Noname manuscript No.
(will be inserted by the editor)

Simple proofs of SCH from reflection principles without using better scales

Hiroshi Sakai

Received: date / Accepted: date

Abstract We give simple proofs of the Singular Cardinal Hypothesis from the Weak Reflection Principle and the Fodor-type Reflection Principle which do not use better scales.

Keywords Singular Cardinal Hypothesis · Weak Reflection Principle · Fodor-type Reflection Principle

Mathematics Subject Classification (2000) 03E55 · 03E65

1 Introduction

The Singular Cardinal Hypothesis (SCH) below is a restriction of the Generalized Continuum Hypothesis to singular cardinals and has been studied extensively by many set theorists:

$$\mathsf{SCH} \;\equiv\; \lambda^{\mathrm{cf}(\lambda)} = \lambda^+ \text{ for every singular cardinal } \lambda \text{ with } 2^{\mathrm{cf}(\lambda)} < \lambda.$$

It is known that compactness properties tend to imply SCH. First it was proved by Solovay [14] that if κ is a strongly compact cardinal, then SCH holds above κ. Strong forcing axioms also imply SCH. Foreman-Magidor-Shelah [4] showed that Martin's Maximum (MM) implies SCH, and Viale [15] showed that so does the Proper Forcing Axiom (PFA). Moreover several reflection principles, which follow from these forcing axioms, are also known to imply SCH. For example, the Mapping Reflection Principle (MRP), the Weak Reflection Principles (WRP) and the Fodor-type Reflection Principle (FRP) were shown to imply SCH by Viale [15], Shelah [11] and Fuchino-Rinot [6], respectively.

This research was supported by JSPS and FWF under the Japan - Austria Research Cooperative Program "New Developments regarding Forcing in Set Theory".

H. Sakai
Graduate School of System Informatics, Kobe University
1-1 Rokkodai-cho, Nada, Kobe 657-8501, Japan
E-mail: hsakai@people.kobe-u.ac.jp

In this paper we present new proofs of SCH from WRP and FRP. First we briefly review these reflection principles:

WRP is the assertion that WRP(λ) below holds for every cardinal $\lambda \geq \omega_2$:

WRP(λ) $\equiv$ For any stationary $X \subseteq [\lambda]^\omega$ there is $R \in [\lambda]^{\omega_1}$ such that $\omega_1 \subseteq R$ and $X \cap [R]^\omega$ is stationary in $[R]^\omega$.

It was proved in [4] that WRP follows from MM. Moreover, besides SCH, WRP has many interesting consequences such as Chang's Conjecture ([4]), the presaturation of NS_{ω_1} ([4] and Feng-Magidor [3]) and $2^\omega \leq \omega_2$ (Todorčević).

Next we recall FRP. For regular cardinals λ and μ with $\mu < \lambda$ let

$$E^\lambda_\mu := \{\alpha < \lambda \mid \mathrm{cf}(\alpha) = \mu\} .$$

FRP is the assertion that FRP(λ) below holds for every regular cardinal $\lambda \geq \omega_2$:

FRP(λ) $\equiv$ For any stationary $S \subseteq E^\lambda_\omega$ and any sequence $\langle b_\alpha \mid \alpha \in S\rangle$ with $b_\alpha \in [\alpha]^\omega$, there is $\gamma \in E^\lambda_{\omega_1}$ such that for any function f on $S \cap \gamma$ with $f(\alpha) \in b_\alpha$ there is β with $f^{-1}[\{\beta\}]$ stationary in γ.

FRP also follows from MM and is known to have equivalent reflection principles in terms of topological spaces, infinite graphs and boolean algebras. For details, see Fuchino-Juhász-Soukup-Szentmiklóossy-Usuba [5], Fuchino-Rinot [6] and Fuchino-Soukup-Sakai-Usuba [7].

Now we turn our attention to the original proofs of SCH from WRP and FRP in [11] and [6]: Both proofs use the following theorem stating that the failure of SCH implies the existence of a better scale:

Theorem 1.1 (Shelah [9] Ch. II Claim 1.3 & Ch. IX Conclusion 5.10)**.** *Assume* SCH *fails, and let λ be the least singular cardinal at which* SCH *fails. (So $\mathrm{cf}(\lambda) = \omega$ by Silver's theorem [13].) Then we have the following:*

($\star$) *There is an increasing sequence $\langle \lambda_n \mid n < \omega\rangle$ of regular cardinals converging to λ such that $\langle \Pi_{n<\omega}\lambda_n, <^*\rangle$ has a better scale of length λ^+.*

In fact, what was proved in [11] and [6] is that ($\star$) implies the failure of WRP(λ^+) and FRP(λ^+).

Theorem 1.1 is a profound theorem in PCF Theory and is quite useful. But its proof is long and complicated. In this paper we present proofs of SCH from WRP and FRP without using better scales. In our proofs, Theorem 1.1 is replaced with some easy lemmata, and, as for WRP, the rest of the proof is simpler than the original one. Our proofs use some ideas of the proof by [15] of the fact that MRP implies SCH.

There is another motivation to give a proof of SCH from WRP without using better scales. This is relevant to the Tree Property TP(κ, λ) and the Ineffable Tree Property ITP(κ, λ) on $\mathcal{P}_\kappa(\lambda)$ introduced by Weiß [18].

First we briefly review known facts on these principles: ITP(κ, λ) is stronger than TP(κ, λ). Let TP(κ) or ITP(κ) denotes the statement that TP(κ, λ) or ITP(κ, λ) holds for every $\lambda \geq \kappa$, respectively. Then TP(κ) and ITP(κ) characterize the strong compactness and the supercompactness for an inaccessible κ.

Namely, an inaccessible cardinal κ is strongly compact or supercompact if and only if $\mathsf{TP}(\kappa)$ or $\mathsf{ITP}(\kappa)$ holds, respectively. On the other hand, [18] proved that $\mathsf{TP}(\kappa)$ and $\mathsf{ITP}(\kappa)$ for small cardinals κ are also consistent. In particular, the case when $\kappa = \omega_2$ is interesting. It was proved in [18] that PFA implies $\mathsf{ITP}(\omega_2)$, and Sakai-Veličković [8] proved that $\mathsf{WRP} + \mathsf{MA}_{\omega_1}$ implies $\mathsf{ITP}(\omega_2)$. These results show that ω_2 is similar to a supercompact cardinal under PFA or $\mathsf{WRP} + \mathsf{MA}_{\omega_1}$. Moreover a result in Viale-Weiß [16] show that we need a supercompact cardinal to force PFA or $\mathsf{WRP} + \mathsf{MA}_{\omega_1}$ by standard forcing notions.

In consideration of the fact that SCH holds above a strongly compact cardinal, it is natural to conjecture that $\mathsf{TP}(\kappa)$ and $\mathsf{ITP}(\kappa)$ imply SCH above κ. But this is still open. Here note that $\mathsf{ITP}(\omega_2)$ is consistent with $(\star)$ for every singular cardinal λ of cofinality ω. This follows from the well-known facts below, due to Magidor and Cummings-Foreman-Magidor [4] respectively, together with the above mentioned fact that PFA implies $\mathsf{ITP}(\omega_2)$:

- PFA is consistent with $\square_{\lambda,\omega_2}$ for every uncountable cardinal λ.
- $\square_{\lambda,\omega_2}$ implies $(\star)$ for any singular cardinal λ of cofinality ω.

So we cannot prove the above conjecture by way of $(\star)$ as in the original proofs of SCH from WRP and FRP. Then, because $\mathsf{ITP}(\omega_2)$ follows from $\mathsf{WRP}+\mathsf{MA}_{\omega_1}$, it is natural to ask whether we can prove SCH from WRP $(+\mathsf{MA}_{\omega_1})$ without using better scales. We hope that our new proof has some contribution to the solution of the above conjecture.

We give proofs of SCH from FRP and WRP in Section 2 and 3, respectively. By modifying a proof from WRP slightly, we can also prove that the Semi-stationary Reflection Principle (SSR) implies SCH, which was originally proved in [8] using better scales. We give an outline of our proof in Section 4.

We will use Silver's theorem in all of our proofs. Let us recall it before proceeding to our proofs:

Theorem 1.2 (Silver [13]). *Assume* SCH *fails, and let λ be the least singular cardinal at which* SCH *fails. Then* $\mathrm{cf}(\lambda) = \omega$.

2 FRP

Here we give a proof of the following without using better scales:

Theorem 2.1 (Fuchino-Rinot [6]). FRP *implies* SCH.

We prove the contraposition. Assume SCH fails, and let λ be the least singular cardinal at which SCH fails. We prove that $\mathsf{FRP}(\lambda^+)$ fails. Note that $\mathrm{cf}(\lambda) = \omega$ by Theorem 1.2. Note also that $\mu^\omega < \lambda$ for all $\mu < \lambda$. In fact, by induction on cardinals μ with $2^\omega \leq \mu < \lambda$, we can easily prove that $\mu^\omega = \mu$ if $\mathrm{cf}(\mu) > \omega$ and that $\mu^\omega = \mu^+$ if $\mathrm{cf}(\mu) = \omega$.

In [5] it was proved that the following combinatorial principle on an almost disjoint sequence implies the failure of $\mathsf{FRP}(\lambda^+)$:

$\mathsf{ADS}^-(\lambda)$ $\equiv$ There is a sequence $\langle b_\alpha \mid \alpha < \lambda^+ \rangle$ in $[\lambda]^\omega$ such that for any $\gamma < \lambda^+$ of uncountable cofinality there is a club $c \subseteq \gamma$ and a function g on c with the following properties:
(i) $g(\alpha) \in [b_\alpha]^{<\omega}$ for each $\alpha \in c$.
(ii) $\langle b_\alpha \setminus g(\alpha) \mid \alpha \in c \rangle$ is pairwise disjoint.

Lemma 2.2 (Fuchino et al. [5])**.** *If* $\mathsf{ADS}^-(\lambda)$ *holds, then* $\mathsf{FRP}(\lambda^+)$ *fails.*

Proof. Suppose that $\langle b_\alpha \mid \alpha < \lambda^+ \rangle$ witnesses $\mathsf{ADS}^-(\lambda)$. We show that $\mathsf{FRP}(\lambda^+)$ fails for $S := E^{\lambda^+}_\omega \setminus \lambda$ and $\langle b_\alpha \mid \alpha \in S \rangle$. For this it suffices to show that for any $\gamma \in E^{\lambda^+}_{\omega_1} \setminus \lambda$ there is a club $c \subseteq \gamma$ and an injection f on c with $f(\alpha) \in b_\alpha$. Suppose that $\gamma \in E^{\lambda^+}_{\omega_1}$. Then we can take a club $c \subseteq \gamma$ and a function g on c satisfying (i) and (ii) of $\mathsf{ADS}^-(\lambda)$. Let f be a function on c such that $f(\alpha) \in b_\alpha \setminus g(\alpha)$. Then c and f are as desired. □

So it suffices to prove the following:

Lemma 2.3. $\mathsf{ADS}^-(\lambda)$ *holds.*

For this we use the following lemma:

Lemma 2.4. *For any* $\mathcal{A} \subseteq [\lambda]^{<\lambda}$ *with* $|\mathcal{A}| \leq \lambda^+$ *there is* $b \in [\lambda]^\omega$ *such that* $b \cap A$ *is finite for any* $A \in \mathcal{A}$*.*

Proof. First take a bijection $h : \lambda \to [\lambda]^{<\omega}$. By increasing if necessary, we may assume that each element of $\mathcal{A}$ is closed under h.

Note that $|\bigcup_{A \in \mathcal{A}} [A]^\omega| \leq \lambda^+ < \lambda^\omega$ because $\mu^\omega < \lambda$ for all $\mu < \lambda$. Then we can take $b' \in [\lambda]^\omega$ such that $b' \not\subseteq A$ for any $A \in \mathcal{A}$. Let $\langle \beta'_n \mid n < \omega \rangle$ be an enumeration of b', and for each $n < \omega$ take $\beta_n < \lambda$ with $h(\beta_n) = \{\beta'_m \mid m < n\}$. Then $b := \{\beta_n \mid n < \omega\}$ is as desired: Take an arbitrary $A \in \mathcal{A}$. Then there is $m < \omega$ with $\beta'_m \notin A$. Then, because A is closed under h, it holds that $\beta_n \notin A$ for any $n > m$. □

Proof of Lemma 2.3. Let E be the set of all $\gamma < \lambda^+$ of uncountable cofinality, and fix a club $c_\gamma \subseteq \gamma$ of order type $\mathrm{cf}(\gamma)$ for each $\gamma \in E$. By induction on $\alpha < \lambda^+$, take $b_\alpha \in [\lambda]^\omega$ as follows: Suppose that b_β has been taken for each $\beta < \alpha$. For each $\gamma \in E$ let

$$A^\alpha_\gamma := \bigcup \{ b_\beta \mid \beta \in c_\gamma \cap \alpha \} \ .$$

By Lemma 2.4 let $b_\alpha \in [\lambda]^\omega$ be such that $b_\alpha \cap A^\alpha_\gamma$ is finite for any $\gamma \in E$.

Now it is easy to see that $\langle b_\alpha \mid \alpha < \lambda^+ \rangle$ witnesses $\mathsf{ADS}^-(\lambda)$: Suppose that $\gamma \in E$. Recall that c_γ is club in γ. For each $\alpha \in c_\gamma$ let $g(\alpha) := b_\alpha \cap A^\alpha_\gamma$. Then $g(\alpha) \in [b_\alpha]^{<\omega}$ for each $\alpha \in c_\gamma$, and $\langle b_\alpha \setminus g(\alpha) \mid \alpha \in c_\gamma \rangle$ is pairwise disjoint. □

3 WRP

In this section we give a simple proof of the following without using better scales:

Theorem 3.1 (Shelah [11]). WRP *implies* SCH.

As in the proof of Theorem 2.1 and the original proof of Theorem 3.1 in [11], we prove the contraposition. Assume that SCH fails, and let λ be the least singular cardinal at which SCH fails. We construct a non-reflecting stationary $X^* \subseteq [\lambda^+]^\omega$. Note that $\mathrm{cf}(\lambda) = \omega$ by Theorem 1.2.

First we present X^*: Fix a function $h : E^{\lambda^+}_\omega \to \omega_1$ such that $h^{-1}[\{\rho\}]$ is stationary in λ^+ for all $\rho < \omega_1$. Moreover, for each $\gamma \in E^{\lambda^+}_{\omega_1}$ fix a partition $\langle A_{\gamma,n} \mid n < \omega \rangle$ of γ such that $|A_{\gamma,n}| < \lambda$. Then let

X^* := the set of all $x \in [\lambda^+]^\omega$ such that
(I) $\sup(x) \notin x$,
(II) $x \cap \omega_1 \subseteq h(\sup(x))$,
(III) $x \cap A_{\gamma,n}$ is bounded in x for any $\gamma \in E^{\lambda^+}_{\omega_1}$ and any $n < \omega$.

We claim the following:

Lemma 3.2. X^* *is stationary in* $[\lambda^+]^\omega$.

Lemma 3.3. X^* *is non-reflecting, that is,* $X^* \cap [R]^\omega$ *is non-stationary for any* $R \in [\lambda^+]^{\omega_1}$ *with* $\omega_1 \subseteq R$.

Before proving these lemmata, we mention the roles of the properties (I)–(III) in the definition of X^* and the difference between our proof and the original proof in [11]: First note that there are three types of elements R of $[\lambda^+]^{\omega_1}$: $\sup(R) \in R$, $\mathrm{cf}(\sup(R)) = \omega$ or $\mathrm{cf}(\sup(R)) = \omega_1$. X^* does not reflect to R with $\sup(R) \in R$ by (I). Moreover, as we will show below, (II) and (III) assure that X^* does not reflect to R with $\mathrm{cf}(\sup(R)) = \omega$ and $\mathrm{cf}(\sup(R)) = \omega_1$, respectively. [11] also constructed a non-reflecting stationary subset, say Y^*, of $[\lambda^+]^\omega$ for the same λ as ours, and (II) was used there for the same purpose. A similar property as (II) can be found also in Shelah-Shioya [12]. What is new in our proof is (III). In [11], another property employing a better scale was used to assure that Y^* does not reflect to R with $\mathrm{cf}(\sup(R)) = \omega_1$. (III) replaces it.

Now we start to prove Lemmata 3.2 and 3.3. First we prove Lemma 3.3:

Proof of Lemma 3.3. Take an arbitrary $R \in [\lambda^+]^{\omega_1}$ with $\omega_1 \subseteq R$. If $\sup(R) \in R$, then clearly $X^* \cap [R]^\omega$ is non-stationary by (I) in the definition of X^*. So assume that $\sup(R) \notin R$. Then $\mathrm{cf}(\sup(R))$ is ω or ω_1.

First suppose that $\mathrm{cf}(\sup(R)) = \omega$. Let Y_0 be the set of all $x \in [R]^\omega$ with $\sup(x) = \sup(R)$ and $x \cap \omega_1 \not\subseteq h(\sup(R))$. Then Y_0 is club in $[R]^\omega$, and $X^* \cap Y_0 = \emptyset$ by (II).

Next suppose that $\mathrm{cf}(\sup(R)) = \omega_1$. Let $\gamma := \mathrm{cf}(\sup(R))$. Then we can take $n < \omega$ such that $A_{\gamma,n} \cap R$ is unbounded in R. Let Y_1 be the set of all $x \in [R]^\omega$ such that $x \cap A_{\gamma,n}$ is unbounded in x. Then Y_1 is club in $[R]^\omega$, and $X^* \cap Y_1 = \emptyset$ by (III). □

Next we prove Lemma 3.2. For this we need some preparations. The first one is a variant of Lemma 2.4. We call a sequence $\langle I_\xi \mid \xi < \lambda^+ \rangle$ an *interval partition* of λ^+ if $I_\xi = \delta_{\xi+1} \setminus \delta_\xi$ for some increasing continuous cofinal sequence $\langle \delta_\xi \mid \xi < \lambda^+ \rangle$ in λ^+ with $\delta_0 = 0$.

Lemma 3.4. *For any $\mathcal{A} \subseteq [\lambda^+]^{<\lambda}$ with $|\mathcal{A}| \leq \lambda^+$, any stationary $S \subseteq E^{\lambda^+}_\omega$ and any interval partition $\langle I_\xi \mid \xi < \lambda^+ \rangle$ of λ^+, there is a strictly increasing sequence $\langle \xi_n \mid n < \omega \rangle$ in λ^+ such that $B := \bigcup_{n<\omega} I_{\xi_n}$ has the following properties:*

(i) $\sup(B) \in S$.
(ii) *$B \cap A$ is bounded in B for any $A \in \mathcal{A}$.*

Proof. Let $\mathcal{A}$, S and $\langle I_\xi \mid \xi < \lambda^+ \rangle$ be as in the lemma. Take a surjection $f : \lambda^+ \to \lambda$ such that $f^{-1}[\{\beta\}]$ is unbounded in λ^+ for each $\beta < \lambda$, and let $g : \lambda^+ \to \lambda$ be the function such that $g(\alpha) = f(\xi)$ for each $\alpha \in I_\xi$. By increasing if necessary, we may assume that each $A \in \mathcal{A}$ is closed under g.

By the choice of f, we can take $\eta \in E^{\lambda^+}_\omega$ such that $\bigcup_{\xi<\eta} I_\xi \in S$ and such that $f^{-1}[\{\beta\}] \cap \eta$ is unbounded in η for every $\beta < \lambda$. Moreover, by Lemma 2.4, we can take $b \in [\lambda]^\omega$ such that $b \cap A$ is finite for any $A \in \mathcal{A}$. Let $\langle \beta_n \mid n < \omega \rangle$ be an injective enumeration of b. Then we can take an increasing cofinal sequence $\langle \xi_n \mid n < \omega \rangle$ in η such that $f(\xi_n) = \beta_n$.

We show that $\langle \xi_n \mid n < \omega \rangle$ is as desired: Let $B := \bigcup_{n<\omega} I_{\xi_n}$. (i) holds because $\langle \xi_n \mid n < \omega \rangle$ is cofinal in η, and $\bigcup_{\xi<\eta} I_\xi \in S$. To check (ii), take an arbitrary $A \in \mathcal{A}$. Then there is $m < \omega$ such that $\beta_n \notin A$ for all $n \geq m$. Here note that $g(\alpha) = f(\xi_n) = \beta_n$ if $\alpha \in I_{\xi_n}$. So $I_{\xi_n} \cap A = \emptyset$ for all $n \geq m$ because A is closed under g, and $\beta_n \notin A$. Thus $B \cap A$ is bounded in B. □

To prove Lemma 3.2 we use a game, which is a combination of games introduced in [12] and Veličković [17]. For a function $F : [\lambda^+]^{<\omega} \to \lambda^+$ and an ordinal $\rho < \omega_1$ let $\Game(F, \rho)$ be the following two players game of length ω:

I	J_0	J_1	J_2	$\cdots$	J_n	$\cdots$
II	α_0	α_1	α_2	$\cdots$	α_n	$\cdots$

At the n-th stage, first I chooses a non-empty bounded interval $J_n \subseteq \lambda^+$, and then II chooses $\alpha_n < \lambda^+$. If $n \geq 1$, then I must choose J_n so that $\alpha_{n-1} < \min(J_n)$. Suppose that $\langle J_n, \alpha_n \mid n < \omega \rangle$ is a play of $\Game(F, \rho)$, and let x be the closure of the set $\{\min(J_n) \mid n < \omega\}$ under F. I wins if $x \subseteq \bigcup_{n<\omega} J_n$, and $x \cap \omega_1 \subseteq \rho$. Otherwise, II wins.

The following lemma can be proved by combining the proofs of the corresponding lemmata in [12] and [17]:

Lemma 3.5. *For any function $F : [\lambda^+]^{<\omega} \to \lambda^+$ there is $\rho < \omega_1$ such that I has a winning strategy for $\Game(F, \rho)$.*

Proof. Fix a function $F : [\lambda^+]^{<\omega} \to \lambda^+$. For the contradiction assume that I does not have a winning strategy for $\Game(F, \rho)$ for any $\rho < \omega_1$. First note that

$\Game(F,\rho)$ is a closed game for I for each $\rho < \omega_1$. So it is determined. Hence II has a winning strategy τ_ρ for $\Game(F,\rho)$ for each $\rho < \omega_1$.

Take a strictly increasing sequence $\langle \beta_n \mid 1 \leq n < \omega \rangle$ in $E^{\lambda^+}_{\omega_1}$ such that each β_n is closed under F and τ_ρ for all $\rho < \omega_1$. Here we say that β is closed under τ_ρ if $\tau_\rho(\langle J_m \mid m \leq n \rangle) < \beta$ whenever each J_m is a bounded interval in β. We can take such $\langle \beta_n \mid 1 \leq n < \omega \rangle$ because λ^+ is a regular cardinal $> \omega_1$. Let $\beta_0 := 0$, and let x be the closure of the set $\{\beta_n \mid n < \omega\}$ under F. Then $x \subseteq \sup_{n<\omega} \beta_n$, and x is countable. Take $\rho^* < \omega_1$ with $x \cap \omega_1 \subseteq \rho^*$. Moreover for each $n < \omega$ let γ_n, J_n and α_n be as follows:

$$\gamma_n := \sup(x \cap \beta_{n+1}) + 1 \ , \quad J_n := \gamma_n \setminus \beta_n \ , \quad \alpha_n := \tau_{\rho^*}(\langle J_m \mid m \leq n \rangle) \ .$$

Note that $\beta_n < \gamma_n < \beta_{n+1}$ because $\beta_n \in x$, x is countable, and $\mathrm{cf}(\beta_{n+1}) = \omega_1$. So J_n is a non-empty interval bounded in β_{n+1}. Hence $\alpha_n < \beta_{n+1}$ by the closure of β_{n+1} under τ_{ρ^*}. Then $\langle J_n, \alpha_n \mid n < \omega \rangle$ is a legal play of $\Game(F,\rho^*)$ in which II has moved according to the winning strategy τ_{ρ^*}. So II wins with this play. But x is the closure of $\{\min(J_n) \mid n < \omega\}$ under F, $x \cap \omega_1 \subseteq \rho^*$, and $x \subseteq \bigcup_{n<\omega} J_n$. Therefore I wins with this play. This is a contradiction. □

Now we prove Lemma 3.2:

Proof of Lemma 3.2. Take an arbitrary function $F : [\lambda^+]^{<\omega} \to \lambda^+$. It suffices to find $x \in X^*$ closed under F.

By Lemma 3.5 take $\rho < \omega_1$ such that I has a winning strategy τ for $\Game(F,\rho)$. Let D be the set of all $\delta < \lambda^+$ which is closed under τ. Note that D is club in λ^+. Let $\langle \delta_\xi \mid \xi < \lambda^+ \rangle$ be the increasing enumeration of $D \cup \{0\}$, and let $I_\xi := \delta_{\xi+1} \setminus \delta_\xi$ for each $\xi < \lambda^+$.

By Lemma 3.4 we can take a strictly increasing sequence $\langle \xi_n \mid n < \omega \rangle$ such that, letting $B := \bigcup_{n<\omega} I_{\xi_n}$, we have the following:

(i) $\sup(B) \in h^{-1}[\{\rho\}]$.
(ii) $B \cap A_{\gamma,n}$ is bounded in B for any $\gamma \in E^{\lambda^+}_{\omega_1}$ and any $n < \omega$.

We may assume that $\xi_0 = 0$. For each $n < \omega$ let $\alpha_n := \delta_{\xi_{n+1}}$ and $J_n := \tau(\langle \alpha_m \mid m < n \rangle)$. Then let x be the closure of the set $\{\min(J_n) \mid n < \omega\}$ under F. It suffices to show $x \in X^*$.

For this we claim the following:

(iii) $x \subseteq B$.
(iv) $\sup(x) = \sup(B)$.
(v) $x \cap \omega_1 \subseteq \rho$.

First note that $x \subseteq \bigcup_{n<\omega} J_n =: C$ and $x \cap \omega_1 \subseteq \rho$ because τ is a winning strategy of I for $\Game(F,\rho)$. In particular, (v) holds. For (iii) and (iv) note that $J_n \subseteq \delta_{\xi_n+1} \setminus \delta_{\xi_n} = I_{\xi_n}$ for each $n < \omega$ because $\delta_{\xi_n} = \alpha_{n-1} < \min(J_n)$ if $n \geq 1$, and δ_{ξ_n+1} is closed under τ. So $C \subseteq B$. Then (iii) holds because $x \subseteq C$. Moreover (iv) follows from (iii) and the fact that $\min(J_n) \in x$ for all $n < \omega$.

Now we check that x satisfies (I)–(III) in the definition of X^*. First x satisfies (I) by (iii), (iv) and the fact that $\sup(B) \notin B$ by the construction of B. Next x satisfies (II) by (i), (iv) and (v). Finally x satisfies (III) by (ii), (iii) and (iv). □

4 SSR

By modifying the proof of SCH from WRP slightly, we can prove that the Semi-Stationary Reflection Principle (SSR) implies SCH. Here we give an outline of the proof.

First we recall SSR: For countable sets x and y we let $x \sqsubseteq y$ denote that $x \subseteq y$ and $x \cap \omega_1 = y \cap \omega_1$. For a set $W \supseteq \omega_1$, a subset X of $[W]^\omega$ is said to be *semi-stationary* in $[W]^\omega$ if the set $\{y \in [W]^\omega \mid (\exists x \in X)\ x \sqsubseteq y\}$ is stationary. SSR is the assertion that $\mathsf{SSR}(\lambda)$ below holds for every cardinal $\lambda \geq \omega_2$:

$\mathsf{SSR}(\lambda) \equiv$ For any semi-stationary $X \subseteq [\lambda]^\omega$ there is $R \in [\lambda]^{\omega_1}$ such that $\omega_1 \subseteq R$ and $X \cap [R]^\omega$ is semi-stationary in $[R]^\omega$.

It is easy to see that SSR follows from WRP, and SSR is known, due to Shelah [10], to be equivalent to the assertion that every ω_1-stationary preserving forcing notion is semi-proper. It is also known, due to Doebler-Schindler [2], that SSR is equivalent to a variant of the Strong Chang's Conjecture CC^{**}.

As we promised above, we give an outline of the proof of the following theorem which was proved in [8] using better scales:

Theorem 4.1 (Sakai-Veličković [8]). SSR *implies* SCH.

We will use a lemma in [8]. First we recall it: For sets x and y of ordinals, we write $x \sqsubseteq^* y$ if $x \sqsubseteq y$, x is unbounded in y, and $x \cap \gamma$ is unbounded in $y \cap \gamma$ for all $\gamma \in x$ of cofinality ω_1.

Lemma 4.2 (Sakai-Veličković [8] Lemma 2.2). *Let μ be a cardinal $\geq \omega_2$, and assume $\mathsf{SSR}(\mu)$. Then for any stationary $X \subseteq [\mu]^\omega$ which is upward closed under $\sqsubseteq^*$ there is $R \in [\mu]^{\omega_1}$ such that $\omega_1 \subseteq R$ and $X \cap [R]^\omega$ is stationary in $[R]^\omega$.*

Now we start an outline of the proof of Theorem 4.1. Assume SCH fails, and let λ, h and $\langle A_{\gamma,n} \mid \gamma \in E^{\lambda^+}_{\omega_1},\ n < \omega\rangle$ be as in the proof of Theorem 3.1. By modifying X^* in the proof of Theorem 3.1, define Y^* as follows:

Y^* := the set of all $x \in [\lambda^+]^\omega$ with the following properties:

(I) $\sup(x) \notin x$.

(II) $x \cap \omega_1 \subseteq h(\sup(x))$.

(III) For any $\gamma \in E^{\lambda^+}_{\omega_1}$ and any $n < \omega$, there is $\beta < \sup(x)$ such that $x \cap A_{\gamma,n} \subseteq \beta$ and such that $\mathrm{cf}(\min(x \setminus \alpha)) = \omega_1$ for any $\alpha \in A_{\gamma,n} \setminus \beta$ with $\alpha < \sup(x)$.

We claim that Y^* is a counter-example of $\mathsf{SSR}(\lambda^+)$ in the sense of Lemma 4.2. First note that $Y^* \subseteq X^*$. So $Y^* \cap [R]^\omega$ is non-stationary for any $R \in [\lambda^+]^{\omega_1}$ with $\omega_1 \subseteq R$ by Lemma 3.3. Moreover it is easy to check that Y^* is upward closed under $\sqsubseteq^*$. So it suffices to prove that Y^* is stationary. This can be proved by almost the same way as Lemma 3.2:

Take an arbitrary $F : [\lambda^+]^{<\omega} \to \lambda^+$. We must find $x \in Y^*$ closed under F. For each $\rho < \omega_1$ let $\Game'(F, \rho)$ be the game obtained from $\Game(F, \rho)$ by additionally requiring I to choose J_n with $\mathrm{cf}(\min(J_n)) = \omega_1$ for $n \geq 1$. The proof of

Lemma 3.5 actually shows that I has a winning strategy for $\Game'(F,\rho)$ for some $\rho < \omega_1$. Let ρ be such an ordinal and τ' be a winning strategy of I for $\Game'(F,\rho)$. Construct x as in the proof of Lemma 3.2 using τ' instead of τ. Then it is easy to see that x is as desired.

References

1. J. Cummings, M. Foreman and M. Magidor, *Squares, scales and stationary reflection*, J. Math. Log. **1** (2001), no. 1, 35–98.
2. P. Doebler and R. Schindler, *Π_2 consequences of* $\mathrm{BMM} + \mathrm{NS}_{\omega_1}$ *is precipitous and the semiproperness of stationary set preserving forcings*, Math. Res. Lett. **16** (2009), no. 5, 797–815.
3. Q. Feng and M. Magidor, *On reflection of stationary sets*, Fund. Math. **140** (1992), no. 2, 175–181.
4. M. Foreman, M. Magidor and S. Shelah, *Martin's maximum, saturated ideals, and non-regular ultrafilters I*, Ann. of Math. (2) **127** (1988), no. 1, 1–47.
5. S. Fuchino, I. Juhász, L. Soukup, Z. Szentmiklóossy and T. Usuba, *Fodor-type reflection principle and reflection of metrizability and meta-Lindelöfness*, Topology Appl. **157** (2010), no. 8, 1415–1429.
6. S. Fuchino and A. Rinot, *Openly generated Boolean algebras and the Fodor-type reflection principle*, Fund. Math. **212** (2011), no. 3, 261–283.
7. S. Fuchino, L. Soukup, H. Sakai and T. Usuba, *More on Fodor-type Reflection Principle*, preprint.
8. H. Sakai and B. Veličković, *Stationary reflection principles and two cardinal tree properties*, J. Inst. Math. Jussieu **14** (2015), no. 1, 69–85.
9. S. Shelah, *Cardinal Arithmetic*, Oxford Logic Guides, vol.29, Oxford Univ. Press, London and New York, 1994.
10. S. Shelah, *Proper and Improper Forcing, Second Edition*, Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1998.
11. S. Shelah, *Reflection implies the SCH*, Fund. Math. **198** (2008), no. 2, 95–111.
12. S. Shelah and M. Shioya, *Nonreflecting stationary sets in $\mathcal{P}_\kappa\lambda$*, Adv. Math. **199** (2006), no. 1, 185–191.
13. J. Silver, *On the singular cardinals problem*, Proceedings of the International Congress of Mathematicians (Vancouver, B.C., 1974), Vol. 1, pp.265–268, Canad. Math. Congress, Montreal, Que., 1975.
14. R. M. Solovay, *Strongly compact cardinals and the GCH*, Proceedings of the Tarski Symposium (Proc. Sympos. Pure Math., Vol. XXV, Univ. California, Berkeley, Calif., 1971), pp.365–372, Amer. Math. Soc., Providence, R.I., 1974.
15. M. Viale, *The proper forcing axiom and the singular cardinal hypothesis*, J. Symbolic Logic **71** (2006), no. 2, 473–479.
16. M. Viale and C. Weiß, *On the consistency strength of the proper forcing axiom*, Adv. Math. **228** (2011), no. 5, 2672–2687.
17. B. Veličković, *Forcing axioms and stationary sets*, Adv. Math. **94** (1992), no. 2, 256–284.
18. C. Weiß, *The combinatorial essence of supercompactness*, Ann. Pure Appl. Logic **163** (2012), no. 11, 1710–1717.