in\omega_1} \nabla_{\xi<l(\beta)}\bar{X}_\xi^\beta$ is stationary in $\mathcal{P}_\kappa\mathcal{H}_\theta$.

First we show Theorem 3.4 using Lemma 3.5. After that we show Lemma 3.5.

Proof of Theorem. Let I be $NS_{\kappa,\mathcal{H}_\lambda} \restriction IA_\omega^{\kappa,\lambda}$. Fix a well-ordering Δ_θ of $\mathcal{H}_\theta$. To show the theorem, take arbitrary X, $\dot{E}$ and D such that

- $X \subseteq IA_\omega^{\kappa,\lambda}$ and X is stationary in $\mathcal{P}_\kappa\mathcal{H}_\lambda$,
- $\dot{E}$ is a $\mathbb{P}_I$-name such that $X \Vdash$ "$\dot{E}$ is a club subset of ω_1^V",
- D is a stationary subset of ω_1.

We want to show that there are an $\alpha \in D$ and a stationary $Y \subseteq X$ such that $Y \Vdash$ "$\alpha \in \dot{E}$".

For each $\beta < \omega_1$, let $l(\beta)$, $\langle X_\xi^\beta \mid \xi < l(\beta)\rangle$ and $\langle \gamma_\xi^\beta \mid \xi < l(\beta)\rangle$ be such that

- $X_\xi^\beta \subseteq X$ for every $\beta < \omega_1$ and $\xi < l(\beta)$,
- $\langle X_\xi^\beta \mid \xi < l(\beta)\rangle$ is a maximal antichain below X in $\mathbb{P}_I$,
- $\gamma_\xi^\beta < \omega_1$ and $X_\xi^\beta \Vdash$ "$\gamma_\xi^\beta = min(\dot{E} \setminus \beta)$" for each $\xi < l(\beta)$.

Then it suffices to show that there are an $\alpha \in D$ and a stationary $Y \subseteq X$ such that

($\star$) for each $\beta < \alpha$ and each $\xi < l(\beta)$, if $Y \cap X^\beta_\xi$ is stationary in $\mathcal{P}_\kappa\mathcal{H}_\lambda$ then $\gamma^\beta_\xi < \alpha$.

To use Lemma 3.5, let $\bar{X} := \{\bar{x} \in \mathcal{P}_\kappa\mathcal{H}_\theta \mid \bar{x} \cap \mathcal{H}_\lambda \in X\}$ and $\bar{X}^\beta_\xi := \{\bar{x} \in \mathcal{P}_\kappa\mathcal{H}_\theta \mid \bar{x} \cap \mathcal{H}_\lambda \in X^\beta_\xi\}$ for each $\beta < \omega_1$ and $\xi < l(\beta)$. Moreover let $\bar{Y} := \bigcap_{\beta<\omega_1} \nabla_{\xi<l(\beta)} \bar{X}^\beta_\xi$ and let $\bar{Y}^\alpha := \bigcap_{\beta<\alpha} \nabla\langle \bar{X}^\beta_\xi \mid \xi < l(\beta) \wedge \gamma^\beta_\xi < \alpha\rangle$ for each $\alpha < \omega_1$. By Lemma 3.5, $\bar{Y}$ is stationary in $\mathcal{P}_\kappa\mathcal{H}_\theta$. We claim the following:

Claim 3.4.1. *There are club many $\alpha < \omega_1$ such that $\bar{Y}^\alpha \cap IA^{\kappa,\theta}_\omega$ is stationary in $\mathcal{P}_\kappa\mathcal{H}_\theta$.*

Proof of Claim. First we show that $\bar{Y}^\alpha$ is stationary for club many $\alpha < \omega_1$. After that we show that if $\bar{Y}^\alpha$ is stationary then $\bar{Y}^\alpha \cap IA^{\kappa,\theta}_\omega$ is stationary.

Assume that $F := \{\alpha < \omega_1 \mid \bar{Y}^\alpha \text{ is nonstationary in } \mathcal{P}_\kappa\mathcal{H}_\theta\}$ is stationary in ω_1. Then $\bigcup_{\alpha\in F} \bar{Y}^\alpha$ is nonstationary in $\mathcal{P}_\kappa\mathcal{H}_\theta$ because $\kappa > \omega_1$. Hence there is an $\bar{N} \in \bar{Y} \setminus \bigcup_{\alpha\in F} \bar{Y}^\alpha$. For each $\beta < \omega_1$, let $\xi(\beta) < l(\beta)$ be such that $\xi(\beta) \in \bar{N} \in \bar{X}^\beta_{\xi(\beta)}$. Such a $\xi(\beta)$ exists because $\bar{N} \in \bar{Y} \subseteq \nabla_{\xi<l(\beta)}\bar{X}^\beta_\xi$. Now, because F is stationary in ω_1, there is an $\alpha \in F$ such that $\gamma^\beta_{\xi(\beta)} < \alpha$ for each $\beta < \alpha$. Then, by the definition of $\bar{Y}^\alpha$, $\bar{N} \in \bar{Y}^\alpha$. This contradicts $\bar{N} \notin \bigcup_{\alpha\in F} \bar{Y}^\alpha$. Thus there are club many $\alpha < \omega_1$ with $\bar{Y}^\alpha$ stationary.

Next assume that $\bar{Y}^\alpha$ is stationary in $\mathcal{P}_\kappa\mathcal{H}_\theta$. We show that $\bar{Y}^\alpha \cap IA^{\kappa,\theta}_\omega$ is also stationary. It suffices to show that for every $\bar{A} \subseteq \mathcal{H}_\theta$, there is an $\bar{M} \in \bar{Y}^\alpha \cap IA^{\kappa,\theta}_\omega$ such that $\bar{M} \prec \langle \mathcal{H}_\theta, \in, \bar{A}\rangle$ and $\bar{M} \cap \kappa \in \kappa$.

Take an arbitrary $\bar{A} \subseteq \mathcal{H}_\theta$. Because $\bar{Y}^\alpha$ is stationary we can take an $\bar{N} \in \bar{Y}^\alpha$ such that $\bar{N} \prec \langle \mathcal{H}_\theta, \in, \Delta_\theta, \bar{A}\rangle$ and $\bar{N} \cap \kappa \in \kappa$. Because $\bar{Y}^\alpha \subseteq \bar{X}$ note that $\bar{N} \cap \mathcal{H}_\lambda \in X \subseteq IA^{\kappa,\lambda}_\omega$. For each $\beta < \alpha$, let $\xi(\beta) < l(\beta)$ be such that $\xi(\beta) \in \bar{N} \in \bar{X}^\beta_{\xi(\beta)}$ and $\gamma^\beta_{\xi(\beta)} < \alpha$. Now let $\bar{M}$ be the Skolem hull of $(\bar{N} \cap \mathcal{H}_\lambda) \cup \{\xi(\beta) \mid \beta < \alpha\}$ in $\langle \mathcal{H}_\theta, \in, \Delta_\theta, \bar{A}\rangle$. We show $\bar{M}$ is the one desired. First note that $\bar{M} \cap \mathcal{H}_\lambda = \bar{N} \cap \mathcal{H}_\lambda$ because $\bar{N} \cap \mathcal{H}_\lambda \subseteq \bar{M} \subseteq \bar{N}$. Hence $\xi(\beta) \in \bar{M} \in \bar{X}^\beta_\xi$ for each $\beta < \alpha$ and so $\bar{M} \in \bar{Y}^\alpha$. Moreover $\bar{M} \cap \kappa = \bar{N} \cap \kappa \in \kappa$. Next recall that $\bar{N} \cap \mathcal{H}_\lambda \in IA^{\kappa,\lambda}_\omega$. Hence $\bar{M} \in IA^{\kappa,\lambda}_\omega$ by Lemma 3.3 (2). Finally it is clear that $\bar{M} \prec \langle \mathcal{H}_\theta, \in, \bar{A}\rangle$. $\square$.*Claim*

We return to the proof of the theorem. We must show that there is an $\alpha \in D$ and a stationary $Y \subseteq X$ satisfying ($\star$).

By Claim 3.4.1, there is an $\alpha \in D$ such that $\bar{Y}^\alpha \cap IA^{\kappa,\theta}_\omega$ is stationary. Then by the assumption of the lemma, there is a W such that $\mathcal{H}_\lambda \subseteq W$, $|W| = |\mathcal{H}_\lambda|$ and $\bar{Y}^\alpha \cap \mathcal{P}_\kappa W$ is stationary in $\mathcal{P}_\kappa W$. Let $\pi : W \to \mathcal{H}_\lambda$ be a bijection and let $Y := \{\bar{M} \cap \mathcal{H}_\lambda \mid \bar{M} \in \bar{Y}^\alpha \cap \mathcal{P}_\kappa W \wedge \bar{M}$ is closed under $\pi, \pi^{-1}\}$. We show that α and Y satisfy $(\star)$.

Assume $\beta < \alpha$, $\xi < l(\beta)$ and $Y \cap X^\beta_\xi$ is stationary. Note that $Y \cap X^\beta_\xi = \{\bar{M} \cap \mathcal{H}_\lambda \mid \bar{M} \in \bar{Y}^\alpha \cap \mathcal{P}_\kappa W \cap \bar{X}^\beta_\xi \wedge \bar{M}$ is closed under $\pi, \pi^{-1}\}$. Hence $\bar{Y}^\alpha \cap \mathcal{P}_\kappa W \cap \bar{X}^\beta_\xi = (\bar{Y}^\alpha \cap \mathcal{P}_\kappa W) \cap (\bar{X}^\beta_\xi \cap \mathcal{P}_\kappa W)$ is stationary in $\mathcal{P}_\kappa W$ by Fact 1.6 (1). Here note that $\bar{Y}^\alpha \cap \mathcal{P}_\kappa W = \nabla\langle \bar{X}^\beta_\eta \cap \mathcal{P}_\kappa W \mid \eta \in l(\beta) \cap W \wedge \gamma^\beta_\eta < \alpha\rangle$. Therefore $\gamma^\beta_\xi < \alpha$. This shows that α and Y satisfy $(\star)$. □.*Theorem*

We must prove Lemma 3.5. Lemma 3.5 can be shown by the catching antichain argument due to Foreman, Magidor and Shelah [2]. The point is that by restricting to $IA^{\kappa,\lambda}_\omega$, we can iterate catching antichain procedures for ω_1 times.

Proof of Lemma. Let X, $\langle X^\beta_\xi \mid \xi < l(\beta)\rangle$, $\bar{X}$, $\langle \bar{X}^\beta_\xi \mid \xi < l(\beta)\rangle$ be as in Lemma 3.5 and fix a well-ordering Δ_θ of $\mathcal{H}_\theta$. We may assume $X^\beta_\xi \subseteq X$ for each $\beta < \omega_1$ and $\xi < l(\beta)$.

Claim 3.5.2. *$\bar{X} \cap IA^{\kappa,\theta}_\omega$ is stationary in $\mathcal{P}_\kappa \mathcal{H}_\theta$. $\bar{X}^\beta_\xi \cap IA^{\kappa,\theta}_\omega$ is also stationary for every $\beta < \omega_1$ and $\xi < l(\beta)$.*

Proof of Claim. We only show the former. The proof of the latter is the same. It suffices to show that for every $\bar{A} \subseteq \mathcal{H}_\theta$, there is an $\bar{M} \in \bar{X} \cap IA^{\kappa,\lambda}_\omega$ such that $\bar{M} \prec \langle \mathcal{H}_\theta, \in, \bar{A}\rangle$ and $\bar{M} \cap \kappa \in \kappa$.

Take an arbitrary $\bar{A} \subseteq \mathcal{H}_\theta$. Then, because $\bar{X}$ is stationary, there is an $\bar{N} \in \bar{X}$ such that $\bar{N} \prec \langle \mathcal{H}_\theta, \in, \Delta_\theta, \bar{A}\rangle$ and $\bar{N} \cap \kappa \in \kappa$. Let $\bar{M}$ be the Skolem hull of $\bar{N} \cap \mathcal{H}_\lambda$ in $\langle \mathcal{H}_\theta, \in, \Delta_\theta, \bar{A}\rangle$. We show that $\bar{M}$ satisfies the above properties. First note that $\bar{N} \cap \mathcal{H}_\lambda \subseteq \bar{M} \subseteq \bar{N}$ and so $\bar{M} \cap \mathcal{H}_\lambda = \bar{N} \cap \mathcal{H}_\lambda$ Therefore $\bar{M} \cap \kappa \in \kappa$ and $\bar{M} \in \bar{X}$. Moreover, because $\bar{N} \cap \mathcal{H}_\lambda \in X \subseteq IA^{\kappa,\lambda}_\omega$, $\bar{M} \in IA^{\kappa,\theta}_\omega$ by Lemma 3.3 (2). Finally it is clear that $\bar{M} \prec \langle \mathcal{H}_\theta, \in, \bar{A}\rangle$.

□.*Claim*

The following is the key claim.

Claim 3.5.3. *There is a club $\bar{C} \subseteq \mathcal{P}_\kappa \mathcal{H}_\theta$ such that for every $\beta < \omega_1$ and every $\bar{M} \in \bar{C} \cap \bar{X} \cap IA^{\kappa,\theta}_\omega$, there is an $\bar{N} \in \mathcal{P}_\kappa \mathcal{H}_\theta$ with the following properties.*

(1) $\bar{M} \subseteq \bar{N} \prec \langle \mathcal{H}_\theta, \in, \Delta_\theta\rangle$.
(2) $\bar{M} \cap \mathcal{H}_\lambda = \bar{N} \cap \mathcal{H}_\lambda$.
(3) $\bar{N} \in \nabla_{\xi < l(\beta)} \bar{X}^\beta_\xi$.

Proof of Claim. Assume not. Then, because $\kappa \geq \omega_2$ and $\bar{X} \cap IA^{\kappa,\theta}_\omega$ is stationary, there is a $\beta < \omega_1$ and a stationary $\bar{S} \subseteq \bar{X} \cap IA^{\kappa,\theta}_\omega$ such that for every $\bar{M} \in \bar{S}$ there is no $\bar{N}$ satisfying (1)-(3). Then, by the assumption of the lemma, there is a W such that $\mathcal{H}_\lambda \subseteq W$, $|W| = |\mathcal{H}_\lambda|$ and $\bar{S} \cap \mathcal{P}_\kappa W$ is stationary in $\mathcal{P}_\kappa W$.

Recall that $\{X^\beta_\xi \mid \xi \in l(\beta)\}$ is predense below X in $\mathbb{P}_{NS_{\kappa,\mathcal{H}_\lambda}}$. So $\{\bar{X}^\beta_\xi \cap \mathcal{P}_\kappa W \mid \xi < l(\beta)\}$ is predense below $\bar{X} \cap \mathcal{P}_\kappa W$ in $\mathbb{P}_{NS_{\kappa,W}}$ by Fact 1.6 (3). Moreover $\bar{S} \cap \mathcal{P}_\kappa W \subseteq \bar{X} \cap \mathcal{P}_\kappa W$ is stationary. Hence there is a $\xi < l(\beta)$ such that $\bar{X}^\beta_\xi \cap \bar{S} \cap \mathcal{P}_\kappa W$ is stationary in $\mathcal{P}_\kappa W$.

Now we can take an $\bar{N} \in \mathcal{P}_\kappa \mathcal{H}_\theta$ such that $\xi \in \bar{N} \prec \langle \mathcal{H}_\theta, \in, \Delta_\theta \rangle$ and $\bar{N} \cap W \in \bar{X}^\beta_\xi \cap \bar{S}$. Let $\bar{M} := \bar{N} \cap W$. Then $\bar{M} \in \bar{S}$ and clearly $\bar{N}$ satisfies (1) and (2) for this $\bar{M}$. Moreover, by the choice of $\bar{N}$, $\xi \in \bar{N} \in \bar{X}^\beta_\xi$. Hence $\bar{N}$ satisfies (3). This is a contradiction. $\square$.*Claim*

We return to the proof of the lemma. Let $\bar{Y} := \bigcap_{\beta<\omega_1} \nabla_{\xi<l(\beta)} \bar{X}^\beta_\xi$.

Take an arbitrary club $\bar{C} \subseteq \mathcal{P}_\kappa \mathcal{H}_\theta$. We show that $\bar{C} \cap \bar{Y} \neq \emptyset$. By reducing $\bar{C}$ if necessary, we may assume that $\bar{C}$ witnesses Claim 3.5.3 and there is an $\bar{A} \subseteq \mathcal{H}_\theta$ such that $\bar{C} = \{\bar{M} \in \mathcal{P}_\kappa \mathcal{H}_\theta \mid \bar{M} \prec \langle \mathcal{H}_\theta, \in, \Delta_\theta, \bar{A} \rangle \wedge \bar{M} \cap \kappa \in \kappa\}$.

By induction on $\beta < \omega_1$, we define $\subseteq$-increasing $\langle \bar{M}_\beta \mid \beta < \omega_1 \rangle$ so that for each $\beta \in \omega_1$,

(a) $\bar{M}_\beta \in \bar{C} \cap \bar{X} \cap IA^{\kappa,\theta}_\omega$,
(b) $\bar{M}_\beta \cap \mathcal{H}_\lambda = \bar{M}_0 \cap \mathcal{H}_\lambda$,
(c) $\bar{M}_{\beta+1} \in \nabla_{\xi<l(\beta)} \bar{X}^\beta_\xi$.

First let $\bar{M}_0$ be an element of $\bar{C} \cap \bar{X} \cap IA^{\kappa,\theta}_\omega$ with $2^{2^{<\lambda}} \in \bar{M}_0$. Next assume $\beta < \omega_1$ is limit and $\bar{M}_{\beta'}$ has been defined to satisfy (a)-(c) for each $\beta' < \beta$. Then let $\bar{M}_\beta := \bigcup_{\beta'<\beta} \bar{M}_{\beta'}$. Note that $\bar{M}_\beta \in IA^{\kappa,\theta}_\omega$ by 3.3 (2). It is clear that $\bar{M}_\beta$ satisfies other conditions in (a)-(c).

Finally assume $\beta < \omega_1$ and $\bar{M}_\beta$ has been defined to satisfy (a)-(c). Then because $\bar{M}_\beta \in \bar{C} \cap \bar{X} \cap IA^{\kappa,\theta}_\omega$ there is an $\bar{N} \in \mathcal{P}_\kappa \mathcal{H}_\theta$ satisfying (1)-(3) of Claim 3.5.3 for β and $\bar{M}_\beta$. Let $\xi < l(\beta)$ be the one witnessing (3), i.e. be such that $\xi \in \bar{N} \in \bar{X}^\beta_\xi$. Now let $\bar{M}_{\beta+1} := \{f(\xi) \mid f : 2^{2^{<\lambda}} \to \mathcal{H}_\theta \wedge f \in \bar{M}_\beta\}$. We must check $\bar{M}_{\beta+1}$ satisfies (a)-(c). Note that

(i) $\bar{M}_{\beta+1}$ is the Skolem hull of $\bar{M}_\beta \cup \{\xi\}$ in $\langle \mathcal{H}_\theta, \in, \Delta_\theta, \bar{A} \rangle$,
(ii) $\bar{M}_\beta \cap \mathcal{H}_\lambda = \bar{M}_{\beta+1} \cap \mathcal{H}_\lambda = \bar{N} \cap \mathcal{H}_\lambda$.

For (ii), note that $\bar{M}_\beta \subseteq \bar{M}_{\beta+1} \subseteq \bar{N}$ and $\bar{M}_\beta \cap \mathcal{H}_\lambda = \bar{N} \cap \mathcal{H}_\lambda$.

(a): It is easy to check $\bar{M}_{\beta+1} \in \bar{C} \cap \bar{X}$. Because $\bar{M}_\beta \in IA^{\kappa,\theta}_\omega$, $\bar{M}_{\beta+1} \in IA^{\kappa,\theta}_\omega$ by (2) and Lemma 3.3. (b): Clear from (ii). (c): By (ii) and the fact that

$\bar{N} \in \bar{X}^{\beta}_{\xi}$, $\bar{M}_{\beta+1} \in \bar{X}^{\beta}_{\xi}$. Then $\xi \in \bar{M}_{\beta+1} \in \bar{X}^{\beta}_{\xi}$ and so $\bar{M}_{\beta+1}$ satisfies (c). This completes the inductive definition of $\langle \bar{M}_\beta \mid \beta < \omega_1 \rangle$.

Now let $\bar{M}^* := \bigcup_{\beta<\omega_1} \bar{M}_\beta$. Then clearly $\bar{M}^* \in \bar{C}$. Moreover $\bar{M}^* \in \nabla_{\xi<l(\beta)} \bar{X}^{\beta}_{\xi}$ for each $\beta < \omega_1$ because $\bar{M}_{\beta+1} \cap \mathcal{H}_\lambda = \bar{M}^* \cap \mathcal{H}_\lambda$ and $\bar{M}_{\beta+1} \in \nabla_{\xi<l(\beta)} \bar{X}^{\beta}_{\xi}$. Hence $\bar{M}^* \in \bar{Y}$. Therefore $\bar{M}^* \in \bar{C} \cap \bar{Y} \neq \emptyset$. $\square$.*Lemma*

Corollary 3.6. *Assume κ and λ are regular cardinals such that $\omega_2 \leq \kappa \leq \lambda$ and δ is a supercompact cardinal $> \lambda$. Then the following hold in $V^{Col(\lambda,<\delta)}$:*

1. *$NS_{\kappa,\mathcal{H}_\lambda} \restriction IA^{\kappa,\lambda}_{\omega}$ is ω_1-stationary preserving.*
2. *There is a stationary $S \subseteq E^{\kappa,\lambda}_{\omega}$ such that $NS_{\kappa,\lambda} \restriction S$ is ω_1-stationary preserving.*
3. *If $\kappa = \lambda$ then $NS_\kappa \restriction E^{\kappa}_{\omega}$ is ω_1-stationary preserving.*

Proof. (1). In $V^{Col(\lambda,<\delta)}$, the assumption of Theorem 3.4 holds. Hence (1) holds.

(2). First note that in $V^{Col(\lambda,<\delta)}$, $|\mathcal{H}_\lambda| = \lambda$. Let $\pi : \mathcal{H}_\lambda \to \lambda$ be a bijection and let $S := \{M \cap \lambda \mid M \in IA^{\kappa,\lambda}_{\omega},\ M \cap \kappa \in \kappa,\ M \text{ is closed under } \pi, \pi^{-1}\}$. Then by (1) and Fact 1.6 (2), $NS_{\kappa,\lambda} \restriction S$ is ω_1-preserving. Moreover it is easy to see $S \subseteq E^{\kappa,\lambda}_{\omega}$.

(3). Assume $\kappa = \lambda$. Let S be as in the proof of (2). Then $S \subseteq E^{\kappa}_{\omega}$ and $NS_\kappa \restriction S$ is ω_1-stationary preserving. We show that $E^{\kappa}_{\omega} \setminus S$ is nonstationary. First note that if $M \prec \langle \mathcal{H}_\kappa, \in, \pi \rangle$ and $M \cap \kappa \in E^{\kappa}_{\omega}$ then M is the Skolem hull of $M \cap \kappa$ in $\langle \mathcal{H}_\kappa, \in, \pi \rangle$ and so $M \in IA^{\kappa,\kappa}_{\omega}$ by Lemma 3.3. Hence $S \supseteq \{M \cap \kappa \mid M \prec \langle \mathcal{H}_\kappa, \in, \pi \rangle \ \wedge\ M \cap \kappa \in E^{\kappa}_{\omega}\}$. Therefore $E^{\kappa}_{\omega} \setminus S$ is nonstationary. $\square$

Note that ideals in Corollary 3.6 are not necessarily semiproper. If $\square_\gamma$ fails in V for some γ with $\kappa \leq \gamma^+ \leq \lambda$ then $\square_\gamma$ still fails in $V^{Col(\lambda,<\delta)}$. Therefore, by Corollary 2.12, there is no semiproper κ-ideal on $\mathcal{P}_\kappa\lambda$ in $V^{Col(\lambda,<\delta)}$.

On the other hand, recall that $MA^+(\sigma\text{-}closed)$ is preserved by ω_2-closed poset and $MA^+(\sigma\text{-}closed)$ implies (†). For the former see König-Yoshinobu [7] and for the latter see Foreman-Magidor-Shelah [2].

Corollary 3.7. *Let κ and λ be regular cardinals such that $\omega_2 \leq \kappa \leq \lambda$ and assume δ is a supercompact cardinal $> \lambda$. Moreover assume $MA^+(\sigma\text{-}closed)$ holds in V. Then in $V^{Col(\lambda,<\delta)}$ the following hold:*

1. *$NS_{\kappa,\mathcal{H}_\lambda} \restriction IA^{\kappa,\lambda}_{\omega}$ is semiproper.*
2. *There is a stationary $S \subseteq E^{\kappa,\lambda}_{\omega}$ such that $NS_{\kappa,\lambda} \restriction S$ is semiproper.*
3. *If $\kappa = \lambda$ then $NS_\kappa \restriction E^{\kappa}_{\omega}$ is semiproper.*

We end this paper with two remarks.

First, using the method developed in Goldring [4], we can replace "a supercompact cardinal" in Corollary 3.6 and 3.7 by "a Woodin cardinal".

Next it is shown in Larson [8] that MM is preserved by ω_2-directed closed posets. Hence, in the theorem above, if we assume MM holds in V then it still holds in $V^{Col(\lambda,<\delta)}$ and therefore we can apply MM for the corresponding posets of ideals in Corollary 3.7.

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