Partial square at ω_1 is implied by MM but not by PFA

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Abstract

We prove the results stated in the title.

1 Introduction

The square principle, introduced by Jensen [9], and its weak versions, play important roles in Set Theory. Using these square principles, we can construct incompact objects such as Souslin trees and non-reflecting stationary sets. Thus propositions asserting some compactness tend to imply the failure of square principles. For example, it was shown by Magidor [12] that the stationary reflection principle at δ^+ implies the failure of $\square_\delta$. It is also known, from work of Todorčević [17], that PFA implies the failure of $\square(\delta)$ for any regular $\delta \geq \omega_2$.

In this paper we study consequences of forcing axioms for the partial square principle at ω_1. In particular we study those of Martin's Maximum, MM, and the Proper Forcing Axiom, PFA. First we recall the partial square principle. Below, for a set A of ordinals, $\mathrm{otp}(A)$ denotes the order type of A, and $\mathrm{Lim}(A)$ denotes the set of all limit points in A, i.e. $\mathrm{Lim}(A) = \{\alpha \in A \mid \sup(A \cap \alpha) = \alpha\}$:

Definition 1.1. Let δ be an uncountable cardinal. For $S \subseteq \mathrm{Lim}(\delta^+)$ let

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$\square_\delta(S) \equiv$ there exists a sequence $\langle c_\alpha \mid \alpha \in S\rangle$ such that

(i) c_α is a club in α with $\mathrm{otp}(c_\alpha) \leq \delta$ for each $\alpha \in S$,

(ii) if $\alpha \in S$ and $\beta \in \mathrm{Lim}(c_\alpha)$, then $\beta \in S$ and $c_\beta = c_\alpha \cap \beta$.

A sequence $\langle c_\alpha \mid \alpha \in S\rangle$ satisfying (i) and (ii) is called a $\square_\delta(S)$-*sequence.*

The above partial square was used in [2], [7], [10], [13], [15], etc.

Note that $\square_\delta(\mathrm{Lim}(\delta^+))$ is equivalent to Jensen's $\square_\delta$ introduced in [9]. In fact it is easy to see that $\square_\delta$ holds if and only if $\square_\delta(C)$ holds for some club $C \subseteq \mathrm{Lim}(\delta^+)$. On the other hand it is not hard to see that if S is a nonstationary subset of δ^+, then $\square_\delta(S)$ holds.

As for $\square_\delta(S)$ for a stationary $S \subseteq \mathrm{Lim}(\delta^+)$, the following was shown by Shelah:

Fact 1.2 (Shelah [15]). *Suppose that δ and ρ are regular cardinals with $\rho < \delta$. Then there exists $S \subseteq \mathrm{Lim}(\delta^+)$ such that*

(i) $\square_\delta(S)$ holds,

(ii) the set $\{\alpha \in S \mid \mathrm{cf}(\alpha) = \rho\}$ is stationary in δ^+.

On the other hand it is known that the following partial square principle $\square^{\mathrm{p}}_\delta$ is independent of ZFC for a regular uncountable cardinal δ (See §6):

Definition 1.3. For a regular uncountable cardinal δ let

$\square^{\mathrm{p}}_\delta \equiv$ there exists $S \subseteq \mathrm{Lim}(\delta^+)$ such that

(i) $\square_\delta(S)$ holds,

(ii) the set $\{\alpha \in S \mid \mathrm{cf}(\alpha) = \delta\}$ is stationary in δ^+.

(The superscript "p" in $\square^{\mathrm{p}}_\delta$ stands for "partial".)

We study consequences of MM and PFA for $\square^{\mathrm{p}}_{\omega_1}$. For simplicity of our notation we omit the subscript ω_1 in $\square_{\omega_1}(S)$ and $\square^{\mathrm{p}}_{\omega_1}$:

Notation. *Let $\square(S)$ and $\square^{\mathrm{p}}$ denote $\square_{\omega_1}(S)$ and $\square^{\mathrm{p}}_{\omega_1}$, respectively.*

It is not hard to see that MM does not imply the failure of $\square^{\mathrm{p}}$ (See Thm.6.4). Our first result is the following:

Theorem 1.4. MM *implies $\square^{\mathrm{p}}$.*

On the other hand, we also prove that PFA does not imply $\square^{\mathrm{p}}$:

Theorem 1.5. *If there exists a supercompact cardinal, then there exists a forcing extension in which* PFA *holds but* $\square^{\mathrm{p}}$ *fails.*

Thm.1.4 will be proved in §3, and Thm.1.5 will be proved in §5. In §6 we make a remark that $\square^{\mathrm{p}}_{\delta}$ for a regular cardinal $\delta \geq \omega_2$ is independent of MM.

In §5 and §6 we discuss the consistency of the failure of the partial square. For this we use a strong stationary reflection principle, which was introduced by Magidor [12] and implies the failure of the partial square. In §4 we present facts on this strong stationary reflection principle which we use in §5 and §6.

2 Preliminaries

Here we present our notation and basic facts used in this paper. For those which are not presented below, consult Jech [8].

For a function f and $X \subseteq \mathrm{dom}(f)$ we let $f[\,X\,] := \{f(x) \mid x \in X\}$. For a regular cardinal δ and an ordinal $\kappa > \delta$ let $\mathcal{E}^{\kappa}_{\delta} := \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \delta\}$. Moreover for ordinals δ and κ with $\delta \leq \kappa$ let $\mathcal{E}^{\kappa}_{<\delta} := \{\alpha < \kappa \mid \mathrm{cf}(\alpha) < \delta\}$. For $i = 0, 1$ let $E^2_i := \mathcal{E}^{\omega_2}_{\omega_i}$.

For a regular cardinal δ and a set W, $[W]^{\delta}$ denotes the set of all $x \subseteq W$ with $|x| = \delta$. $C \subseteq [W]^{\delta}$ is said to be a *club* in $[W]^{\delta}$ if for some function $F : {}^{<\omega}W \to W$, C is the set of all $x \in [W]^{\delta}$ closed under F. $X \subseteq [W]^{\delta}$ is said to be *stationary* in $[W]^{\delta}$ if $X \cap C \neq \emptyset$ for any club C in $[W]^{\delta}$, i.e. for any function $F : {}^{<\omega}W \to W$ there exists $x \in X$ which is closed under F.

Let $\mathcal{M}$ be a structure such that there exists a well-ordering of its universe definable over $\mathcal{M}$, and suppose that $x \subseteq \mathcal{M}$. Then $\mathrm{Sk}^{\mathcal{M}}(x)$ denotes the Skolem hull of x in $\mathcal{M}$, i.e. the smallest M with $x \subseteq M \prec \mathcal{M}$.

For a limit ordinal δ, a set M is said to be *internally approachable (i.a.) of length* δ if there exists a $\subseteq$-increasing sequence $\langle M_{\xi} \mid \xi < \delta \rangle$ such that $\bigcup_{\xi<\delta} M_{\xi} = M$ and such that $\langle M_{\xi} \mid \xi < \delta' \rangle \in M$ for any $\delta' < \delta$. A sequence $\langle M_{\xi} \mid \xi < \delta \rangle$ witnessing the internally approachability of M is called an *internally approaching (i.a.) sequence to* M.

We use an ideal $I[\lambda]$ over a regular cardinal $\lambda \geq \omega_2$, which was introduced by Shelah [14]. First we recall the definition of $I[\lambda]$. Suppose that $\lambda \geq \omega_2$ and that $E \subseteq \lambda$. Then $E \in I[\lambda]$ if and only if there exist a sequence $\langle b_{\alpha} \mid \alpha < \lambda \rangle$ of bounded subsets of λ and a club $C \subseteq \lambda$ such that for any limit ordinal $\alpha \in C \cap E$ we can take an unbounded $b \subseteq \alpha$ with $\mathrm{otp}(b) = \mathrm{cf}(\alpha)$ and $\{b \cap \beta \mid \beta < \alpha\} \subseteq \{b_{\beta} \mid \beta < \alpha\}$. We use the following fact:

Fact 2.1 (Shelah [14]). *Let δ be a regular uncountable cardinal.*

(1) Suppose that λ is a regular cardinal $> \delta$ and that E is a stationary subset of $\mathcal{E}^{\lambda}_{<\delta}$ with $E \in I[\lambda]$. Then E remains stationary in $V^{\mathbb{P}}$ for any $<\delta$-closed poset $\mathbb{P}$.

(2) Suppose that $2^{<\delta} = \delta$. Then $\mathcal{E}^{\delta^+}_{<\delta} \in I[\delta^+]$.

Next we give our notation on forcing.

Let $\mathbb{P}$ be a poset. We let $\mathbb{P}$ denote also the base set of $\mathbb{P}$. The order of $\mathbb{P}$ is denoted as $\leq_{\mathbb{P}}$, but we usually omit the subscript $\mathbb{P}$. A poset $\mathbb{Q}$ is said to be a *suborder* of $\mathbb{P}$ if $\mathbb{Q} \subseteq \mathbb{P}$ and $\leq_{\mathbb{Q}} = \leq_{\mathbb{P}} \cap (\mathbb{Q} \times \mathbb{Q})$.

A $\mathbb{P}$-*name* is a set consisting of pairs $(\dot{x}, p)$ such that $\dot{x}$ is a $\mathbb{P}$-name of lower rank and such that $p \in \mathbb{P}$. If $(\dot{x}, p)$ belongs to a $\mathbb{P}$-name $\dot{X}$, then p forces that $\dot{x} \in \dot{X}$. For an ordinal κ we say that $\dot{S}$ is a *nice* $\mathbb{P}$-*name* for a subset of κ if there exists a sequence $\langle A_\alpha \mid \alpha < \kappa \rangle$ of antichains in $\mathbb{P}$ such that $\dot{S} = \{(\check{\alpha}, p) \mid p \in A_\alpha\}$.

For $A \subseteq \mathbb{P}$ and $p \in \mathbb{P}$ we say that p *meets* A if there is $q \in A$ with $q \geq p$. For $A_0, A_1 \subseteq \mathbb{P}$ we say that A_0 *refines* A_1 if all elements of A_0 meet A_1.

$\mathbb{P}$ is said to be ω_1-*stationary preserving* if ${\omega_1}^{V^{\mathbb{P}}} = {\omega_1}^V$, and every stationary subset of ω_1 in V remains stationary in $V^{\mathbb{P}}$.

Let δ be a regular uncountable cardinal. We say that $\mathbb{P}$ has the δ-*chain condition (δ-c.c.)* if there is no antichain A in $\mathbb{P}$ with $|A| = \delta$.

A subset $A \subseteq \mathbb{P}$ is said to be *directed* if for any $p, q \in A$ there exists $r \in A$ with $r \leq p, q$. $\mathbb{P}$ is said to be $<\delta$-*directed closed* if every directed $A \subseteq \mathbb{P}$ with $|A| < \delta$ has a lower bound in $\mathbb{P}$.

$\mathbb{P}$ is said to be $<\delta$-*Baire* if for any $p \in \mathbb{P}$ and any family $\mathcal{A}$ of maximal antichains in $\mathbb{P}$ with $|\mathcal{A}| < \delta$, there exists $p^* \leq p$ which meets all $A \in \mathcal{A}$. $\mathbb{P}$ is $<\delta$-Baire if and only if a forcing extension by $\mathbb{P}$ does not add any new sequences of ordinals of length $< \delta$. $\mathbb{P}$ is said to be σ-*Baire* if $\mathbb{P}$ is $<\omega_1$-Baire.

For a regular cardinal δ and an ordinal $\kappa \geq \delta$ let $\mathrm{Col}(\delta, \kappa)$ denote the poset ${}^{<\delta}\kappa$ ordered by reverse inclusions. Moreover let $\mathrm{Col}(\delta, <\kappa)$ be the $<\delta$-support product of $\langle \mathrm{Col}(\delta, \kappa') \mid \delta \leq \kappa' < \kappa \rangle$. Thus if κ is an inaccessible cardinal, then $\mathrm{Col}(\delta, <\kappa)$ is the Lévy collapse forcing κ to be δ^+. Furthermore for an ordinal $\lambda > \kappa$ let $\mathrm{Col}(\delta, [\kappa, \lambda))$ be the $<\delta$-support product of $\langle \mathrm{Col}(\delta, \kappa') \mid \kappa \leq \kappa' < \lambda \rangle$.

Next we give our notation and a basic fact on projections between posets. Let $\mathbb{P}$ and $\mathbb{Q}$ be a poset. A map $\pi : \mathbb{P} \to \mathbb{Q}$ satisfying the following properties is called a *projection*:

(i) π is surjective and order preserving.

(ii) For any $p \in \mathbb{P}$ and any $q \in \mathbb{Q}$ with $q \leq_{\mathbb{Q}} \pi(p)$ there exists $p^* \in \mathbb{P}$ such that $p^* \leq_{\mathbb{P}} p$ and $\pi(p^*) = q$.

Suppose that $\pi : \mathbb{P} \to \mathbb{Q}$ is a projection and that H is a $\mathbb{Q}$-generic filter. Then, in $V[H]$, $\mathbb{P}/H$ denotes the poset obtained by restricting $\mathbb{P}$ to $\pi^{-1}[\,H\,]$. It is standard that $\mathbb{Q} * (\mathbb{P}/\dot{H})$ is forcing equivalent to $\mathbb{P}$, where $\dot{H}$ is the canonical $\mathbb{Q}$-name for a $\mathbb{Q}$-generic filter. (See Abraham [1] §1.)

Finally we present our notation and a fact on forcing axioms:

For a poset $\mathbb{P}$ and an uncountable cardinal δ let $\mathsf{FA}_\delta(\mathbb{P})$ and $\mathsf{FA}^{++}_\delta(\mathbb{P})$ be the following forcing axioms:

$\mathsf{FA}_\delta(\mathbb{P})$ $\equiv$ For any $p \in \mathbb{P}$ and any family $\mathcal{A}$ of maximal antichains in $\mathbb{P}$ with $|\mathcal{A}| \leq \delta$ there exists a filter $G \subseteq \mathbb{P}$ containing p such that $G \cap A \neq \emptyset$ for all $A \in \mathcal{A}$.

$\mathsf{FA}^{++}_\delta(\mathbb{P})$ $\equiv$ For any $p \in \mathbb{P}$, any family $\mathcal{A}$ of maximal antichains in $\mathcal{P}$ with $|\mathcal{A}| \leq \delta$ and any family $\mathcal{R}$ of $\mathbb{P}$-names for stationary subsets of δ with $|\mathcal{R}| \leq \delta$ there exists a filter $G \subseteq \mathcal{P}$ containing p such that $G \cap A \neq \emptyset$ for all $A \in \mathcal{A}$ and such that $\dot{R}_G := \{\xi < \delta \mid \exists p \in G,\ p \Vdash \text{“}\xi \in \dot{R}\text{”}\}$ is stationary in δ for all $\dot{R} \in \mathcal{R}$.

Recall that PFA is FA_{ω_1} for all proper posets and that MM is FA_{ω_1} for all ω_1-stationary preserving posets. We let PFA^{++} denote $\mathsf{FA}^{++}_{\omega_1}$ for all proper posets. PFA^{++} was introduced by Baumgartner [3], and he observed that if κ is a supercompact cardinal, and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ is the standard iteration for PFA, which was also introduced by him (See Devlin [4]), then this iteration in fact forces PFA^{++}.

Let $\mathbb{P}$ be a poset and M be a set. $g \subseteq \mathbb{P} \cap M$ is called an $(M, \mathbb{P})$-*generic filter* if g is a filter on $\mathbb{P} \cap M$ such that $g \cap A \neq \emptyset$ for every maximal antichain $A \in M$ in $\mathbb{P}$.

We use the following fact. (1) is proved in Woodin [18] (See the proof of Thm.2.53 in [18]):

Lemma 2.2. *Suppose that $\mathbb{P}$ is a poset, that δ is an uncountable cardinal and that $\mathsf{FA}_\delta(\mathbb{P})$ holds. Let $p \in \mathbb{P}$, and let θ be a regular cardinal $> \delta$ with $\mathbb{P} \in \mathcal{H}_\theta$.*

(1) There are stationary many $M \in [\mathcal{H}_\theta]^\delta$ with the following properties:

(i) $\delta \subseteq M$, and $p \in M$.

(ii) There exists an $(M, \mathbb{P})$-generic filter containing p.

(2) If $\mathbb{P}$ is $<\delta$-Baire, then there are stationary many $M \in [\mathcal{H}_\theta]^\delta$ with the properties (i), (ii) above and the following:

(iii) M is internally approachable of length δ.

(3) If $\mathsf{FA}^{++}_\delta(\mathbb{P})$ holds, then there are stationary many $M \in [\mathcal{H}_\theta]^\delta$ with the property (i) above and the following:

(iv) There exists an $(M, \mathbb{P})$-generic filter g containing p such that $\dot{R}_g = \{\xi < \delta \mid \exists q \in g,\ q \Vdash \text{“}\xi \in \dot{R}\text{”}\}$ is stationary in δ for any $\mathbb{P}$-name $\dot{R} \in M$ for a stationary subset of δ.

If $\mathbb{P}$ is $< \delta$-Baire in addition, then there are stationary many $M \in [\mathcal{H}_\theta]^\delta$ with the properties (i), (iii) and (iv).

In the proof of the above lemma we use the following well-known lemma:

Lemma 2.3 (folklore)**.** *Let θ be a regular uncountable cardinal, Δ be a well-ordering of $\mathcal{H}_\theta$, and $\mathcal{M}$ be a structure obtained by adding countable many constants, functions and predicates to $\langle \mathcal{H}_\theta, \in, \Delta\rangle$. Suppose that M is an elementary submodel of $\mathcal{M}$ and that $d \subseteq D \in M$. Then*

$$\mathrm{Sk}^{\mathcal{M}}(M \cup d) \ = \ \{f(b) \mid b \in {}^{<\omega}d,\ f : {}^{|b|}D \to \mathcal{H}_\theta,\ f \in M\ \}\ .$$

Proof. Let N be the set in the right side of the equation. $\mathrm{Sk}^{\mathcal{M}}(M \cup d) \supseteq N$ clearly. We prove the reverse inclusion. Before starting we prepare a notation. For each formula $\varphi(u, v_0, \ldots, v_{m-1}, w_0, \ldots, w_{n-1})$ of the language for $\mathcal{M}$ let $h_\varphi : {}^{m+n}\mathcal{H}_\theta \to \mathcal{H}_\theta$ be the Skolem function for φ in $\mathcal{M}$. That is, for any $a = \langle a_0, \ldots, a_{m-1}\rangle \in {}^m\mathcal{H}_\theta$ and any $b = \langle b_0, \ldots, b_{n-1}\rangle \in {}^n\mathcal{H}_\theta$ if there exists x with $\mathcal{M} \models \varphi(x, a_0, \ldots, a_{m-1}, b_0, \ldots, b_{n-1})$, then $h_\varphi(a, b)$ is the Δ-least such x, otherwise $h_\varphi(a, b) = 0$.

To show that $\mathrm{Sk}^{\mathcal{M}}(M \cup d) \subseteq N$, take an arbitrary $x \in \mathrm{Sk}^{\mathcal{M}}(M \cup d)$. Then there exists a formula $\varphi(u, v_0, \ldots, v_{m-1}, w_1, \ldots, w_{n-1})$, $a^* \in {}^mM$ and $b^* \in {}^nd$ such that $h_\varphi(a^*, b^*) = x$. Let $f : {}^nD \to \mathcal{H}_\theta$ be the function such that $f(b) = h_\varphi(a^*, b)$ for every $b \in {}^nD$.

Then $f \in \mathcal{H}_\theta$, and f is definable in $\mathcal{M}$ from parameters D and a^*, both of which are in M. Hence $f \in M$ by the elementarity of M. Moreover $x = f(b^*)$. Therefore $x \in N$. □

Now we prove Lem.2.2:

Proof of Lem.2.2. Let Δ be a well-ordering of $\mathcal{H}_\theta$, and suppose that $\mathcal{M}$ is a structure obtained by adding countable many constants, functions and predicates to $\langle \mathcal{H}_\theta, \in, \Delta, \mathbb{P}, p, \delta \rangle$. For (1) it suffices to find $M \in [\mathcal{H}_\theta]^\delta$ with the properties (i) and (ii) such that $M \prec \mathcal{M}$. For (2) or (3) it suffices to find such M with (iii) or (iv), respectively.

First we can take $N \in [\mathcal{H}_\theta]^\delta$ such that N is i.a. of length δ and such that $N \prec \mathcal{M}$. Here note that $\delta \subseteq N$. Let $\mathcal{A}$ be the set of all maximal antichains in $\mathbb{P}$ which belong to N, and let $\mathcal{R}$ be the set of all $\mathbb{P}$-names in N for stationary subsets of δ. By $\mathsf{FA}_\delta(\mathbb{P})$ take a filter $G \subseteq \mathbb{P}$ containing p and intersecting all elements of $\mathcal{A}$. Here note that if $\mathsf{FA}_\delta^{++}(\mathbb{P})$ holds, then we can take G so that $\dot{R}_G$ is stationary for all $\dot{R} \in \mathcal{R}$. For each $A \in \mathcal{A}$ let p_A be the unique element of $G \cap A$, and let $d := \{p_A \mid A \in \mathcal{A}\}$. Moreover let $M := \mathrm{Sk}^{\mathcal{M}}(N \cup d)$. Clearly $\delta \cup \{p\} \subseteq M \prec \mathcal{M}$. It suffices to prove the following:

(1) $g := G \cap M$ is an $(M, \mathbb{P})$-generic filter.

(2) If $\mathbb{P}$ is $<\delta$-Baire, then M is i.a. of length δ.

(3) If $\dot{R}_G$ is stationary in δ for every $\dot{R} \in \mathcal{R}$, then $\dot{R}_g$ is stationary in δ for every $\mathbb{P}$-name $\dot{R} \in M$ for a stationary subset of δ.

(1) Let $A^* \in M$ be a maximal antichain in $\mathbb{P}$. We show that $g \cap A^* \neq \emptyset$.

By Lem.2.3 there exist $b^* = \langle p^*_0, \dots, p^*_{n-1} \rangle \in {}^{<\omega}d$ and a function $f : {}^n\mathbb{P} \to \mathcal{H}_\theta$ in N such that $f(b^*) = A^*$. We may assume that $f(b)$ is a maximal antichain in $\mathbb{P}$ for every $b \in {}^n\mathbb{P}$.

For each $i < n$ take $A_i \in \mathcal{A}$ with $p^*_i = p_{A_i}$. Let K be the set of all $b \in \Pi_{i<n} A_i$ which has a lower bound. Here we say that $b = \langle p_0, \dots, p_{n-1} \rangle$ has a lower bound if $\{p_0, \dots, p_{n-1}\}$ has a lower bound in $\mathbb{P}$. Note that if $b, b' \in K$ and $b \neq b'$, then b and b' have no common lower bound. This is because each A_i is an antichain.

For each $b \in K$ let A_b be the Δ-least maximal antichain below b which refines $f(b)$. Let $A^\circ := \bigcup_{b \in K} A_b$. Then it is easy to see that A° is a maximal antichain in $\mathbb{P}$ and that $A^\circ \in N$. That is, $A^\circ \in \mathcal{A}$.

Here note that p_{A° must be in A_{b^*}. Otherwise p_{A° is incompatible with at least one of $p^*_0, \dots, p^*_{n-1}$, and this contradicts that all $p_{A^\circ}, p^*_0, \dots, p^*_{n-1}$ belong to the filter G. Moreover recall that A_{b^*} refines $f(b^*) = A^*$. Let p^* be the unique element of A^* with $p^* \geq p_{A^\circ}$.

Then $p^* \in G$ because $p^* \geq p_{A^\circ} \in G$. Moreover $p^* \in M$ because p^* is definable from $p_{A^\circ}, A^* \in M$. Therefore $p^* \in g \cap A^* \neq \emptyset$.

(2) Let $\langle N_\xi \mid \xi < \delta\rangle$ be an i.a. sequence to N. We may assume that $|N_\xi| < \delta$ for each $\xi < \delta$. For each $\xi < \delta$ let $\mathcal{A}_\xi := \mathcal{A} \cap N_\xi$, $d_\xi := \{p_A \mid A \in \mathcal{A}_\xi\}$ and

$$M_\xi \;:=\; \{f(b) \mid b \in {}^{<\omega}d_\xi,\ f : {}^{|b|}\mathbb{P} \to \mathcal{H}_\theta,\ f \in N_\xi\}$$

We show that $\langle M_\xi \mid \xi < \delta\rangle$ is an i.a. sequence to M. Clearly $\langle M_\xi \mid \xi < \omega_1\rangle$ is $\subseteq$-increasing, and $\bigcup_{\xi<\omega_1} M_\xi = M$ by Lem.2.3. Thus it suffices to show that $\langle M_\xi \mid \xi < \zeta\rangle \in M$ for every $\zeta < \delta$. Here note that $\langle M_\xi \mid \xi < \zeta\rangle$ is definable in $\langle \mathcal{H}_\theta, \in\rangle$ from parameters $\mathbb{P}$, $\langle N_\xi \mid \xi < \zeta\rangle$ and $\langle d_\xi \mid \xi < \zeta\rangle$. Moreover $\mathbb{P}, \langle N_\xi \mid \xi < \zeta\rangle \in N \subseteq M$. Therefore all we have to show is that $\langle d_\xi \mid \xi < \zeta\rangle \in M$ for every $\zeta < \delta$.

Fix $\zeta < \delta$. Because $\mathbb{P}$ is $< \delta$-Baire, there exists a maximal antichain A^* in $\mathbb{P}$ which refines all maximal antichains in $\mathcal{A}_\zeta$. We can take such A^* in N because $\mathcal{A}_\zeta \in N$. Then for each $A \in \mathcal{A}_\zeta$, p_A is the unique $p \in A$ with $p \geq p_{A^*}$. Hence $d_\xi = \{p \in \bigcup \mathcal{A}_\xi \mid p \geq p_{A^*}\}$ for each $\xi < \zeta$. Then $\langle d_\xi \mid \xi < \zeta\rangle \in M$ because $p_{A^*}, \langle \mathcal{A}_\xi \mid \xi < \zeta\rangle \in M \prec \langle \mathcal{H}_\theta, \in\rangle$.

(3) Suppose that $\dot{R}_G$ is stationary in δ for all $\dot{R} \in \mathcal{R}$. Take an arbitrary $\mathbb{P}$-name $\dot{R}^* \in M$ for stationary subsets of δ. We show that $\dot{R}^*_g$ is stationary in δ.

By Lem.2.3 there exist $b^* = \langle p^*_0, \dots, p^*_{n-1}\rangle \in {}^n d$ and a function $f : {}^n\mathbb{P} \to \mathcal{H}_\theta$ in N such that $f(b^*) = \dot{R}^*$. We may assume that $f(b)$ is a $\mathbb{P}$-name for a stationary subset of δ for every $b \in {}^n d$. Moreover take A_i, $i < n$, and K as in the proof of (1) above.

Then we can take a $\mathbb{P}$-name $\dot{R}^\circ \in N$ such that for any $b \in K$ all lower bounds of b force that $\dot{R}^\circ = f(b)$. Recall that $f(b^*) = \dot{R}^*$, that $b^* = \langle p^*_0, \dots, p^*_{n-1}\rangle$ and that $p^*_0, \dots, p^*_{n-1} \in G$. Then it is easy to see that $\dot{R}^\circ_G = \dot{R}^*_g$. Moreover $\dot{R}^\circ_G$ is stationary in δ because $\dot{R}^\circ \in \mathcal{R}$. Therefore $\dot{R}^*_g$ is stationary in δ. $\square$

3 MM implies $\square^{\mathrm{p}}$

In this section we prove Thm.1.4:

Theorem 1.4. MM *implies* $\square^{\mathrm{p}}$.

This will be proved in §3.3. In the preceding subsections (§3.1, 3.2), we make preliminaries for the proof.

3.1 ω_1-stationary preserving σ-Baire posets

In the proof of Thm.1.4 we will construct an ω_1-stationary preserving σ-Baire poset and apply MM to it. Here we present a sufficient condition for a poset to be ω_1-stationary preserving and σ-Baire. For this we use the notions of projectively stationary sets and of strongly generic conditions:

Definition 3.1 (Feng-Jech [5])**.** Let W be a set with $\omega_1 \subseteq W$. $X \subseteq [W]^\omega$ is said to be *projectively stationary* if the set $\{x \in X \mid x \cap \omega_1 \in R\}$ is stationary in $[W]^\omega$ for every stationary $R \subseteq \omega_1$.

Definition 3.2. Let $\mathbb{P}$ be a poset and M be a set. $p \in \mathbb{P}$ is called a *strongly $(M, \mathbb{P})$-generic condition* if $\{q \in \mathbb{P} \cap M \mid q \geq p\}$ is an $(M, \mathbb{P})$-generic filter.

Note that if p is a strongly $(M, \mathbb{P})$-generic condition, then p meets $A \cap M$ for every maximal antichain $A \in M$ in $\mathbb{P}$.

Now we give a sufficient condition:

Lemma 3.3. *Let $\mathbb{P}$ be a poset. Suppose that $\mathbb{P}$ satisfies the following:*

($*$) *For every sufficiently large regular cardinal θ and every $p \in \mathbb{P}$ the following set is projectively stationary:*

$$\{M \in [\mathcal{H}_\theta]^\omega \mid \text{ a strongly } (M, \mathbb{P})\text{-generic condition below } p \text{ exists}\}$$

Then $\mathbb{P}$ is ω_1-stationary preserving and σ-Baire.

Proof. Assume ($*$). Let θ be a sufficiently large regular cardinal.

To show that $\mathbb{P}$ is σ-Baire, suppose that $p \in \mathbb{P}$ and that $\mathcal{A}$ is a countable family of maximal antichains in $\mathbb{P}$. By ($*$) we can take $M \in [\mathcal{H}_\theta]^\omega$ and $p^* \leq p$ such that $\mathcal{A} \cup \{p\} \subseteq M \prec \langle \mathcal{H}_\theta, \in \rangle$ and such that p^* is a strongly $(M, \mathbb{P})$-generic condition. Then $p^* \leq p$, and p^* meets all elements of $\mathcal{A}$. This completes the proof of the σ-Baireness.

Next, to prove that $\mathbb{P}$ is ω_1-stationary preserving, arbitrarily take $p \in \mathbb{P}$, a stationary $R \subseteq \omega_1$ and a $\mathbb{P}$-name $\dot{C}$ for a club in ${\omega_1}^V$. It suffices to find $p^* \leq p$ and $\xi \in R$ such that $p^* \Vdash$ "$\xi \in \dot{C}$".

By ($*$) we can take $M \in [\mathcal{H}_\theta]^\omega$ and $p^* \leq p$ such that $\mathbb{P}, p, R, \dot{C} \in M \prec \langle \mathcal{H}_\theta, \in \rangle$, such that $M \cap \omega_1 \in R$ and such that p^* is a strongly $(M, \mathbb{P})$-generic condition. Let $\xi := M \cap \omega_1$. Then $\xi \in R$, and $p^* \Vdash$ "$\xi \in \dot{C}$" by the standard argument. $\square$

3.2 Variant of diamond principle in $[\omega_2]^\omega$

In the proof of Thm.1.4 we use a certain diamond principle in $[\omega_2]^\omega$. Here we prove that MM implies it:

Notation. *For $X \subseteq [\omega_2]^\omega$ we say that $\sup \restriction X$ is injective if $\sup x \neq \sup y$ for any distinct $x, y \in X$.*

Lemma 3.4. *Assume MM. Let S be a stationary subset of E^2_0. Then there are $X \subseteq [\omega_2]^\omega$ and a sequence $\langle \mathcal{B}_x \mid x \in X \rangle$ with the following properties:*

(i) $\sup x \notin x$ *for each* $x \in X$, $\{\sup x \mid x \in X\} = S$, *and* $\sup \restriction X$ *is injective.*

(ii) $\mathcal{B}_x$ *is a countable family of subsets of x for each $x \in X$.*

(iii) For every sufficiently large regular cardinal θ, the set of all $M \in [\mathcal{H}_\theta]^\omega$ such that

- $M \cap \omega_2 \in X$,
- $\mathcal{B}_{M \cap \omega_2} = \{B \cap M \mid B \in \mathcal{P}(\omega_2) \cap M\}$,

is projectively stationary.

Lem.3.4 follows from Fact 3.5, 3.6 and Lem.3.7 below:

Fact 3.5 (Foreman-Magidor-Shelah [6]). MM *implies that* $2^{\omega_1} = \omega_2$.

Fact 3.6 (Shelah [16]). *If $2^{\omega_1} = \omega_2$, then $\diamondsuit_{\omega_2}(S)$ holds for every stationary $S \subseteq E^2_0$.*

Lemma 3.7. *Suppose that S is a stationary subset of E^2_0 and that $\diamondsuit_{\omega_2}(S)$ holds. Then there exist X and $\langle \mathcal{B}_x \mid x \in X \rangle$ satisfying (i)–(iii) in Lem.3.4.*

We prove Lem.3.7. For this we need the following lemma:

Lemma 3.8. *Suppose that S is a stationary subset of E^2_0 and that $\diamondsuit_{\omega_2}(S)$ holds. Then there exist $X \subseteq [\omega_2]^\omega$ and a sequence $\langle b_x \mid x \in X \rangle$ such that*

(i) $\sup x \notin x$ *for each* $x \in X$, $\{\sup x \mid x \in X\} = S$, *and* $\sup \restriction X$ *is injective,*

(ii) $b_x \subseteq x$ *for each* $x \in X$,

(iii) for every $B \subseteq \omega_2$ the set $\{x \in X \mid b_x = B \cap x\}$ is projectively stationary.

Proof. We may assume that $S \subseteq E^2_0 \setminus \omega_1$. By $\diamondsuit_{\omega_2}(S)$ there exists a sequence $\langle R_\alpha, f_\alpha, b'_\alpha \mid \alpha \in S\rangle$ with the following properties:

- For each $\alpha \in S$, R_α is a stationary subset of ω_1, f_α is a function from ${}^{<\omega}\alpha$ to α, and $b'_\alpha \subseteq \alpha$.
- If R is a stationary subset of ω_1, F is a function from ${}^{<\omega}\omega_2$ to ω_2, and $B \subseteq \omega_2$, then there exists $\alpha \in S$ such that $R_\alpha = R$, $f_\alpha = F \restriction {}^{<\omega}\alpha$ and $b'_\alpha = B \cap \alpha$.

For each $\alpha \in S$, take $x_\alpha \in [\alpha]^\omega$ such that $\sup x_\alpha = \alpha$, $x_\alpha \cap \omega_1 \in R_\alpha$ and x_α is closed under f_α. We can take such x_α because $\alpha \in E^2_0 \setminus \omega_1$ and R_α is stationary. Let $X := \{x_\alpha \mid \alpha \in S\}$. Moreover let $b_x := b'_{\sup x} \cap x$ for each $x \in X$. (Hence $b_{x_\alpha} = b'_\alpha \cap x_\alpha$ for each $\alpha \in S$.)

We show that these X and $\langle b_x \mid x \in X\rangle$ witness the lemma. Clearly they satisfy (i) and (ii). We check (iii).

Fix $B \subseteq \omega_2$. It suffices to show that for every stationary $R \subseteq \omega_1$ and every function $F : {}^{<\omega}\omega_2 \to \omega_2$ there exists $x \in X$ such that $x \cap \omega_1 \in R$, x is closed under F and $b_x = B \cap x$.

Take an arbitrary stationary $R \subseteq \omega_1$ and an arbitrary function $F : {}^{<\omega}\omega_2 \to \omega_2$. Then there exists $\alpha \in S$ with $R_\alpha = R$, $f_\alpha = F \restriction {}^{<\omega}\alpha$ and $b'_\alpha = B \cap \alpha$. Then $x_\alpha \cap \omega_1 \in R$, x_α is closed under F, and $b_{x_\alpha} = b'_\alpha \cap x_\alpha = B \cap x_\alpha$ by the choice of x_α. Moreover $x_\alpha \in X$. Hence x_α is as desired. □

Now we prove Lem.3.7:

Proof of Lem.3.7. Before starting we prepare a notation. For each $D \subseteq \mathrm{On} \times \mathrm{On}$ and each $\gamma \in \mathrm{On}$, let $D(\gamma)$ denote the set $\{\beta \in \mathrm{On} \mid \langle \gamma, \beta\rangle \in D\}$.

Now we start the proof. By Lem.3.8 we can easily take $X \subseteq [\omega_2]^\omega$ and a sequence $\langle d_x \mid x \in X\rangle$ such that

(i′) $\sup x \notin x$ for each $x \in X$, $\{\sup x \mid x \in X\} = S$, and $\sup \restriction X$ is injective.

(ii′) $d_x \subseteq x \times x$,

(iii′) for every $D \subseteq \omega_2 \times \omega_2$ the set $\{x \in X \mid d_x = D \cap (x \times x)\}$ is projectively stationary.

For each $x \in X$ let $\mathcal{B}_x = \{d_x(\gamma) \mid \gamma \in x\}$. We show that X and $\langle \mathcal{B}_x \mid x \in X\rangle$ witness Lem.3.7. Clearly (i) and (ii) in Lem.3.4 hold. We check (iii).

Let θ be a sufficiently large regular cardinal. Take an arbitrary stationary $R \subseteq \omega_1$ and an arbitrary function $F : {}^{<\omega}\mathcal{H}_\theta \to \mathcal{H}_\theta$. It suffices to find

$M \in [\mathcal{H}_\theta]^\omega$ such that $M \cap \omega_1 \in R$, M is closed under F, $M \cap \omega_2 \in X$ and $\mathcal{B}_{M\cap\omega_2} = \{B \cap M \mid B \in \mathcal{P}(\omega_2) \cap M\}$

First take $N \subseteq \mathcal{H}_\theta$ such that $|N| = \omega_2 \subseteq N$, $\mathcal{P}(\omega_2) \cap N \neq \emptyset$ and N is closed under F. Moreover take an enumeration $\langle B_\gamma \mid \gamma \in \omega_2 \rangle$ of $\mathcal{P}(\omega_2) \cap N$. For each $x \in [\omega_2]^\omega$ let

$$M_x := \mathrm{cl}_F(x \cup \{B_\gamma \mid \gamma \in x\}) \subseteq N ,$$

where $\mathrm{cl}_F(a)$ denotes the closure of a under F. Then let C be the set of all $x \in [\omega_2]^\omega$ with $M_x \cap \omega_2 = x$ and $\mathcal{P}(\omega_2) \cap M_x = \{B_\gamma \mid \gamma \in x\}$. Finally let D be a subset of $\omega_2 \times \omega_2$ such that $D(\gamma) = B_\gamma$ for each $\gamma \in \omega_2$.

Note that C is a club in $[\omega_2]^\omega$. Hence, by (iii′), there exists $x \in X \cap C$ such that $x \cap \omega_1 \in R$ and $d_x = D \cap (x \times x)$. Then $M_x \in [\mathcal{H}_\theta]^\omega$, $M_x \cap \omega_2 = x \in X$, $M_x \cap \omega_1 = x \cap \omega_1 \in R$, and M_x is closed under F. Moreover

$$\begin{aligned}
\mathcal{B}_{M_x\cap\omega_2} &= \mathcal{B}_x = \{d_x(\gamma) \mid \gamma \in x\} = \{D(\gamma) \cap x \mid \gamma \in x\} \\
&= \{B_\gamma \cap x \mid \gamma \in x\} = \{B \cap x \mid B \in \mathcal{P}(\omega_2) \cap M_x\} \\
&= \{B \cap M_x \mid B \in \mathcal{P}(\omega_2) \cap M_x\} .
\end{aligned}$$

Thus M_x is as desired. This completes the proof. □

3.3 Proof of Thm.1.4

Before proving Thm.1.4 we present a poset to which we apply MM:

Definition 3.9. Suppose that $S \subseteq E^2_0$ and that $\vec{c} = \langle c_\alpha \mid \alpha \in S \rangle$ is a $\square(S)$-sequence. (S may be bounded in ω_2.) Then let $\mathbb{P}(\vec{c})$ be the following poset:

- $\mathbb{P}(\vec{c}) = S$.
- $\alpha \leq_{\mathbb{P}(\vec{c})} \beta$ if and only if $\beta \in \mathrm{Lim}(c_\alpha) \cup \{\alpha\}$ for each $\alpha, \beta \in S$.

For $g \subseteq \mathbb{P}(\vec{c})$ let

$$c_g := \bigcup_{\alpha\in g} c_\alpha .$$

Note that $\alpha \leq_{\mathbb{P}(\vec{c})} \beta$ if and only if c_β is an initial segment of c_α. The following lemma can be easily proved. The proof is left to the reader:

Lemma 3.10. *Let S be a subset of E^2_0 and $\vec{c} = \langle c_\alpha \mid \alpha \in S \rangle$ be a $\square(S)$-sequence.*

(1) Suppose that g is a filter on $\mathbb{P}(\vec{c})$. Then c_g is a club in $\sup c_g$ of order type $\leq \omega_1$. Moreover if $\beta \in \mathrm{Lim}(c_g)$, then $\beta \in S$, and $c_\beta = c_g \cap \beta$.

(2) Suppose that the following $(**)$ *holds:*

$(**)$ $\mathbb{P}(\vec{c}) \setminus \gamma$ *is dense in* $\mathbb{P}(\vec{c})$ *for every* $\gamma < \omega_2$.

Let θ *be a sufficiently large regular cardinal and* M *be an elementary submodel of* $\langle \mathcal{H}_\theta, \in, \vec{c} \rangle$. *Suppose also that* g *is an* $(M, \mathbb{P}(\vec{c}))$*-generic filter. Then* $\sup c_g = \sup(M \cap \omega_2)$.

Now we prove Thm.1.4:

Proof of Thm.1.4. Assume MM. We want to prove that $\square(S)$ holds for some $S \subseteq \mathrm{Lim}(\omega_2)$ with $S \cap E^2_1$ stationary.

Our proof is composed of two steps. First we construct a $\square(E^2_0)$-sequence $\vec{c} = \langle c_\alpha \mid \alpha \in E^2_0 \rangle$ so that $\mathbb{P}(\vec{c})$ satisfies $(*)$ in Lem.3.3 and $(**)$ in Lem.3.10. Then, using Lem.2.2, we show that $\vec{c}$ can be extended to a $\square(S)$-sequence for some $S \subseteq \mathrm{Lim}(\omega_2)$ with $S \cap E^2_1$ stationary.

(Step 1) Construction of $\vec{c} = \langle c_\alpha \mid \alpha \in E^2_0 \rangle$

First take a stationary partition $\langle T_\beta \mid \beta \in E^2_0 \rangle$ of E^2_0, i.e. $\langle T_\beta \mid \beta \in E^2_0 \rangle$ is a pairwise disjoint sequence of stationary subsets of E^2_0 such that $\bigcup \{ T_\beta \mid \beta \in E^2_0 \} = E^2_0$. By Lem.3.4, for each $\beta \in E^2_0$ we can take $X_\beta \subseteq [\omega_2]^\omega$ and $\langle \mathcal{B}^\beta_x \mid x \in X_\beta \rangle$ with the following properties:

(i) $\sup x \notin x$ for each $x \in X_\beta$, $\{ \sup x \mid x \in X_\beta \} = T_\beta$, and $\sup \restriction X_\beta$ is injective.

(ii) $\mathcal{B}^\beta_x$ is a countable family of subsets of x for each $x \in X_\beta$.

(iii) For every sufficiently large regular cardinal θ the set of all $M \in [\mathcal{H}_\theta]^\omega$ such that

- $M \cap \omega_2 \in X_\beta$,
- $\mathcal{B}^\beta_{M \cap \omega_2} = \{ B \cap M \mid B \in \mathcal{P}(\omega_2) \cap M \}$,

is projectively stationary.

By induction on $\alpha \in E^2_0$ we construct a $\square(E^2_0)$-sequence $\vec{c} = \langle c_\alpha \mid \alpha \in E^2_0 \rangle$. Suppose that $\alpha \in E^2_0$ and that $\langle c_\beta \mid \beta \in E^2_0 \cap \alpha \rangle$ has been defined to be a $\square(E^2_0 \cap \alpha)$-sequence. Then take c_α as follows:

First let

$\beta_\alpha :=$ the unique element of E^2_0 with $\alpha \in T_{\beta_\alpha}$,

$x_\alpha :=$ the unique element of X_{β_α} with $\sup x_\alpha = \alpha$.

If $\beta_\alpha \notin x_\alpha$ or there exists $\beta \in E^2_0 \cap x_\alpha$ with $\mathrm{Lim}(c_\beta) \not\subseteq x_\alpha$, then let c_α be an arbitrary unbounded subset of α of order type ω.

Suppose that $\beta_\alpha \in x_\alpha$ and that $\mathrm{Lim}(c_\beta) \subseteq x_\alpha$ for every $\beta \in E^2_0 \cap x_\alpha$. Then note that $\langle c_\beta \mid \beta \in E^2_0 \cap x_\alpha \rangle$ is a $\Box(E^2_0 \cap x_\alpha)$-sequence. Let

$$\mathbb{P}_\alpha \ := \ \mathbb{P}(\langle c_\beta \mid \beta \in E^2_0 \cap x_\alpha \rangle).$$

Note also that $\beta_\alpha \in \mathbb{P}_\alpha \subseteq x_\alpha$.

Recall that $\mathcal{B}^{\beta_\alpha}_{x_\alpha}$ is a countable family of subsets of x_α. Hence we can take a filter g_α on $\mathbb{P}_\alpha$ such that

(iv) $\beta_\alpha \in g_\alpha$,

(v) $g_\alpha \cap b \neq \emptyset$ for every $b \in \mathcal{B}^{\beta_\alpha}_{x_\alpha}$ which is a maximal antichain in $\mathbb{P}_\alpha$.

If $\sup c_{g_\alpha} = \alpha$, then let $c_\alpha := c_{g_\alpha}$. Otherwise, take an unbounded $c \subseteq \alpha$ such that $\mathrm{otp}(c) = \omega$ and $\beta_\alpha = \min c$, and let $c_\alpha := c_{\beta_\alpha} \cup c$.

This completes the choice of c_α. Using Lem.3.10 (1), it is easy to check that $\langle c_\beta \mid \beta \in E^2_0 \cap \alpha+1 \rangle$ is a $\Box(E^2_0 \cap \alpha+1)$-sequence. Note that if $\beta_\alpha \in x_\alpha$ and $\mathrm{Lim}(c_\beta) \subseteq x_\alpha$ for every $\beta \in E^2_0 \cap x_\alpha$, then $\beta_\alpha \in \mathrm{Lim}(c_\alpha)$.

Now we have constructed a $\Box(E^2_0)$-sequence $\vec{c} = \langle c_\alpha \mid \alpha \in E^2_0 \rangle$. We show that $\mathbb{P}(\vec{c})$ satisfies $(*)$ and $(**)$:

Claim 1. $\mathbb{P}(\vec{c})$ *satisfies* $(**)$ *in Lem.3.10.*

Proof of Claim 1. Take an arbitrary $\beta^* \in E^2_0$ and an arbitrary $\gamma^* < \omega_2$. We must find $\alpha^* \in E^2_0 \setminus \gamma^*$ with $\alpha^* \leq_{\mathbb{P}(\vec{c})} \beta^*$.

Let θ be a sufficiently large regular cardinal. Because X_{β^*} is stationary in $[\omega_2]^\omega$, we can take $M \prec \langle \mathcal{H}_\theta, \in, \vec{c} \rangle$ such that $\beta^*, \gamma^* \in M$ and $M \cap \omega_2 \in X_{\beta^*}$. Let $\alpha^* := \sup(M \cap \omega_2)$. Clearly $\alpha^* \in E^2_0 \setminus \gamma^*$.

Note that $\alpha^* \in T_{\beta^*}$. So $\beta_{\alpha^*} = \beta^*$. Note also that $x_{\alpha^*} = M \cap \omega_2$. Hence $\beta_{\alpha^*} \in x_{\alpha^*}$ by the choice of M. Moreover $\mathrm{Lim}(c_\beta) \subseteq x_{\alpha^*}$ for every $\beta \in E^2_0 \cap x_{\alpha^*}$ because $M \prec \langle \mathcal{H}_\theta, \in, \vec{c} \rangle$ and each c_β is a countable set. Then $\beta^* = \beta_{\alpha^*} \in \mathrm{Lim}(c_{\alpha^*})$ by the choice of c_{α^*}. Thus $\alpha^* \leq_{\mathbb{P}(\vec{c})} \beta^*$. $\Box_{\text{Claim1}}$

Claim 2. $\mathbb{P}(\vec{c})$ *satisfies* $(*)$ *in Lem.3.3.*

Proof of Claim 2. Suppose that θ is a sufficiently large regular cardinal and that $\beta^* \in E^2_0 = \mathbb{P}(\vec{c})$. We prove that there are projectively stationary many $M \in [\mathcal{H}_\theta]^\omega$ such that a strongly $(M, \mathbb{P}(\vec{c}))$-generic condition below β^* exists.

Let Y be the set of all $M \in [\mathcal{H}_\theta]^\omega$ such that

(vi) $\beta^*, \vec{c} \in M \prec \langle \mathcal{H}_\theta, \in \rangle$,

(vii) $M \cap \omega_2 \in X_{\beta^*}$

(viii) $\mathcal{B}^{\beta^*}_{M\cap\omega_2} = \{B \cap M \mid B \in \mathcal{P}(\omega_2) \cap M\}$.

Then Y is projectively stationary in $[\mathcal{H}_\theta]^\omega$ by (iii). It suffices to show that $\sup(M \cap \omega_2)$ is a strongly $(M, \mathbb{P}(\vec{c}))$-condition below β^* for each $M \in Y$.

Fix $M \in Y$, and let $\alpha^* := \sup(M \cap \omega_2)$. Then $\beta_{\alpha^*} = \beta^*$, and $x_{\alpha^*} = M \cap \omega_2$. Hence $\beta_{\alpha^*} \in x_{\alpha^*}$, and $\mathrm{Lim}(c_\beta) \subseteq x_{\alpha^*}$ for each $\beta \in E^2_0 \cap x_{\alpha^*}$. Note also that $\mathbb{P}_{\alpha^*} = \mathbb{P}(\vec{c}) \cap M$. Then g_{α^*} is a filter on $\mathbb{P}(\vec{c}) \cap M$ containing β^* by (iv). Moreover g_{α^*} is an $(M, \mathbb{P}(\vec{c}))$-generic filter by (v) and (viii).

Here note that $\sup c_{g_{\alpha^*}} = \sup(M \cap \omega_2) = \alpha^*$ by Lem.3.10 (2) and Claim 1. Hence $c_{\alpha^*} = c_{g_{\alpha^*}}$, and so $g_{\alpha^*} = \{\beta \in \mathbb{P}(\vec{c}) \cap M \mid \beta \geq_{\mathbb{P}(\vec{c})} \alpha^*\}$. Therefore α^* is a strongly $(M, \mathbb{P}(\vec{c}))$-generic condition below β^*. $\square_{\text{Claim2}}$

$\dashv_{\text{Step1}}$

(Step 2) Extension of $\vec{c}$.

Let θ be a sufficiently large regular cardinal, and let Z be the set of all $N \in [\mathcal{H}_\theta]^{\omega_1}$ such that

(ix) $N \prec \langle \mathcal{H}_\theta, \in, \vec{c}\rangle$,

(x) N is i.a. of length ω_1,

(xi) there exists an $(N, \mathbb{P}(\vec{c}))$-generic filter.

By Claim 2 and Lem.3.3, $\mathbb{P}(\vec{c})$ is ω_1-stationary preserving and σ-Baire. Hence Z is stationary in $[\mathcal{H}_\theta]^{\omega_1}$ by MM and Lem.2.2.

Note that $\sup(N \cap \omega_2) \in E^2_1$ for each $N \in Z$ because N is i.a. of length ω_1. Hence $S' := \{\sup(N \cap \omega_2) \mid N \in Z\}$ is a stationary subset of E^2_1.

For each $\alpha \in S'$ choose $N_\alpha \in Z$ with $\sup(N_\alpha \cap \omega_2) = \alpha$ and an $(N_\alpha, \mathbb{P}(\vec{c}))$-generic filter g_α. Moreover let $c_\alpha := c_{g_\alpha}$ for each $\alpha \in S'$. Note that $\sup c_\alpha = \alpha$ by Claim 1 and Lem.3.10 (2). Then, by Lem.3.10 (1), c_α is a club of α of order type ω_1, and if $\beta \in \mathrm{Lim}(c_\alpha)$, then $\beta \in E^2_0$, and $c_\beta = c_\alpha \cap \beta$.

Now let $S := E^2_0 \cup S'$. Then $S \cap E^2_1 = S'$ is stationary. Moreover $\langle c_\alpha \mid \alpha \in S\rangle$ is a $\square(S)$-sequence. $\dashv_{\text{Step2}}$

This completes the proof of Thm.1.4. $\square$

4 Strong stationary reflection principle

In §5 and §6 we will discuss the consistency of the failure of the partial square. For this we will use the following stationary reflection principle, which implies the failure of the partial square:

Definition 4.1. Let κ be a successor cardinal of some regular uncountable cardinal δ. Then let

$$\mathsf{OSR}^*_\kappa \equiv \text{For any stationary } S \subseteq \mathcal{E}^\kappa_{<\delta} \text{ there exists a club } C \subseteq \kappa \text{ such that } S \cap \alpha \text{ is stationary in } \alpha \text{ for any } \alpha \in C \cap \mathcal{E}^\kappa_\delta.$$

$\mathsf{OSR}^*_{\omega_2}$ is introduced by Magidor [12], and OSR^*_κ is its straight forward generalization.

First we prove that $\mathsf{OSR}^*_{\delta+}$ implies the failure of $\square^{\mathrm{p}}_\delta$. This can be shown by the same argument as the fact that the usual stationary reflection principle for stationary subsets of E^2_0 implies the failure of $\square_{\omega_1}$, which is also shown by [12]:

Lemma 4.2. *Let δ be a regular uncountable cardinal, and let $\kappa := \delta^+$. Then OSR^*_κ implies the failure of $\square^{\mathrm{p}}_\delta$.*

Proof. Assume $\square^{\mathrm{p}}_\delta$. We prove that OSR^*_κ fails.

Let $\langle c_\alpha \mid \alpha \in S\rangle$ be a witness of $\square^{\mathrm{p}}_\delta$. First note that $S \cap \mathcal{E}^\kappa_{<\delta}$ is stationary in κ because $S \cap \mathcal{E}^\kappa_\delta$ is stationary in κ, and $S \cap \alpha$ contains a club $c_\alpha \subseteq \alpha$ for any $\alpha \in S \cap \mathcal{E}^\kappa_\delta$. Moreover the correspondence $\beta \mapsto \mathrm{otp}(c_\beta)$ is regressive on $S \cap \mathcal{E}^\kappa_{<\delta}$. Hence by Fodor's lemma there exist ξ^* and a stationary $S^* \subseteq S \cap \mathcal{E}^\kappa_{<\delta}$ such that $\mathrm{otp}(c_\beta) = \xi^*$ for all $\beta \in S^*$.

Note that $\mathrm{Lim}(c_\alpha) \cap S^*$ contains at most one element, the ξ^*-th element of c_α, for any $\alpha \in S \cap \mathcal{E}^\kappa_\delta$. Thus $S^* \cap \alpha$ is nonstationary in α for all $\alpha \in S \cap \mathcal{E}^\kappa_\delta$. Here recall that S^* is a stationary subset of $\mathcal{E}^\kappa_{<\delta}$ and that $S \cap \mathcal{E}^\kappa_\delta$ is stationary. Therefore S^* witnesses the failure of OSR^*_κ. $\square$

In the rest of this section we discuss the construction of models of OSR^*_κ.

Magidor [12] proved that, after a weakly compact cardinal is Lévy collapsed to ω_2, there exists a $< \omega_2$-Baire ω_3-c.c. poset forcing $\mathsf{OSR}^*_{\omega_2}$. If a weakly compact cardinal is Lévy collapsed to ω_2, then $\mathsf{FA}^{++}_{\omega_1}$ holds for all σ-closed posets of size $\leq \omega_2 = 2^{\omega_1}$ (See Lem.6.2). In fact this forcing axiom implies the existence of $< \omega_2$-Baire ω_3-c.c. poset forcing $\mathsf{OSR}^*_{\omega_2}$. Here we prove the following proposition which generalizes this fact:

Proposition 4.3. *Let δ be a regular uncountable cardinal. Assume that FA^{++}_δ holds for all $< \delta$-directed closed posets of size $\leq 2^\delta$ and that $\mathcal{E}^{\delta^+}_{<\delta} \in I[\delta^+]$. Then there is a $(2^\delta)^+$-c.c. $<\delta^+$-Baire $<\delta$-directed closed poset which forces $\mathsf{OSR}^*_{\delta+}$.*

Note that if $\delta = \omega_1$, then $\mathcal{E}^{\delta^+}_{<\delta} = E^2_0 \in I[\omega_2] = I[\delta^+]$. Thus the second assumption of Prop.4.3 holds if $\delta = \omega_1$.

The rest of this section is devoted to the proof of Prop.4.3. In the rest of this section fix a regular uncountable cardinal δ, and let $\kappa := \delta^+$.

The poset forcing OSR^*_κ will be one of the following OSR^*_κ-iterations, which are essentially δ-support iterations of club shootings through κ:

Definition 4.4. We say that $\langle \mathbb{B}_\mu, \dot{S}_\nu \mid \mu \leq \upsilon, \nu < \upsilon \rangle$ is a OSR^*_κ*-iteration of length* υ if it satisfies the following:

(i) Each $\mathbb{B}_\mu$ is a poset, and each $\dot{S}_\nu$ is a nice $\mathbb{B}_\nu$-name for a stationary subset of $(\mathcal{E}^\kappa_{<\delta})^V$.

(ii) Each $\mathbb{B}_\mu$ is the poset of all partial functions p on μ such that

- $|\operatorname{dom}(p)| < \kappa$,
- $p \restriction \nu \in \mathbb{B}_\nu$ for all $\nu < \mu$,
- if $\nu \in \operatorname{dom}(p)$, then $p(\nu)$ is a closed bounded subset of κ, and $p \restriction \nu \Vdash$ "$\dot{S}_\nu \cap \alpha$ is stationary" for any $\alpha \in p(\nu) \cap \mathcal{E}^\kappa_\delta$.

(iii) For $p, q \in \mathbb{B}_\mu$, $p \leq q$ if and only if $\operatorname{dom}(p) \supseteq \operatorname{dom}(q)$, and $p(\nu)$ end-extends $q(\nu)$ for every $\nu \in \operatorname{dom}(q)$.

If $\langle \mathbb{B}_\mu, \dot{S}_\nu \mid \mu \leq \upsilon, \nu < \upsilon \rangle$ is an OSR^*_κ-iteration, then $\mathbb{B}_\upsilon$ is also called an OSR^*_κ-iteration.

First we present basic properties of OSR^*_κ-iterations:

Lemma 4.5. *All* OSR^*_κ*-iterations have the* $(2^\delta)^+$*-c.c.*

Proof. Suppose that A is a subset of $\mathbb{B}_\upsilon$ with $|A| = (2^\delta)^+$. We find distinct $p, q \in A$ which are compatible.

By the Δ-system lemma we may assume that there is $e \subseteq \upsilon$ of size $\leq \delta$ such that $\operatorname{dom}(p) \cap \operatorname{dom}(q) = e$ for all distinct $p, q \in A$. Note that $|\{p \restriction e \mid p \in A\}| \leq 2^\delta < |A|$. Hence we can take distinct $p, q \in A$ such that $p \restriction e = q \restriction e$. Then $p \cup q$ is a common extension of p and q. $\square$

Next we examine the closure property of OSR^*_κ-iterations. It is easy to see that OSR^*_κ-iterations are $<\delta$-directed closed. Below we prove a more general fact for the later use. More precisely, we prove that appropriate complete suborders of OSR^*_κ-iterations and their quotients of OSR^*_κ-iterations are both $<\delta$-directed closed:

Definition 4.6. Suppose that $\vec{\mathbb{B}} := \langle \mathbb{B}_\mu, \dot{S}_\nu \mid \mu \leq \upsilon, \nu < \upsilon \rangle$ is an OSR^*_κ-iteration. $U \subseteq \upsilon$ is said to be $\vec{\mathbb{B}}$-*complete subset of* υ if for any $\nu \in U$ and any $(\check{\alpha}, p) \in \dot{S}_\nu$ we have $\mathrm{dom}(p) \subseteq U$. For a $\mathbb{B}$-complete $U \subseteq \upsilon$ and each $\mu \leq \upsilon$ let $\mathbb{B}_{\mu,U}$ be the suborder of $\mathbb{B}_\mu$ such that

$$\mathbb{B}_{\mu,U} \;=\; \{p \in \mathbb{B}_\mu \mid \mathrm{dom}(p) \subseteq U\} \; .$$

Lemma 4.7. *Suppose that $\vec{\mathbb{B}} := \langle \mathbb{B}_\mu, \dot{S}_\nu \mid \mu \leq \upsilon, \nu < \upsilon \rangle$ is an OSR^*_κ-iteration and that U is a $\vec{\mathbb{B}}$-complete subset of υ. Then the following hold for every $\mu \leq \upsilon$:*

(1) $p \restriction U \in \mathbb{B}_{\mu,U}$ for every $p \in \mathbb{B}_\mu$.

(2) The map $p \mapsto p \restriction U$ is a projection from $\mathbb{B}_\mu$ to $\mathbb{B}_{\mu,U}$.

(3) $\mathbb{B}_{\mu,U}$ is $<\delta$-directed closed.

(4) $\Vdash_{\mathbb{B}_{\mu,U}}$ "$\mathbb{B}_\mu / \dot{H}_{\mu,U}$ is $<\delta$-directed closed", where $\dot{H}_{\mu,U}$ is the canonical $\mathbb{B}_{\mu,U}$-name for a $\mathbb{B}_{\mu,U}$-generic filter.

Proof. We show the lemma by induction on $\mu \leq \upsilon$. Suppose that $\mu \leq \upsilon$ and that the lemma holds for all $\nu < \mu$. We prove the lemma for μ.

(1) Take an arbitrary $p \in \mathbb{B}_\mu$. It suffices to show that $r := p \restriction U \in \mathbb{B}_\mu$.

First suppose that μ is a limit ordinal. To see that $r \in \mathbb{B}_\mu$, it suffices to show that $r \restriction \nu \in \mathbb{B}_\nu$ for all $\nu < \mu$ and that $|\mathrm{dom}(r)| < \kappa$. The latter is clear because r is a restriction of p which belongs to $\mathbb{B}_\mu$. The former easily follows from the induction hypothesis (1) for $\nu < \mu$.

Next suppose that μ is a successor ordinal, and let $\nu := \mu - 1$. Then note that $r \restriction \nu \in \mathbb{B}_\nu$. This implies that $r \in \mathbb{B}_\mu$ if $\nu \notin \mathrm{dom}(r)$. So suppose that $\nu \in \mathrm{dom}(r)$. It suffices to show that $r \restriction \nu \Vdash_{\mathbb{B}_\nu}$ "$\dot{S}_\nu \cap \alpha$ is stationary" for all $\alpha \in r(\nu) \cap \mathcal{E}^\kappa_\delta$.

Fix $\alpha \in r(\nu) \cap \mathcal{E}^\kappa_\delta$. First note that $\dot{S}_\nu$ is a $\mathbb{B}_{\nu,U}$-name because $\nu \in U$, and U is $\vec{\mathbb{B}}$-complete. Moreover $p \restriction \nu \Vdash_{\mathbb{B}_\nu}$ "$\dot{S}_\nu \cap \alpha$ is stationary" because $p \in \mathbb{B}_\mu$. Then

$$r \restriction \nu = (p \restriction \nu) \restriction U \Vdash_{\mathbb{B}_{\nu,U}} \text{“}\dot{S}_\nu \cap \alpha \text{ is stationary”}$$

because the restriction to U is a projection from $\mathbb{B}_\nu$ to $\mathbb{B}_{\nu,U}$ by the induction hypothesis (2). But $\mathrm{cf}(\alpha) = \delta$, and $V^{\mathbb{B}_\nu}$ is a $<\delta$-closed forcing extension of $V^{\mathbb{B}_{\nu,U}}$ by the induction hypothesis (4) for ν. Therefore $r \restriction \nu \Vdash_{\mathbb{B}_\nu}$ "$\dot{S}_\nu \cap \alpha$ is stationary".

(2) Clearly the map $p \mapsto p \restriction U$ from $\mathbb{B}_\mu$ to $\mathbb{B}_{\mu,U}$ is order preserving and surjective. We show that if $p \in \mathbb{B}_\mu$ and $q \leq p \restriction U$ in $\mathbb{B}_{\mu,U}$, then there exists $p^* \leq p$ in $\mathbb{B}_\mu$ with $p^* \restriction U = q$.

Suppose that $p \in \mathbb{B}_\mu$ and that $q \leq p \restriction U$ in $\mathbb{B}_{\mu,U}$. Let $p^* := q \cup (p \restriction \mu \setminus U)$. Then by induction on $\nu \leq \mu$ we can easily prove that $p^* \restriction \nu \in \mathbb{B}_\nu$ and that $p^* \restriction \nu \leq p \restriction \nu, q \restriction \nu$ in $\mathbb{B}_\nu$. Then p^* is as desired.

(3) Suppose that $A \subseteq \mathbb{B}_{\mu,U}$ is directed and that $|A| < \delta$. We must find a lower bound p^* of A.

Let $c_\nu := \bigcup\{p(\nu) \mid p \in A \wedge \nu \in \mathrm{dom}(p)\}$ for each $\nu \in \bigcup\{\mathrm{dom}(p) \mid p \in A\}$, and let p^* be a function on $\bigcup\{\mathrm{dom}(p) \mid p \in A\}$ such that $p^*(\nu) = c_\nu \cup \{\sup c_\nu\}$. First note that each $p^*(\nu)$ is a closed bounded subset of κ. Note also that if $\sup c_\nu \notin c_\nu$, then $\sup c_\nu \notin \mathcal{E}^\kappa_\delta$ because $|A| < \delta$. Then by induction on $\nu \leq \mu$ we can easily prove that $p^* \restriction \nu \in \mathbb{B}_\nu$ and that $p^* \restriction \nu$ is a lower bound of $\{p \restriction \nu \mid p \in A\}$. Therefore p^* is as desired.

(4) Suppose that $H_{\mu,U}$ is a $\mathbb{B}_{\mu,U}$-generic filter over V. In $V[H_{\mu,U}]$ suppose that $A \subseteq \mathbb{B}_\mu / H_{\mu,U}$ is directed and that $|A| < \delta$. We find a lower bound p^* of A in $\mathbb{B}_\mu / H_{\mu,U}$. Here recall that $\mathbb{B}_\mu / H_{\mu,U} = \{p \in \mathbb{B}_\mu \mid p \restriction U \in H_{\mu,U}\}$.

First note that A is directed in $\mathbb{B}_\mu$ by the definition of $\mathbb{B}_\mu / H_{\mu,U}$. Note also that $A \in V$ because $\mathbb{B}_{\mu,U}$ is $<\delta$-closed by (3). In V construct p^* from A as in the proof of (3). That is, let $c_\nu := \bigcup\{p(\nu) \mid p \in A \wedge \nu \in \mathrm{dom}(p)\}$ for each $\nu \in \bigcup\{\mathrm{dom}(p) \mid p \in A\}$, and let p^* be a function on $\bigcup\{\mathrm{dom}(p) \mid p \in A\}$ such that $p^*(\nu) = c_\nu \cup \{\sup c_\nu\}$. Then p^* is a lower bound of A in $\mathbb{B}_\mu$. It suffices to show that $p^* \in \mathbb{B}_\mu / H_{\mu,U}$.

Note that $p^* \restriction U$ is the greatest lower bound of $\{p \restriction U \mid p \in A\}$ in $\mathbb{B}_{\mu,U}$. Moreover $\{p \restriction U \mid p \in A\} \in V$, and $\{p \restriction U \mid p \in A\} \subseteq H_{\mu,U}$ because $A \subseteq \mathbb{B}_\mu / H_{\mu,U}$. Therefore $p^* \restriction U \in H_{\mu,U}$ by the genericity of $H_{\mu,U}$. That is, $p^* \in \mathbb{B}_\mu / H_{\mu,U}$. $\square$

Note that if $\vec{\mathbb{B}} = \langle \mathbb{B}_\mu, \dot{S}_\nu \mid \mu \leq \upsilon, \nu < \upsilon \rangle$ is an OSR^*_κ-iteration, then every $\mu \leq \upsilon$ is a $\vec{\mathbb{B}}$-complete subset of υ. Thus $\mathbb{B}_\mu$ is $<\delta$-directed closed for every $\mu \leq \upsilon$ by Lem.4.7 (3). Moreover if we let $\dot{H}_\mu$ be the canonical $\mathbb{B}_\mu$-name for a $\mathbb{B}_\mu$-generic filter, then $\mathbb{B}_\mu$ forces that $\mathbb{B}_\upsilon / \dot{H}_\mu$ is $<\delta$-directed closed by Lem.4.7 (4).

Hence OSR^*_κ-iterations preserve all cardinals $\leq \delta$. They also preserve all cardinals $\geq (2^\delta)^+$ by Lem.4.5. But in general OSR^*_κ-iterations do not preserve the cardinality of κ. We prove that under the assumption of Prop.4.3 they are $<\kappa$-Baire and so preserve the cardinality of κ:

Lemma 4.8. *Assume that $\mathsf{FA}^{++}_{\delta}$ holds for all $<\delta$-directed closed posets of size $\leq 2^{\delta}$ and that $\mathcal{E}^{\kappa}_{<\delta} \in I[\kappa]$. Then all $\mathsf{OSR}^{*}_{\kappa}$-iterations are $<\kappa$-Baire.*

This easily follows from Lem.4.9 and 4.10 below:

Lemma 4.9. *Let $\mathbb{P}$ be a poset which preserves all cofinalities $\leq \delta$ and all stationary subsets of $\mathcal{E}^{\kappa}_{<\delta}$. Assume that $\mathsf{FA}^{++}_{\delta}(\mathbb{P} * \mathrm{Col}(\delta, \kappa))$ holds and that $\mathcal{E}^{\kappa}_{<\delta} \in I[\kappa]$. Then for any sufficiently large regular cardinal θ and $p \in \mathbb{P}$ there are stationarily many $M \in [\mathcal{H}_{\theta}]^{\delta}$ with the following property:*

(i) $M \cap \kappa \in \kappa$, and $p \in M$.

(ii) There is an $(M, \mathbb{P})$-generic filter g containing p such that

$$\dot{S}_{g,M} := \{\alpha < M \cap \kappa \mid \exists p' \in g,\ p' \Vdash \text{“}\alpha \in \dot{S}\,\text{”}\}$$

is stationary in $M \cap \kappa$ for any $\mathbb{P}$-name $\dot{S} \in M$ for a stationary subset of $\mathcal{E}^{\kappa}_{<\delta}$.

Proof. Suppose that θ is a sufficiently large regular cardinal and that $p \in \mathbb{P}$. Let $\mathcal{M}$ be a structure obtained by adding countable many constants, functions and predicates to $\langle \mathcal{H}_{\theta}, \in, \mathbb{P}, p\rangle$. It suffices to find $M \in [\mathcal{H}_{\theta}]^{\delta}$ such that $M \prec \mathcal{M}$ and such that M satisfies the properties (i) and (ii).

By Lem.2.2 we can take $M \in [\mathcal{H}_{\theta}]^{\delta}$ and an $(M, \mathbb{P}*\mathrm{Col}(\delta, \kappa))$-generic filter $\bar{g}$ containing $p * \check{\emptyset}$ such that $\delta \subseteq M \prec \mathcal{M}$ and such that $\dot{R}_{\bar{g}}$ is stationary in δ for any $\mathbb{P} * \mathrm{Col}(\delta, \kappa)$-name $\dot{R} \in M$ for a stationary subset of δ. Note that $M \cap \kappa \in \kappa$ because $\delta \subseteq M \prec \mathcal{M}$. So M satisfies the property (i) of the lemma.

We show that $g := \{q \in \mathbb{P} \cap M \mid q * \check{\emptyset} \in \bar{g}\}$ witnesses the property (ii) for M. Note that g is an $(M, \mathbb{P})$-generic filter containing p. Take an arbitrary $\mathbb{P}$-name $\dot{S} \in M$ for a stationary subset of $\mathcal{E}^{\kappa}_{<\delta}$. We show that $\dot{S}_{g,M}$ is stationary.

First note that $\mathrm{cf}(\kappa) = \delta$ in $V^{\mathbb{P}*\mathrm{Col}(\delta,\kappa)}$. Let $\dot{f} \in M$ be a $\mathbb{P}*\mathrm{Col}(\delta, \kappa)$-name for an increasing continuous cofinal map from δ to κ, and let $\dot{R} \in M$ be a $\mathbb{P} * \mathrm{Col}(\delta, \kappa)$-name for $\dot{f}^{-1}[\dot{S}]$.

Here note that $\dot{S}$ remains stationary in $V^{\mathbb{P}*\mathrm{Col}(\delta,\kappa)}$ by the $<\delta$-closure of $\mathrm{Col}(\delta, \kappa)$ and the fact that $\mathcal{E}^{\kappa}_{<\delta} \in I[\kappa]$ in $V^{\mathbb{P}}$ (See Fact 2.1). Hence $\dot{R}$ is a $\mathbb{P} * \mathrm{Col}(\delta, \kappa)$-name for a stationary subset of δ. Then $\dot{R}_{\bar{g}}$ is stationary in δ by the choice of M and $\bar{g}$. Moreover $\dot{f}_{\bar{g}} := \{\langle \xi, \alpha\rangle \in \delta \times \kappa \mid \exists r \in \bar{g},\ r \Vdash \text{“}\dot{f}(\xi) = \alpha\text{”}\}$ is an increasing continuous cofinal map from δ to $M \cap \kappa$, and $\dot{f}_{\bar{g}}[\dot{R}_{\bar{g}}] = \dot{S}_{g,M}$. Therefore $\dot{S}_{g,M}$ is stationary in $M \cap \kappa$. □

Lemma 4.10. *Assume that $\mathcal{E}^\kappa_{<\delta} \in I[\kappa]$. Let $\vec{\mathbb{B}} = \langle \mathbb{B}_\mu, \dot{S}_\nu \mid \mu < \upsilon, \nu < \upsilon \rangle$ be an OSR^*_κ-iteration, U be a $\vec{\mathbb{B}}$-complete subset of υ and p be a condition in $\mathbb{B}_{\upsilon,U}$. Moreover let θ be a sufficiently large regular cardinal. Assume that $M \in [\mathcal{H}_\theta]^\delta$, that $M \prec \langle \mathcal{H}_\theta, \in, \vec{\mathbb{B}}, U, p \rangle$ and that M satisfies the properties (i) and (ii) in Lem.4.9 for $\mathbb{P} = \mathbb{B}_{\upsilon,U}$. Let g be an $(M, \mathbb{B}_{\upsilon,U})$-generic filter witnessing the property (ii). Then g has a lower bound in $\mathbb{B}_{\upsilon,U}$.*

Proof. For each $\nu \in M \cap U$ let $c_\nu := \bigcup\{q(\nu) \mid q \in g\}$. Then c_ν is club in $M \cap \kappa$ for every $\nu \in M \cap U$ because g is an $(M, \mathbb{B}_{\upsilon,U})$-generic filter. Let $\gamma^* := M \cap \kappa$, and let p^* be a function on $M \cap U$ such that $p^*(\nu) = c_\nu \cup \{\gamma^*\}$. It suffices to show that $p^* \in \mathbb{B}_\upsilon$. (Then $p^* \in \mathbb{B}_{\upsilon,U}$, and p^* is a lower bound of g clearly.)

By induction on $\mu \leq \upsilon$ we prove that $p^* \restriction \mu \in \mathbb{B}_\mu$. Suppose that $\mu \leq \upsilon$ and that $p^* \restriction \nu \in \mathbb{B}_\nu$ for every $\nu < \mu$. We show that $p^* \restriction \mu \in \mathbb{B}_\mu$. If μ is a limit ordinal, then this follows from the fact that $|\operatorname{dom}(p^*)| \leq |M| = \delta$.

Suppose that μ is a successor ordinal, and let $\nu := \mu - 1$. If $\nu \notin M \cap U$, then $p^* \restriction \mu \in \mathbb{B}_\nu$ clearly. Thus we also assume that $\nu \in M \cap U$. It suffices to show that if $\alpha \in p^*(\nu) \cap \mathcal{E}^\kappa_\delta$, then $p^* \restriction \nu \Vdash_{\mathbb{B}_\nu}$ "$\dot{S}_\nu \cap \alpha$ is stationary". Take an arbitrary $\alpha \in p^*(\nu) \cap \mathcal{E}^\kappa_\delta$.

First suppose that $\alpha \in c_\nu$. Then there exists $q \in g$ such that $\alpha \in q(\nu)$. Then $q \restriction \nu \Vdash_{\mathbb{B}_\nu}$ "$\dot{S}_\nu \cap \alpha$ is stationary" because $q \in \mathbb{B}_\upsilon$. But $p^* \restriction \nu \leq q \restriction \nu$. Hence $p^* \restriction \nu \Vdash_{\mathbb{B}_\nu}$ "$\dot{S}_\nu \cap \alpha$ is stationary".

Next suppose that $\alpha = \gamma^*$. Note that $\dot{S}_\nu$ can be seen as a $\mathbb{B}_{\upsilon,U}$-name because $\nu \in U$. Moreover $\dot{S}_\nu$ is stationary in $\mathcal{E}^\kappa_{<\delta}$ in $V^{\mathbb{B}_{\upsilon,U}}$ by Lem.4.7, Fact.2.1 and the fact that $\mathcal{E}^\kappa_{<\delta} \in I[\kappa]$ (in $V^{\mathbb{B}_\nu}$). Then $s := (\dot{S}_\nu)_{g,M}$ is stationary in γ^* by the assumption on M and g. Note that s is the set of all $\beta < \gamma^*$ such that $q \restriction \nu \Vdash_{\mathbb{B}_\nu}$ "$\beta \in \dot{S}_\nu$" for some $q \in g$. Moreover $p^* \restriction \nu$ is a lower bound of $\{q \restriction \nu \mid q \in g\}$. Hence $p^* \restriction \nu \Vdash_{\mathbb{B}_\nu}$ "$\dot{S}_\nu \cap \gamma^* \supseteq s$". Here note that $\mathbb{B}_\nu$ preserves stationary subsets of γ^* because $\mathbb{B}_\nu$ is $<\delta$-closed, and $\operatorname{cf}(\gamma^*) \leq \delta$. So s remains stationary in $V^{\mathbb{B}_\nu}$. Therefore $p^* \restriction \nu \Vdash_{\mathbb{B}_\nu}$ "$\dot{S}_\nu \cap \gamma^*$ is stationary". □

Proof of Lem.4.8. Let $\vec{\mathbb{B}} = \langle \mathbb{B}_\mu, \dot{S}_\nu \mid \mu \leq \upsilon, \nu < \upsilon \rangle$ be an OSR^*_κ-iteration. Suppose that $p \in \mathbb{B}_\upsilon$ and that $\mathcal{A}$ is a family of maximal antichains in $\mathbb{B}_\upsilon$ with $|\mathcal{A}| \leq \delta$. We will find $p^* \leq p$ meeting all maximal antichains in $\mathcal{A}$.

First take a $\vec{\mathbb{B}}$-complete $U \subseteq \upsilon$ such that $|U| \leq 2^\delta$ and such that $\{p\} \cup (\bigcup \mathcal{A}) \subseteq \mathbb{B}_{\upsilon,U}$. We can easily take such U by the $(2^\delta)^+$-c.c. of $\mathbb{B}_\upsilon$. Then $|\mathbb{B}_{\upsilon,U}| \leq 2^\delta$, and $\mathbb{B}_{\upsilon,U}$ is $<\delta$-directed closed by Lem.4.7. Then, by the assumptions in Lem.4.8, $\mathbb{B}_{\upsilon,U}$ satisfies the assumptions on $\mathbb{P}$ in Lem.4.9. Thus we can take $M \in [\mathcal{H}_\theta]^\delta$ such that $\mathcal{A} \subseteq M \prec \langle \mathcal{H}_\theta, \in, \vec{\mathbb{B}}, U, p \rangle$ and such that M satisfies the properties (i) and (ii) in Lem.4.9 for $\mathbb{P} = \mathbb{B}_{\upsilon,U}$. Let g be an $(M, \mathbb{B}_{\upsilon,U})$-generic filter witnessing the property (ii). Then there is a lower bound p^* of g in $\mathbb{B}_{\upsilon,U}$ by Lem.4.10.

Here note that p^* is a lower bound of g also in $\mathbb{B}_\upsilon$. Hence $p^* \leq p$ because $p \in g$. Note also that each $A \in \mathcal{A}$ is a maximal antichain in $\mathbb{B}_{\upsilon,U}$ which belongs to M. Thus p^* meets all $A \in \mathcal{A}$ by the $(M, \mathbb{B}_{\upsilon,U})$-genericity of g. Therefore p^* is as desired. $\square$

Now Prop.4.3 easily follows from what we have proved:

Proof of Prop.4.3. Let $\upsilon := 2^\kappa \cdot (2^\delta)^+$. Then by the $(2^\delta)^+$-c.c. of OSR^*_κ-iterations we can construct an OSR^*_κ-iteration $\langle \mathbb{B}_\mu, \dot{S}_\nu \mid \mu \leq \upsilon, \nu < \upsilon \rangle$ such that for any $\mathbb{B}_\upsilon$-name $\dot{S}$ for a stationary subset of $\mathcal{E}^\kappa_{<\delta}$ there exists $\nu < \upsilon$ with $\Vdash_{\mathbb{B}_\upsilon}$ "$\dot{S} = \dot{S}_\nu$". $\mathbb{B}_\upsilon$ is $<\delta$-directed closed by Lem.4.7 and $<\kappa$-Baire by Lem.4.8. Moreover $\mathbb{B}_\upsilon$ forces OSR^*_κ by the construction of $\mathbb{B}_\upsilon$. $\square$

5 PFA does not imply $\square^{\mathrm{p}}$

In this section we prove Thm.1.5:

Theorem 1.5. *If there exists a supercompact cardinal, then there exists a forcing extension in which* PFA *holds but* $\square^{\mathrm{p}}$ *fails.*

In fact we prove the following which implies the above theorem by Lem.4.2:

Theorem 5.1. PFA *is consistent with* $\mathsf{OSR}^*_{\omega_2}$.
Precise Statement: If there exists a supercompact cardinal, then there exists a forcing extension in which both PFA *and* $\mathsf{OSR}^*_{\omega_2}$ *hold.*

Recall that if κ is a supercompact cardinal, and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ is the standard iteration for PFA, then PFA^{++} and $2^{\omega_1} = \kappa = \omega_2$ hold in $V^{\mathbb{P}_\kappa}$. Hence, by results in the previous section, in $V^{\mathbb{P}_\kappa}$ we can construct a $\mathsf{OSR}^*_{\omega_2}$-iteration forcing $\mathsf{OSR}^*_{\omega_2}$. Thus the following lemma implies Thm.5.1:

Lemma 5.2. *If* PFA^{++} *holds, then* $\mathbb{B}_\upsilon$ *forces* PFA *for any* $\mathsf{OSR}^*_{\omega_2}$*-iteration* $\langle \mathbb{B}_\mu, \dot{S}_\nu \mid \mu \leq \upsilon, \nu < \upsilon \rangle$.

Proof. Assume that PFA^{++} holds and that $\vec{\mathbb{B}} = \langle \mathbb{B}_\mu, \dot{S}_\nu \mid \mu \leq \upsilon, \nu < \upsilon \rangle$ is an $\mathsf{OSR}^*_{\omega_2}$-iteration. Suppose that $p \in \mathbb{B}_\upsilon$, that $\dot{\mathbb{Q}}$ is a $\mathbb{B}_\upsilon$-name for a proper poset and that $\mathcal{A}$ is a family of $\mathbb{B}_\upsilon$-names for maximal antichains in $\dot{\mathbb{Q}}$ with $|\mathcal{A}| \leq \omega_1$. It suffices to find $p^* \leq p$ and a $\mathbb{B}_\upsilon$-name $\dot{H}$ such that p^* forces that $\dot{H}$ is a filter on $\dot{\mathbb{Q}}$ and that $\dot{H} \cap \dot{A} \neq \emptyset$ for all $\dot{A} \in \mathcal{A}$. We work in V.

Take a sufficiently large regular cardinal θ. Note that $\mathbb{B}_\upsilon * \dot{\mathbb{Q}}$ is proper and that FA^{++} holds for $\mathbb{B}_\upsilon * \dot{\mathbb{Q}} * \mathrm{Col}(\omega_1, {\omega_2}^V)$ by PFA^{++}. Note also that $\mathcal{E}^{\omega_2}_{<\omega_1} = E^2_0 \in I[\omega_2]$. Then by Lem.4.9 we can take $M \in [\mathcal{H}_\theta]^{\omega_1}$ and an $(M, \mathbb{B}_\upsilon * \dot{\mathbb{Q}})$-generic filter k such that

(i) $M \cap \omega_2 \in \omega_2$, and $\mathcal{A} \subseteq M \prec \langle \mathcal{H}_\theta, \in, \vec{\mathbb{B}}, p \rangle$,

(ii) $p * 1_{\dot{\mathbb{Q}}} \in k$,

(iii) $\dot{S}_{k,M}$ is stationary in $M \cap \omega_2$ for any $\mathbb{B}_\upsilon * \dot{\mathbb{Q}}$-name $\dot{S} \in M$ for a stationary subset of $(E^2_0)^V$.

Let $g := \{p' \in \mathbb{B}_\upsilon \cap M \mid p' * 1_{\dot{\mathbb{Q}}} \in k\}$. Then g is an $(M, \mathbb{B}_\upsilon)$-generic filter, and $\dot{S}_{g,M}$ is stationary in $M \cap \omega_2$ for any $\mathbb{B}_\upsilon$-name $\dot{S} \in M$ for a stationary subset of $(E^2_0)^V$. (For the latter note that if $\dot{S}$ is a $\mathbb{B}_\upsilon$-name for a stationary subset of $(E^2_0)^V$, then $\mathbb{B}_\upsilon * \dot{\mathbb{Q}}$ forces $\dot{S}$ to remain stationary because $\dot{\mathbb{Q}}$ is proper.) So by Lem.4.10 we can take a lower bound $p^* \in \mathbb{B}_\upsilon$ of g.

Let $\dot{h} := \{(\dot{q}, p') \mid p' * \dot{q} \in k\}$. Then $\dot{h}$ is a $\mathbb{B}_\upsilon$-name for a subset of $\dot{\mathbb{Q}}$. Moreover it is easy to see that p^* forces $\dot{h}$ to be an $(M[\dot{G}], \dot{\mathbb{Q}})$-generic filter, where $\dot{G}$ is the canonical name for $\mathbb{B}_\upsilon$-generic filter, and $M[G]$ denotes the set $\{\dot{x}_G \mid \dot{x}$ is a $\mathbb{B}_\upsilon$-name in $M\}$ for a $\mathbb{B}_\upsilon$-generic filter G. In particular, p^* forces that $\dot{h} \cap \dot{A} \neq \emptyset$ for all $\dot{A} \in \mathcal{A}$. Let $\dot{H}$ be a $\mathbb{B}_\upsilon$-name for the filter on $\dot{\mathbb{Q}}$ generated by $\dot{h}$. Then p^* and $\dot{H}$ is as desired. $\square$

6 $\square^{\mathrm{p}}_\delta$ for regular $\delta \geq \omega_2$

At the end of this paper we make a remark that $\square^{\mathrm{p}}_\delta$ for a regular $\delta \geq \omega_2$ is independent of MM.

First we prove that MM is consistent with the failure of $\square^{\mathrm{p}}_\delta$. In fact we prove the following stronger fact:

Theorem 6.1. MM *is consistent with* $\mathsf{OSR}^*_{\delta^+}$ *for a regular* $\delta \geq \omega_2$.
Precise Statement: In V *suppose that* MM *holds, that* δ *is a regular cardinal* $\geq \omega_2$ *and that there exists a weakly compact cardinal* $> \delta$. *Then there exists a* $< \delta$*-directed closed* $< \delta^+$*-Baire forcing extension in which both* MM *and* $\mathsf{OSR}^*_{\delta^+}$ *hold.*

For this we use the following well-known lemma:

Lemma 6.2 (folklore)**.** *Suppose that* δ *is a regular uncountable cardinal and that* κ *is a weakly compact cardinal* $> \delta$. *Then* FA^{++}_δ *for all* $<\delta$*-closed poset of size* $\leq 2^\delta$ *holds in* $V^{\mathrm{Col}(\delta, <\kappa)}$.

Proof. Let H be a $\mathrm{Col}(\delta, {<}\kappa)$-generic filter over V. In $V[H]$ suppose that $\mathbb{P}$ is a ${<}\delta$-closed poset of size $\leq 2^\delta$, that $\mathcal{A}$ is a family of maximal antichains in $\mathbb{P}$ with $|\mathcal{A}| \leq \delta$ and that $\mathcal{R}$ is a family of $\mathbb{P}$-names for stationary subsets of δ with $|\mathcal{R}| \leq \delta$. In $V[H]$ we find a filter G^* on $\mathbb{P}$ such that $G^* \cap A \neq \emptyset$ for any $A \in \mathcal{A}$ and such that $\dot{R}_{G^*}$ is stationary in δ for all $\dot{R} \in \mathcal{R}$. Note that $2^\delta = \kappa$ in $V[H]$. So we may assume that $\mathbb{P} \subseteq \kappa$.

Let $\dot{\mathbb{P}}$, $\dot{\mathcal{A}}$ and $\dot{\mathcal{R}}$ be $\mathrm{Col}(\delta, {<}\kappa)$-names for $\mathbb{P}$, $\mathcal{A}$ and $\mathcal{R}$, respectively. We may assume that $\dot{\mathbb{P}}, \dot{\mathcal{A}}, \dot{\mathcal{R}} \in \mathcal{H}_{\kappa^+}$ in V. Then in V we can take a transitive $M \prec \langle \mathcal{H}_{\kappa^+}, \in, \delta, \kappa, \dot{\mathbb{P}}, \dot{\mathcal{A}}, \dot{\mathcal{R}} \rangle$ of size κ. Moreover in V we can take a transitive N and an elementary embedding $j : M \to N$ whose critical point is κ. This is because κ is weakly compact in V.

Note that $M \subseteq N$ because $\mathcal{P}(\kappa) \cap M \subseteq N$. Hence $\mathbb{P}, \mathcal{A}, \mathcal{R} \in M[H] \subseteq N[H]$. Moreover in $N[H]$ it is easy to see that $\mathbb{P}$ is ${<}\delta$-closed, that $\mathcal{A}$ is a family of maximal antichains in $\mathbb{P}$ with $|\mathcal{A}| \leq \delta$ and that $\mathcal{R}$ is a family of $\mathbb{P}$-names for stationary subsets of δ with $|\mathcal{R}| \leq \delta$.

Here take a $\mathbb{P} \times \mathrm{Col}(\delta, [\kappa, j(\kappa)))^{N[H]}$-generic filter $G \times I$ over $V[H]$. Below we work in $V[H][G \times I]$.

Note that $\mathbb{P} \times \mathrm{Col}(\delta, [\kappa, j(\kappa)))^{N[H]}$ is isomorphic to $\mathrm{Col}(\delta, [\kappa, j(\kappa)))^{N[H]}$ in $N[H]$ because $\mathbb{P}$ is a ${<}\delta$-closed poset of size $\leq \kappa$. Thus $H * (G \times I)$ can be seen as a $\mathrm{Col}(\delta, {<}j(\kappa))$-generic filter over N. Hence $j : M \to N$ can be naturally extended to an elementary embedding $j : M[H] \to N[H][G][I]$. In $N[H][G][I]$ let $\bar{G}$ be a filter on $j(\mathbb{P})$ generated by $j[G] = G \in N[H][G][I]$.

Then $\bar{G} \cap j(A) \neq \emptyset$ for all $A \in \mathcal{A}$. Moreover for each $\dot{R} \in \mathcal{R}$, $j(\dot{R})_{\bar{G}} = \dot{R}_G$, and $\dot{R}_G$ is stationary in δ in $N[H][G][I]$. The latter is because $\dot{R}$ is a $\mathbb{P}$-name for a stationary subset of δ in $N[H]$, and $N[H][G][I]$ is a ${<}\delta$-closed forcing extension of $N[H][G]$. Note also that $j(\mathcal{A}) = j[\mathcal{A}]$ and $j(\mathcal{R}) = j[\mathcal{R}]$ because $|\mathcal{A}| = |\mathcal{R}| \leq \delta < \kappa$ in $M[H]$. Therefore in $N[H][G][I]$ we have that $\bar{G} \cap A' \neq \emptyset$ for all $A' \in j(\mathcal{A})$ and that $\dot{R}'_{\bar{G}}$ is stationary in δ for all $\dot{R}' \in j(\mathcal{R})$.

Then by the elementarity of j, in $M[H]$ we can take a filter G^* on $\mathbb{P}$ such that $G^* \cap A \neq \emptyset$ for all $A \in \mathcal{A}$ and such that $\dot{R}_{G^*}$ is stationary in δ for all $\dot{R} \in \mathcal{R}$. Here note that $\mathcal{P}(\delta) \cap V[H] \subseteq M[H]$. Hence $\dot{R}_{G^*}$ is stationary also in $V[H]$ for each $\dot{R} \in \mathcal{R}$. Thus G^* is as desired. $\square$

In the proof of Thm.6.1 we also use the following folklore. Its proof is found in Larson [11]:

Fact 6.3 (folklore)**.** MM *is preserved by ${<}\omega_2$-directed closed forcing extensions.*

Now we can easily prove Thm.6.1:

Proof of Thm.6.1. In V assume that MM holds, that δ is a regular cardinal $\geq \omega_2$ and that κ is a weakly compact cardinal $> \delta$.

Let G be a $\mathrm{Col}(\delta, < \kappa)$-generic filter over V. Then $\kappa = \delta^+$ in $V[G]$. Moreover FA^{++}_δ holds for all $<\delta$-directed closed posets of size $\leq 2^\delta$ in $V[G]$ by Lem.6.2. Furthermore $\mathcal{E}^\kappa_{<\delta} \in I[\kappa]$ in $V[G]$ because $2^{<\delta} = \delta$ (See Fact 2.1). Then by Prop.4.3 in $V[G]$ there exists a $<\kappa$-Baire $<\delta$-directed closed poset $\mathbb{B}$ forcing OSR^*_κ. Let H be a $\mathbb{B}$-generic filter over $V[G]$.

Then $\mathsf{OSR}^*_{\delta^+}$ holds in $V[G][H]$. Moreover MM remains to hold in $V[G][H]$ by Fact 6.3 and the fact that $V[G][H]$ is a $<\delta$-directed closed forcing extension of V. $\square$

From Thm.6.1 and Lem.4.2 it follows that MM is consistent with the failure of $\square^{\mathrm{p}}_\delta$ for a regular $\delta \geq \omega_2$. Next we prove that MM is consistent with $\square^{\mathrm{p}}_\delta$:

Theorem 6.4. MM *is consistent with* $\square^{\mathrm{p}}_\delta$ *for an uncountable cardinal* δ. *Precise Statement: In* V *suppose that* MM *holds and that* δ *is an uncountable cardinal. Then there exists a* $<\delta^+$*-directed closed forcing extension in which both* MM *and* $\square^{\mathrm{p}}_\delta$ *hold.*

This follows from Fact 6.3 and the following lemma:

Lemma 6.5. *Let* δ *be an uncountable cardinal. Then there exists a* $<\delta^+$*-directed closed forcing extension in which* $\square^{\mathrm{p}}_\delta$ *holds.*

Proof. Let $\mathbb{P}$ be the following poset:

- $\mathbb{P} := \{p \mid p \text{ is a } \square_\delta(s)\text{-sequence for some bounded } s \subseteq \mathrm{Lim}(\delta^+)\}$.
- For $p = \langle c_\alpha \mid \alpha \in s \rangle$ and $p' = \langle c'_\alpha \mid \alpha \in s' \rangle$ in $\mathbb{P}$, $p \leq p'$ if p is an end-extension of p', that is, $s' = s \cap \sup\{\alpha + 1 \mid \alpha \in s'\}$, and $c_\alpha = c'_\alpha$ for all $\alpha \in s'$.

If $p = \langle c_\alpha \mid \alpha \in s \rangle \in \mathbb{P}$, then c_α and s are denoted as $c_{p,\alpha}$ and s_p, respectively.

It is easy to see that $\mathbb{P}$ is $<\delta^+$-directed closed. We show that $\Vdash_{\mathbb{P}} \square^{\mathrm{p}}_\delta$. Note that if G is a $\mathbb{P}$-generic filter, then clearly $\bigcup G$ is a $\square_\delta(S_G)$-sequence, where $S_G = \bigcup_{p \in G} s_p$. Thus all we have to show is that $S_G \cap \mathcal{E}^{\delta^+}_\delta$ is stationary in $V[G]$.

In V take an arbitrary $p \in \mathbb{P}$ and an arbitrary $\mathbb{P}$-name $\dot{C}$ for a club subset of δ^+. It suffices to find $p^* \leq p$ and $\alpha^* \in \mathcal{E}^{\delta^+}_\delta$ such that $\alpha^* \in s_{p^*}$ and such that $p^* \Vdash$ "$\alpha^* \in \dot{C}$". We work in V.

By induction on ξ we can easily construct a strictly descending sequence $\langle p_\xi \mid \xi \leq \delta \rangle$ below p so that

- s_{p_ξ} has the greatest element α_ξ for each ξ,
- if ξ is successor, then $p_\xi \Vdash$ "$\dot{C} \cap [\alpha_{\xi-1}, \alpha_\xi) \neq \emptyset$",
- if ξ is limit, then $\alpha_\xi = \sup_{\eta<\xi} \alpha_\eta$, and $c_{p_\xi, \alpha_\xi} = \{\alpha_\eta \mid \eta < \xi\}$.

Then it is easy to see that $p^* := p_\delta$ and $\alpha^* := \alpha_\delta$ are as desired. □

References

[1] U. Abraham, *Proper forcing*, Handbook of set theory Vol. 1, 333–394, Springer, Dordrecht, 2010.

[2] A. W. Apter and J. Cummings, *A global version of a theorem of Ben-David and Magidor*, Ann. Pure. Appl. Logic **102** (2000), no.3, 199–222.

[3] J. E. Baumgartner, *PFA++*, in Handbook of set theoretic topology.

[4] K. J. Devlin, *The Yorkshireman's guide to proper forcing*, Surveys in set theory, 60–115, London Math. Soc. Lecture Note Ser., 87, Cambridge Univ. Press, Cambridge, 1983.

[5] Q. Feng and T. Jech, *Projective stationary sets and a strong reflection principle*, J. London Math. Soc. (2) 58 (1998), no.2, 271–283.

[6] M. Foreman, M. Magidor and S. Shelah, *Martin's maximum, saturated ideals, and nonregular ultrafilters I*, Ann. of Math. (2) **127** (1988), no.1, 1–47.

[7] M. Foreman and T. Todorčević, *A new Löwenheim-Skolem Theorem*, Trans. Amer. Math. Soc. **357** (2005), no.5, 1693–1715.

[8] T. Jech, *Set Theory*, 3rd edition, Springer-Verlag, Berlin, 2002.

[9] R. B. Jensen, *The fine structure of the constructible hierarchy*, Ann. Math. Logic **4** (1972), 229–308.

[10] J. Krueger, *Strong compactness and stationary sets*, J. Symbolic Logic **70** (2005), no.3, 767-777.

[11] P. Larson, *Separating Stationary Reflection Principles*, J. Symbolic Logic **65** (2000), 247-258.

[12] M. Magidor, *Reflecting stationary sets*, J. Symbolic Logic **47** (1982), no.4, 755–771.

[13] H. Sakai, *Semistationary and stationary reflection*, J. Symbolic Logic **73** (2008), no.1, 181–192.

[14] S. Shelah, *On successors of singular cardinals*, Logic Colloquium '78 (Mons, 1978), pp.357–380, Stud. Logic Foundations Math., 97, North-Holland, Amsterdam-New York, 1979.

[15] S. Shelah, *Reflecting stationary sets and successors of singular cardinals*, Arch. Math. Logic **31** (1991), no.1, 25–53.

[16] S. Shelah, *Diamonds*, Proc. Amer. Math. Soc. **138** (2010), no.6, 2151–2161.

[17] S. Todorčvić, *A note on proper forcing axiom*, Axiomatic set theory, Boulder, Colo., 1983, 209–218, Contemp. Math. 31, Amer. Math. Soc., Providence, RI, 1984.

[18] H. Woodin, *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal*, De Gruyter Series in Logic and its Applications; 1, Walter de Gruyter, Berlin, New York, 1999.