On preservation and reflection of stationary subsets of $\mathcal{P}_\kappa\lambda$ when $\mathrm{cf}(\lambda) < \kappa$

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Abstract

We study the preservation under $< \kappa$-closed forcing extensions and the reflection of stationary subsets of $\mathcal{P}_\kappa\lambda$ in the case when λ is a singular cardinal with $\mathrm{cf}(\lambda) < \kappa$. In particular we discuss those under GCH.

1 Introduction

The following stationary reflection principle in $\mathcal{P}_\kappa\lambda$, which is often called the weak reflection principle in $\mathcal{P}_\kappa\lambda$, has been studied by many set theorists so far:

Definition 1.1. *For a regular uncountable cardinal κ, a cardinal $\lambda > \kappa$ and a stationary $T \subseteq \mathcal{P}_\kappa\lambda$, let $\mathsf{SR}_{\kappa\lambda}(T)$ be the following assertion:*

$$\mathsf{SR}_{\kappa\lambda}(T) \equiv \text{For every stationary } S \subseteq T \text{ there exists } W \subseteq \lambda \text{ such that } |W| = \kappa \subseteq W \text{ and such that } S \cap \mathcal{P}_\kappa W \text{ is stationary in } \mathcal{P}_\kappa W.$$

We let $\mathsf{SR}_{\kappa\lambda}$ denote $\mathsf{SR}_{\kappa\lambda}(\mathcal{P}_\kappa\lambda)$.

(Nowadays there are two notions of stationary subsets of $\mathcal{P}_\kappa\lambda$. We adopt the one introduced by Jech [8]. More precisely, see Section 2.)

The importance of the above stationary reflection principle was revealed by Foreman-Magidor-Shelah [5]. They showed that $\mathsf{SR}_{\omega_1\lambda}$ is implied by Martin's maximum for every $\lambda > \omega_1$ and holds if a λ-supercompact cardinal is Lévy collapsed to ω_2. Moreover they showed that $\mathsf{SR}_{\omega_1\lambda}$ has interesting consequences such as Chang's conjecture and the presaturation of the nonstationary ideal over ω_1. It is also known that $\mathsf{SR}_{\omega_1\lambda}$ has the influence on the cardinal arithmetic. Todorčević [19] showed that $\mathsf{SR}_{\omega_1\omega_2}$ implies $2^\omega \leq \omega_2$, and Shelah [15] showed that $\mathsf{SR}_{\omega_1\lambda}$ implies SCH below λ. The stationary reflection principle in $\mathcal{P}_{\omega_1}\lambda$ and its strengthening are studied also in [1], [2], [6], [20], etc.

The stationary reflection principle in $\mathcal{P}_\kappa\lambda$ for $\kappa \geq \omega_2$ also have been studied so far, for example, in [3], [4], [6], [16] and [17]. Now, due to Feng-Magidor [3], Foreman-Magidor [4] and Shelah-Shioya [16], it is known that $\mathsf{SR}_{\kappa\lambda}$ is not consistent if $\omega_2 \leq \kappa < \lambda$. On the other hand it was already shown in [5] that the above mentioned facts on $\mathsf{SR}_{\omega_1\lambda}$ can be partially generalized to the case when $\kappa \geq \omega_2$. More precisely, the stationary reflection principle in $\mathcal{P}_\kappa\mathcal{H}_\theta$ below internally approachable sets is consistent for regular cardinals $\theta > \kappa$, and this partial stationary reflection principle implies the precipitousness of the nonstationary ideal over κ. Here we review this more closely in the context of $\mathcal{P}_\kappa\lambda$. (Subsection 3.2 of this paper contains the proof of facts below.)

Let κ be a regular uncountable cardinal, and λ be a cardinal $> \kappa$ with $\lambda^{<\kappa} = \lambda$. Note that $|{}^{<\kappa}(\mathcal{P}_\kappa\lambda)| = \lambda$. Hence we can take a bijection $\psi : {}^{<\kappa}(\mathcal{P}_\kappa\lambda) \to \lambda$. Then $x \in \mathcal{P}_\kappa\lambda$ is said to be *ψ-internally approachable* (ψ-i.a.) if there is a $\subseteq$-increasing sequence $\langle x_\xi \mid \xi < \zeta\rangle$ (ζ: limit ordinal $< \kappa$) such that $\bigcup_{\xi<\zeta} x_\xi = x$ and such that $\psi(\langle x_\xi \mid \xi < \zeta'\rangle) \in x$ for every $\zeta' < \zeta$. (We call such sequence $\langle x_\xi \mid \xi < \zeta\rangle$ a *ψ-internally approaching sequence to x*.) Moreover let $S^\psi_{\kappa\lambda}$ be the set of all ψ-i.a. $x \in \mathcal{P}_\kappa\lambda$.

Note that $S^\psi_{\kappa\lambda}$ is independent of the choice of a bijection ψ modulo the nonstationary ideal over $\mathcal{P}_\kappa\lambda$. In fact, if ψ and φ are both bijections from ${}^{<\kappa}\mathcal{P}_\kappa\lambda$ to λ, then being ψ-i.a. and being φ-i.a. are equivalent for all $x \in \mathcal{P}_\kappa\lambda$ which is closed under $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$. We omit the superscript ψ and write $S_{\kappa\lambda}$ for $S^\psi_{\kappa\lambda}$ because the difference of a nonstationary set does not matter in our context. What is important on $S_{\kappa\lambda}$ is the following:

Fact 1.2 ([5]). *Let κ be a regular uncountable cardinal, and λ be a cardinal $> \kappa$ with $\lambda^{<\kappa} = \lambda$. Then all stationary subsets of $S_{\kappa\lambda}$ remain stationary in all $<\kappa$-closed forcing extensions.*

It was also shown in [4] that the complement of $S_{\kappa\lambda}$ becomes nonstationary in the forcing extension by $\mathrm{Col}(\kappa, \lambda)$. Here $\mathrm{Col}(\kappa, \lambda)$ denotes the forcing notion ${}^{<\kappa}\lambda$ ordered by reverse inclusions. Hence $\mathrm{Col}(\kappa, \lambda)$ is a $<\kappa$-closed forcing notion which forces that $|\lambda| = \kappa$.

We obtain the consistency of $\mathsf{SR}_{\kappa\lambda}(S_{\kappa\lambda})$ from Fact 1.2 and Fact 1.3 below. In Fact 1.3 $\mathrm{Col}(\kappa, <\nu)$ denotes the Lévy collapse forcing ν to be κ^+.

Fact 1.3 ([5]). *Let κ be a regular uncountable cardinal and λ be a cardinal $> \kappa$. Suppose that ν is a λ-supercompact cardinal with $\kappa < \nu < \lambda$. In $V^{\mathrm{Col}(\kappa,<\nu)}$ assume that T is a stationary subset of $\mathcal{P}_\kappa\lambda$ such that all stationary subsets of T remain stationary in all $<\kappa$-closed forcing extensions. Then $\mathsf{SR}_{\kappa\lambda}(T)$ holds in $V^{\mathrm{Col}(\kappa,<\nu)}$.*

Corollary 1.4 ([5]). *Let κ be a regular uncountable cardinal and λ be a cardinal $> \kappa$ with $\lambda^{<\kappa} = \lambda$. Suppose that ν is a λ-supercompact cardinal with $\kappa < \nu < \lambda$. Then $\mathsf{SR}_{\kappa\lambda}(S_{\kappa\lambda})$ holds in $V^{\mathrm{Col}(\kappa,<\nu)}$.*

Furthermore it was shown in [5] that if λ is sufficiently large then $\mathsf{SR}_{\kappa\lambda}(S_{\kappa\lambda})$ implies the precipitousness of the nonstationary ideal over κ.

Above we have briefly reviewed the stationary reflection principle in $\mathcal{P}_\kappa\lambda$ in the case when $\lambda^{<\kappa} = \lambda$. Then the question naturally arises. What about the stationary reflection principle in $\mathcal{P}_\kappa\lambda$ in the case when $\lambda^{<\kappa} > \lambda$? Moreover recall that Fact 1.2 was the key of Cor.1.4. What about the preservation of stationary subsets of $\mathcal{P}_\kappa\lambda$ under $<\kappa$-closed forcing extensions when $\lambda^{<\kappa} > \lambda$? Note that if λ is a singular cardinal with $\mathrm{cf}(\lambda) < \kappa < \lambda$, then $\lambda^{<\kappa} > \lambda$. Shioya [18] asked how is the preservation and the reflection of stationary subsets of $\mathcal{P}_\kappa\lambda$ when λ is a singular cardinal with $\mathrm{cf}(\lambda) < \kappa < \lambda$.

In this paper we discuss the preservation and the reflection of stationary subsets of $\mathcal{P}_\kappa\lambda$ in the case when λ is a singular cardinal with $\mathrm{cf}(\lambda) < \kappa < \lambda$. In particular we study those under GCH.

For this we introduce the notion of the semi-internally approachability:

Notation. *Let κ be a regular uncountable cardinal. A cardinal λ is said to be* $<\kappa$-strong limit *if $\nu^{<\kappa} < \lambda$ for every $\nu < \lambda$. Note that if λ is a $<\kappa$-strong limit cardinal, then $|\bigcup_{\lambda'<\lambda} {}^{<\kappa}(\mathcal{P}_\kappa\lambda')| = \lambda$.*

Definition 1.5. *Let κ be a regular uncountable cardinal, and λ be a $<\kappa$-strong limit cardinal with $\mathrm{cf}(\lambda) < \kappa < \lambda$. Suppose that $\varphi : \bigcup_{\lambda'<\lambda} {}^{<\kappa}(\mathcal{P}_\kappa\lambda') \to \lambda$ is a bijection. Let x be an element of $\mathcal{P}_\kappa\lambda$.*

For a limit ordinal $\zeta < \kappa$, x is said to be φ-semi-internally approachable *(φ-s.i.a.) of length ζ if there exists a $\subseteq$-increasing sequence $\langle x_\xi \mid \xi < \zeta\rangle$ in $\mathcal{P}_\kappa\lambda$ such that*

(i) $\bigcup_{\xi<\zeta} x_\xi = x$,

(ii) $\varphi(\langle x_\xi \cap \lambda' \mid \xi < \zeta'\rangle) \in x$ *for every $\zeta' < \zeta$ and every $\lambda' < \lambda$.*

A sequence $\langle x_\xi \mid \xi < \zeta\rangle$ as above is called a φ-semi-internally approaching sequence to x. *x is said to be* φ-semi-internally approachable *if x is φ-s.i.a. of length ζ for some limit $\zeta < \kappa$.*

It is easy to see that if θ is a regular cardinal $> \lambda$, $M \in \mathcal{P}_\kappa\mathcal{H}_\theta$ is internally approachable of length ζ, and $M \prec \langle \mathcal{H}_\theta, \in \kappa, \lambda, \varphi\rangle$, then $M \cap \lambda$ is φ-s.i.a. of length ζ. (See Lem.3.6.)

As in the case when $\lambda^{<\kappa} = \lambda$, we consider the set of all φ-s.i.a. sets. But we need divide this set according to the length of approaching sequence:

Definition 1.6. *Let κ, λ and φ be as in Def.1.5. Then let*

$$\begin{aligned} T^{\varphi,0}_{\kappa\lambda} &:= \{x \in \mathcal{P}_\kappa\lambda \mid x \text{ is } \varphi\text{-s.i.a. of length } \mathrm{cf}(\lambda)\} \\ T^{\varphi,1}_{\kappa\lambda} &:= \{x \in \mathcal{P}_\kappa\lambda \mid x \text{ is } \varphi\text{-s.i.a. but is not } \varphi\text{-s.i.a. of length } \mathrm{cf}(\lambda)\} \end{aligned}$$

As is the case with $S^\psi_{\kappa\lambda}$, both $T^{\varphi,0}_{\kappa\lambda}$ and $T^{\varphi,1}_{\kappa\lambda}$ do not depend on the choice of φ modulo the nonstationary ideal over $\mathcal{P}_\kappa\lambda$. Hence we often write $T^0_{\kappa\lambda}$ and $T^1_{\kappa\lambda}$ for $T^{\varphi,0}_{\kappa\lambda}$ and $T^{\varphi,1}_{\kappa\lambda}$, respectively.

As we see in Lem.3.7, if $\kappa = \omega_1$, then $T^0_{\omega_1\lambda}$ contains a club in $\mathcal{P}_{\omega_1}\lambda$. Moreover if $\kappa \geq \omega_2$, then $T^0_{\kappa\lambda}$, $T^1_{\kappa\lambda}$ and $\mathcal{P}_\kappa\lambda \setminus (T^0_{\kappa\lambda} \cup T^1_{\kappa\lambda})$ are all stationary in $\mathcal{P}_\kappa\lambda$.

It will be proved in Cor.3.14 that $\mathcal{P}_\kappa\lambda \setminus (T^0_{\kappa\lambda} \cup T^1_{\kappa\lambda})^V$ becomes nonstationary in $V^{\mathrm{Col}(\kappa,\lambda)}$. As for the preservation and the reflection below $T^0_{\kappa\lambda}$ we will obtain the following. In particular $\mathsf{SR}_{\kappa\lambda}(T^0_{\kappa\lambda}) + \mathsf{GCH}$ is consistent:

Theorem 1.7. *Let κ be a regular uncountable cardinal and λ be a $<\kappa$-strong limit cardinal with $\mathrm{cf}(\lambda) < \kappa < \lambda$.*

(1) All stationary subsets of $T^0_{\kappa\lambda}$ remain stationary in all $<\kappa$-closed forcing extensions.

(2) Suppose that ν is a λ-supercompact cardinal with $\kappa < \nu < \lambda$. Then $\mathsf{SR}_{\kappa\lambda}(T^0_{\kappa\lambda})$ holds in $V^{\mathrm{Col}(\kappa,<\nu)}$.

On the other hand we will obtain the following as for the preservation and the reflection below $T^1_{\kappa\lambda}$:

Theorem 1.8. *Assume* GCH. *Let κ be a regular cardinal $\geq \omega_2$ and λ be a singular cardinal with $\mathrm{cf}(\lambda) < \kappa < \lambda$.*

(1) $T^1_{\kappa\lambda}$ (defined in V) remains stationary in all $<\kappa$-closed forcing extensions. But for every stationary $T \subseteq T^1_{\kappa\lambda}$ there exists stationary $S \subseteq T$ which becomes nonstationary in $V^{\mathrm{Col}(\kappa,\lambda)}$.

(2) $\mathsf{SR}_{\kappa\lambda}(T)$ fails for every stationary $T \subseteq T^1_{\kappa\lambda}$.

Thm.1.7 and 1.8 will be proved in Section 4. In Section 3 we study the basic relationship among the notion of s.i.a. and i.a. and the preservation and the reflection of stationary subsets of $\mathcal{P}_\kappa\lambda$. As we mentioned before, Subsection 3.2 contains the proof of Fact 1.2 and 1.3.

2 Preliminaries

Here we present our notation and basic facts used in this paper. For those which are not presented below, consult Jech [9] and Kanamori [10].

First we give our notation and basic facts on subsets of $\mathcal{P}_\kappa W$. For a regular cardinal κ and a set $W \supseteq \kappa$, $\mathcal{P}_\kappa W$ denotes the set $\{x \subseteq W \mid |x| < \kappa\}$. We follow the notion of club and stationary subsets of $\mathcal{P}_\kappa W$ to that introduced by Jech [8]. $C \subseteq \mathcal{P}_\kappa W$ is said to be *club* if C is $\subseteq$-cofinal in $\mathcal{P}_\kappa W$ and closed under unions of $\subseteq$-increasing sequence of length $< \kappa$. $S \subseteq \mathcal{P}_\kappa W$ is said to be *stationary* if $S \cap C \neq$ for every club $C \subseteq \mathcal{P}_\kappa W$.

We use the following facts without any mention. The proof can be found in Jech [9]:

Fact 2.1 (Kueker [11]). *Let κ be a regular uncountable cardinal, and W be a set $\supseteq \kappa$. The following are equivalent for $S \subseteq \mathcal{P}_\kappa W$:*

(I) S is stationary in $\mathcal{P}_\kappa W$.

(II) For every function $F : [W]^{<\omega} \to \mathcal{P}_\kappa W$ there exists $x \in S$ such that $F(a) \subseteq x$ for all $a \in [x]^{<\omega}$.

(III) For every function $F : [W]^{<\omega} \to W$ there exists $x \in S$ such that $F(a) \in x$ for all $a \in [x]^{<\omega}$ and such that $x \cap \kappa \in \kappa$.

(IV) For every structure $\mathcal{M}$ of a countable language with its universe W there exists $x \in S$ such that $x \prec \mathcal{M}$ and $x \cap \kappa \in \kappa$.

Fact 2.2 (Menas [12])**.** *Let κ be a regular uncountable cardinal, and suppose that $\kappa \subseteq W \subseteq \bar{W}$. Then $S \subseteq \mathcal{P}_\kappa W$ is stationary if and only if the set $\{\bar{x} \in \mathcal{P}_\kappa \bar{W} \mid \bar{x} \cap W \in S\}$ is stationary in $\mathcal{P}_\kappa \bar{W}$.*

Next we present notation and a fact on structures. Suppose that $\mathcal{M}$ is a structure for a countable language and that there is a well-ordering of the universe of $\mathcal{M}$ which can be defined over $\mathcal{M}$. For $x \subseteq \mathcal{M}$ let $\mathrm{skull}^{\mathcal{M}}(x)$ denotes the Skolem hull of x in $\mathcal{M}$, i.e. the smallest M with $x \subseteq M \prec \mathcal{M}$.

We often use the following lemma:

Lemma 2.3. *Let κ and θ be regular uncountable cardinals with $\kappa \leq \theta$. Suppose that $M \prec \langle \mathcal{H}_\theta, \in \rangle$ and that $M \cap \kappa \in \kappa$. Then $y \subseteq M$ for every $y \in M \cap \mathcal{P}_\kappa \mathcal{H}_\theta$.*

Proof. There is a bijection $\pi \in M$ from $|y|$ to y by the elementarity of M. Note that $|y| \subseteq M$ because $|y| \in M \cap \kappa \in \kappa$. Thus $y = \pi[|y|] \subseteq M$. □

Next we give our notation on forcing notions for collapsing cardinalities. Let κ be a regular uncountable cardinal. For an ordinal λ let $\mathrm{Col}(\kappa, \lambda)$ denotes the forcing notion ${}^{<\kappa}\lambda$ ordered by reverse inclusions. Note that $\mathrm{Col}(\kappa, \lambda)$ is a $<\kappa$-closed forcing notion and collapses the cardinality of λ to be $\leq \kappa$. For an ordinal ν let $\mathrm{Col}(\kappa, <\nu)$ denotes the Lévy collapse which collapses the cardinality of all ordinals $<\nu$ to be $\leq \kappa$. That is, $\mathrm{Col}(\kappa, <\nu)$ is the $<\kappa$-support product of $\langle \mathrm{Col}(\kappa, \upsilon) \mid \upsilon < \nu \rangle$.

Next we present our notation on scales, which are the central notion of Shelah's PCF theory. Let λ be a singular cardinal, and let $\vec{\lambda} = \langle \lambda_\eta \mid \eta < \mathrm{cf}(\lambda) \rangle$ be an increasing sequence of regular cardinals converging to λ. We let $\Pi\vec{\lambda}$ denote $\Pi_{\eta < \mathrm{cf}(\lambda)} \lambda_\eta$, that is, the set of all functions f on $\mathrm{cf}(\lambda)$ such that $f(\eta) \in \lambda_\eta$ for each $\eta < \mathrm{cf}(\lambda)$. Moreover let J be an ideal over $\mathrm{cf}(\lambda)$. For $g, h \in \Pi\vec{\lambda}$ let

$$\begin{aligned} g <_J h &\Leftrightarrow \{\eta < \mathrm{cf}(\lambda) \mid g(\eta) \geq h(\eta)\} \in J \\ g \leq_J h &\Leftrightarrow \{\eta < \mathrm{cf}(\lambda) \mid g(\eta) > h(\eta)\} \in J \\ g < h &\Leftrightarrow g(\eta) < h(\eta) \text{ for all } \eta < \mathrm{cf}(\lambda) \\ g \leq h &\Leftrightarrow g(\eta) \leq h(\eta) \text{ for all } \eta < \mathrm{cf}(\lambda) \end{aligned}$$

If υ is a regular cardinal and $\vec{h} = \langle h_\alpha \mid \alpha < \upsilon \rangle$ is a $<_J$-increasing cofinal sequence in $\Pi\vec{\lambda}$, then we call $\vec{h}$ a *$(\vec{\lambda}, J)$-scale of length υ*. For a regular cardinal υ let

$$\mathrm{tcf}(\Pi\vec{\lambda}/J) = \upsilon \overset{\text{def}}{\Leftrightarrow} \text{there exists a } (\vec{\lambda}, J)\text{-scale of length } \upsilon.$$

At the end of this section we recall the ideal $I[\kappa]$ over a regular uncountable cardinal κ, introduced by Shelah [13]. Let κ be a regular uncountable cardinal. For a sequence $\vec{b} = \langle b_\xi \mid \xi < \kappa \rangle$ of bounded subsets of κ and a limit ordinal $\zeta < \kappa$, we say that ζ is *approachable with respect to $\vec{b}$* if there exists an unbounded $b \subseteq \zeta$ of order-type $\mathrm{cf}(\zeta)$ such that $b \cap \rho \in \{b_\xi \mid \xi < \zeta\}$ for every $\rho < \zeta$. For each $E \subseteq \kappa$ let

$E \in I[\kappa] \overset{\text{def}}{\Leftrightarrow}$ there exist a sequence $\vec{b} = \langle b_\xi \mid \xi < \kappa \rangle$ of bounded subsets of κ and a club $C \subseteq \kappa$ such that every $\zeta \in E \cap C$ is approachable with respect to $\vec{b}$.

It was shown in [13] that $I[\kappa]$ is a normal ideal over κ. $I[\kappa]$ is not proper, i.e. $\kappa \in I[\kappa]$, in many cases. For example if μ is a regular cardinal with $\mu^{<\mu} = \mu$, then it is easy to see that $I[\mu^+]$ is not proper. On the other hand there may be a regular cardinal κ with $I[\kappa]$ proper. For example it is shown in [13] that if μ is a singular cardinal such that there exists a μ-supercompact cardinal $< \mu$, then $I[\mu^+]$ is proper.

Here note that the set of all regular cardinals $< \kappa$ belongs to $I[\kappa]$. Hence if $I[\kappa]$ is proper, then there exists a regular cardinal $\mu < \kappa$ such that $\{\zeta < \kappa \mid \mathrm{cf}(\zeta) = \mu\} \notin I[\kappa]$ by the normality of $I[\kappa]$.

3 s.i.a, i.a, preservation and reflection

Here we overview the basic relationship among the notion of s.i.a. and i.a. and the preservation and the reflection of stationary subsets of $\mathcal{P}_\kappa\lambda$. (Lem.3.6, 3.11 and Prop.3.15.)

In Subsection 3.1 we present basic facts on s.i.a. and i.a. sets and the relationship between them. In Subsection 3.2 we review the basic relationship among the notion of i.a and the preservation and the reflection of stationary subsets of $\mathcal{P}_\kappa\lambda$.

3.1 s.i.a. and i.a. sets

First we recall the notion of internally approachability:

Definition 3.1. *Let x be a set. For a limit ordinal ζ, x is said to be* internally approachable *(i.a.) of length ζ if there exists an $\subseteq$-increasing sequence $\langle x_\xi \mid \xi < \zeta\rangle$ such that $\bigcup_{\xi<\zeta} x_\xi = x$ and such that $\langle x_\xi \mid x_\xi < \zeta'\rangle \in x$ for every $\zeta' < \zeta$.*

A sequence $\langle x_\xi \mid \xi < \zeta\rangle$ as above is called an internally approaching sequence *to x. x is said to be* internally approachable *if x is internally approachable of length ζ for some limit ordinal ζ.*

Recall that there are stationary many i.a. sets:

Lemma 3.2. *Let μ, κ and θ be regular cardinals with $\mu < \kappa \le \theta$. Then the set $\{M \in \mathcal{P}_\kappa\mathcal{H}_\theta \mid M$ is i.a. of length $\mu\}$ is stationary in $\mathcal{P}_\kappa\mathcal{H}_\theta$.*

Proof. Take an arbitrary function $F : [\mathcal{H}_\theta]^{<\omega} \to \mathcal{H}_\theta$. It suffices to find an i.a. $M \in \mathcal{P}_\kappa\mathcal{H}_\theta$ of length μ such that $M \cap \kappa \in \kappa$ and M is closed under F.

By induction on $\xi < \mu$ we can easily construct a $\subseteq$-increasing sequence $\langle M_\xi \mid \xi < \mu\rangle$ in $\mathcal{P}_\kappa\mathcal{H}_\theta$ such that $\langle M_\eta \mid \eta < \xi\rangle \in M_\xi$, $M_\xi \cap \kappa \in \kappa$ and M_ξ is closed under F for each $\xi < \mu$.

Let $M := \bigcup_{\xi<\mu} M_\xi$. Clearly M is closed under F, and $M \cap \kappa \in \kappa$. Moreover $\langle M_\xi \mid \xi < \mu\rangle$ witnesses that M is i.a. of length μ. □

Next we present easy lemmata on i.a. and s.i.a. sets, which we will use later without any mention:

Lemma 3.3. *Let κ and θ be regular cardinals with $\kappa \leq \theta$. Suppose that $M \in \mathcal{P}_\kappa \mathcal{H}_\theta$ is such that $M \cap \kappa \in \kappa$ and $M \prec \langle \mathcal{H}_\theta, \in, \kappa \rangle$. Suppose also that M is i.a. of length ζ, where ζ is a limit ordinal $< \kappa$. Then* $\mathrm{cf}(\sup(M \cap \nu)) = \mathrm{cf}(\zeta)$ *for every regular cardinal ν with $\kappa \leq \nu \leq \theta$.*

Proof. Fix a regular cardinal ν with $\kappa \leq \nu \leq \theta$. Let $\langle M_\xi \mid \xi < \zeta \rangle$ be an i.a. sequence to M. First note that $M_\xi \in M$ for each $\xi < \zeta$ because M_ξ is the last element of $\langle M_\eta \mid \eta \leq \xi \rangle \in M$. From this, it follows that $\sup(M_\xi \cap \nu) \in M \cap \nu$ for each $\xi < \zeta$: First note that $\sup(M_\xi \cap \nu) < \nu$ because ν is a regular cardinal $\geq \kappa$, and $M_\xi \in \mathcal{P}_\kappa \mathcal{H}_\theta$. We show that $\sup(M_\xi \cap \nu) \in M$. Let $\upsilon := \min(M \setminus \nu)$ if $\nu < \sup(M \cap \theta)$, and $\upsilon := \theta$ otherwise. Note that $M_\xi \cap \nu = M_\xi \cap \upsilon$ because $M_\xi \subseteq M$. Moreover $\sup(M_\xi \cap \upsilon) \in M$ by the elementarity of M, the choice of υ and the fact that $M_\xi \in M$. Thus $\sup(M_\xi \cap \nu) = \sup(M_\xi \cap \upsilon) \in M$.

Hence $\sup(M_\xi \cap \nu) < \sup(M \cap \nu)$ for each $\xi < \zeta$. Moreover $\langle \sup(M_\xi \cap \nu) \mid \xi < \zeta \rangle$ converges to $\sup(M \cap \nu)$ because $\bigcup_{\xi < \zeta} M_\xi = M$. Thus $\mathrm{cf}(\sup(M \cap \nu)) = \mathrm{cf}(\zeta)$. □

Lemma 3.4. *Let κ be a regular uncountable cardinal, let λ be a $<\kappa$-strong limit cardinal with* $\mathrm{cf}(\lambda) < \kappa < \lambda$, *and let θ be a regular cardinal $> \lambda$. Moreover let $\varphi : \bigcup_{\lambda' < \lambda} {}^{<\kappa}(\mathcal{P}_\kappa \lambda') \to \lambda$ be a bijection. Suppose that $M \in \mathcal{P}_\kappa \mathcal{H}_\theta$ is such that $M \cap \kappa \in \kappa$ and $M \prec \langle \mathcal{H}_\theta, \in, \kappa, \lambda, \varphi \rangle$. Let ζ be a limit ordinal $< \kappa$, and suppose that $\vec{x} = \langle x_\xi \mid \xi < \zeta \rangle$ is a $\subseteq$-increasing sequence with $\bigcup_{\xi < \zeta} x_\xi = M \cap \lambda$. Then the following are equivalent:*

(I) $\vec{x}$ is a φ-s.i.a. sequence to $M \cap \lambda$.

(II) $\langle x_\xi \cap \lambda' \mid \xi < \zeta' \rangle \in M$ for every $\zeta' < \zeta$ and $\lambda' < \lambda$.

(III) For every $\lambda' < \lambda$ there exists $\lambda'' \geq \lambda'$ such that $\langle x_\xi \cap \lambda'' \mid \xi < \zeta' \rangle \in M$ for every $\zeta' < \zeta$.

Proof. (I)⇒(II) follows from the fact that M is closed under φ^{-1}. (II)⇒(III) is clear. We prove (III)⇒(I).

Assume (III). It suffices to show that $\varphi(\langle x_\xi \cap \lambda' \mid \xi < \zeta' \rangle) \in M \cap \lambda$ for every $\zeta' < \zeta$ and every $\lambda' < \lambda$. Fix $\zeta' < \zeta$ and $\lambda' < \lambda$.

By (III) take $\lambda'' \geq \lambda'$ such that $\langle x_\xi \cap \lambda'' \mid \xi < \zeta' \rangle \in M$. Moreover let $\nu := \min(M \cap \mathrm{On} \setminus \lambda')$. Note that $x_\xi \cap \lambda' = (x_\xi \cap \lambda'') \cap \nu$ for each $\xi < \zeta'$. Hence $\langle x_\xi \cap \lambda' \mid \xi < \zeta' \rangle \in M$ because $\nu, \langle x_\xi \cap \lambda'' \mid \xi < \zeta' \rangle \in M$. Then $\varphi(\langle x_\xi \cap \lambda' \mid \xi < \zeta' \rangle) \in M \cap \lambda$ because M is closed under φ. □

Lemma 3.5. *Let κ, λ, θ, φ and M be as in Lem.3.4. Suppose also that $M \cap \lambda$ is φ-s.i.a. of length ζ, where ζ is a limit ordinal $< \kappa$. Then* $\mathrm{cf}(\sup(M \cap \nu)) = \mathrm{cf}(M \cap \kappa)$ *for every regular cardinal ν with $\kappa \leq \nu < \lambda$.*

Proof. Similar as the proof of Lem.3.3 using Lem.3.4. □

Here we make a remark on the length of i.a. and s.i.a. sequences:

Let κ and θ be regular cardinals with $\kappa \leq \theta$, and suppose that $M \in \mathcal{P}_\kappa \mathcal{H}_\theta$ and that $M \cap \kappa \in \kappa$. Then the cofinality of the length of i.a. sequences to M is unique. It must be $\mathrm{cf}(M \cap \kappa)$ by Lem.3.3. But the length of i.a. sequences to M may not be uniquely determined.

For example, suppose that $M \cap \kappa$ is not regular and that M is i.a. of length $\mathrm{cf}(M \cap \kappa)$. Then M is i.a. of length ζ for every $\zeta < M \cap \kappa$ with $\mathrm{cf}(\zeta) = \mathrm{cf}(M \cap \kappa)$: Take an arbitrary such ζ. Let $\langle M_\eta \mid \eta < \mathrm{cf}(M \cap \kappa)\rangle$ be an i.a. sequence to M, and take an increasing cofinal sequence $\langle \zeta_\eta \mid \eta < \mathrm{cf}(M \cap \kappa)\rangle \in M$. Moreover for each $\xi < \zeta$, taking the least $\eta < \mathrm{cf}(M \cap \kappa)$ with $\xi \leq \zeta_\eta$, let $M'_\xi := M_\eta$. Then it is easy to see that $\langle M'_\xi \mid \xi < \zeta\rangle$ is an i.a. sequence to M.

The similar is the case for the length of s.i.a. sequences.

Next we present the relationship between the notion of s.i.a. and i.a:

Lemma 3.6. *Let κ, λ, θ, φ and M be as in Lem.3.4. Let ζ and ρ be limit ordinals $< \kappa$.*

(1) If M is i.a. of length ζ, then $M \cap \lambda$ is φ-s.i.a. of length ζ.

(2) If M is i.a. but is not i.a. of length ρ, then $M \cap \lambda$ is φ-s.i.a. but is not φ-s.i.a. of length ρ.

Proof. (1) Let $\langle M_\xi \mid \xi < \zeta\rangle$ be an i.a. sequence to M. Then $\langle M_\xi \cap \lambda \mid \xi < \zeta\rangle$ is a φ-s.i.a. sequence to $M \cap \lambda$ by Lem.3.4.

(2) By (1) it suffices to show that if M is i.a. but is not i.a. of length ρ, then $M \cap \lambda$ is not φ-s.i.a. of length ρ. We prove that if M is i.a. and $M \cap \lambda$ is φ-s.i.a. of length ρ, then M is i.a. of length ρ. Suppose that M is i.a. of length ζ, and let $\langle M_\xi \mid \xi < \zeta\rangle$ be an i.a. sequence to M. Moreover suppose that $M \cap \lambda$ is φ-s.i.a. of length ρ, and let $\langle x_\eta \mid \eta < \rho\rangle$ be a φ-s.i.a. sequence to $M \cap \lambda$.

Note that if $\eta < \rho$, then $\sup(x_\eta \cap \kappa) \in M \cap \kappa$, and so there exists $\xi < \zeta$ with $\sup(x_\eta \cap \kappa) \leq \sup(M_\xi \cap \kappa)$. For each $\eta < \rho$ let ξ_η be the least $\xi < \zeta$ with $\sup(x_\eta \cap \kappa) \leq \sup(M_\xi \cap \kappa)$. Then $\langle \xi_\eta \mid \eta < \rho\rangle$ is increasing clearly. Moreover it is cofinal in ζ because $\bigcup_{\eta<\rho} x_\eta \cap \kappa = M \cap \kappa$. Here note that if $\rho' < \rho$, then $\langle \xi_\eta \mid \eta < \rho'\rangle$ can be recovered from $\langle x_\eta \cap \kappa \mid \eta < \rho'\rangle$ and $\langle M_\xi \mid \xi \leq \xi_{\rho'}\rangle$, both of which belong to M. Hence $\langle \xi_\eta \mid \eta < \rho'\rangle \in M$ for every $\rho' < \rho$.

Then it is easy to see that $\langle M_{\xi_\eta} \mid \eta < \rho\rangle$ is an i.a. sequence to M. Therefore M is i.a. of length ρ. $\square$

Combining above lemmata we obtain the following:

Lemma 3.7. *Let κ be a regular uncountable cardinal and λ be a $< \kappa$-strong limit cardinal with $\mathrm{cf}(\lambda) < \kappa < \lambda$.*

(1) If $\kappa = \omega_1$, then $T^0_{\kappa\lambda}$ contains a club in $\mathcal{P}_\kappa\lambda$.

(2) If $\kappa \geq \omega_2$, then $T^0_{\kappa\lambda}$, $T^1_{\kappa\lambda}$ and $\mathcal{P}_\kappa\lambda \setminus (T^0_{\kappa\lambda} \cup T^1_{\kappa\lambda})$ are all stationary in $\mathcal{P}_\kappa\lambda$.

Proof. Let $\varphi : \bigcup_{\lambda'<\lambda} {}^{<\kappa}(\mathcal{P}_\kappa\lambda') \to \lambda$ be a bijection, and let $T^0_{\kappa\lambda}$ and $T^1_{\kappa\lambda}$ denote $T^{\varphi,0}_{\kappa\lambda}$ and $T^{\varphi,1}_{\kappa\lambda}$, respectively. Moreover let θ be a regular cardinal $> \lambda$, and let $\mathcal{M}$ be the structure $\langle \mathcal{H}_\theta, \in, \kappa, \lambda, \varphi\rangle$.

(1) Suppose $\kappa = \omega_1$. Note that $\mathrm{cf}(\lambda) = \omega$. Let $\bar{T}$ be the set of all $M \in \mathcal{P}_{\omega_1}\mathcal{H}_\theta$ with $M \prec \mathcal{M}$. Note that if $M \in \bar{T}$, then $M \cap \omega_1 \in \omega_1$ and M is i.a. of length ω. Hence $T^0_{\omega_1\lambda} \supseteq \{M \cap \lambda \mid M \in \bar{T}\}$ by Lem.3.6 (1). Then $T^0_{\omega_1\lambda}$ contains a club in $\mathcal{P}_{\omega_1}\lambda$ because $\bar{T}$ is a club in $\mathcal{P}_{\omega_1}\mathcal{H}_\theta$.

(2) Suppose $\kappa \geq \omega_2$. Then there is a regular cardinal $\mu < \kappa$ with $\mu \neq \mathrm{cf}(\lambda)$. Let

$$\bar{T}^0 := \{M \in \mathcal{P}_\kappa\mathcal{H}_\theta \mid M \cap \kappa \in \kappa \ \wedge \ M \prec \mathcal{M} \ \wedge \ M \text{ is i.a. of length } \mathrm{cf}(\lambda)\}\ ,$$
$$\bar{T}^1 := \{M \in \mathcal{P}_\kappa\mathcal{H}_\theta \mid M \cap \kappa \in \kappa \ \wedge \ M \prec \mathcal{M} \ \wedge \ M \text{ is i.a. of length } \mu\}\ .$$

Then both $\bar{T}^0$ and $\bar{T}^1$ are stationary by Lem.3.2. First note that $T^0_{\kappa\lambda} \supseteq \{M \cap \lambda \mid M \in \bar{T}^0\}$ by Lem.3.6 (1). Hence $T^0_{\kappa\lambda}$ is stationary. Next note that every $M \in \bar{T}^1$ is not i.a. of length $\mathrm{cf}(\lambda)$ by Lem.3.3. Thus $T^1_{\kappa\lambda} \supseteq \{M \cap \lambda \mid M \in \bar{T}^1\}$ by Lem.3.6. Therefore $T^1_{\kappa\lambda}$ is stationary.

It is easy to see that the set $S := \{x \in \mathcal{P}_\kappa\lambda \mid \mathrm{cf}(\sup(x \cap \kappa^+)) \neq \mathrm{cf}(\sup(x \cap \kappa))\}$ is stationary in $\mathcal{P}_\kappa\lambda$. But both $S \cap T^0_{\kappa\lambda}$ and $S \cap T^1_{\kappa\lambda}$ are nonstationary by Lem.3.5. Hence $\mathcal{P}_\kappa\lambda \setminus (T^0_{\kappa\lambda} \cup T^1_{\kappa\lambda})$ is stationary. □

At the end of this section we present lemmata on the length of i.a. and s.i.a. sequences:

Lemma 3.8. *Let κ and θ be regular uncountable cardinals with $\kappa \leq \theta$. Suppose that $M \in \mathcal{P}_\kappa\mathcal{H}_\theta$ is such that $M \cap \kappa \in \kappa$ and $M \prec \langle \mathcal{H}_\theta, \in, \kappa\rangle$. Suppose also that M is i.a. Then M is either i.a. of length $M \cap \kappa$ or i.a. of length $\mathrm{cf}(M \cap \kappa)$.*

Proof. We show that if M is not i.a. of length $M \cap \kappa$, then M is i.a. of length $\mathrm{cf}(M \cap \kappa)$. Assume that M is not i.a. of length $M \cap \kappa$.

Suppose that M is i.a. of length ζ and that $\langle M_\xi \mid \xi < \zeta\rangle$ is an i.a. sequence to M. Note that if $\zeta' < \zeta$, then ζ' is the length of the sequence $\langle M_\xi \mid \xi < \zeta'\rangle \in M$, and thus $\zeta' \in M \cap \kappa$. Hence $\zeta \leq M \cap \kappa$. But $\zeta \neq M \cap \kappa$ by our assumption. Therefore $\zeta \in M \cap \kappa$.

Note that $\mathrm{cf}(\zeta) = \mathrm{cf}(M \cap \kappa)$ by Lem.3.3. Because $\zeta \in M$ we can take an increasing sequence $\langle \xi_\eta \mid \eta < \mathrm{cf}(M \cap \kappa)\rangle \in M$ converging to ζ. Then it is easy to see that $\langle M_{\xi_\eta} \mid \eta < \mathrm{cf}(M \cap \kappa)\rangle$ is an i.a. sequence to M. Therefore M is i.a. of length $\mathrm{cf}(M \cap \kappa)$. □

Lemma 3.9. *Let κ, λ, θ, φ and M be as in Lem.3.4. Suppose also that $M \cap \lambda$ is φ-s.i.a. Then $M \cap \lambda$ is either φ-s.i.a. of length $M \cap \kappa$ or φ-s.i.a. of length $\mathrm{cf}(M \cap \kappa)$.*

Proof. Similar as the proof of Lem.3.8 using Lem.3.4. □

Note that there are stationary many $M \in \mathcal{P}_\kappa\mathcal{H}_\theta$ which is i.a. of length $\mathrm{cf}(M \cap \kappa)$ by Lem.3.2 and 3.3. Then there are also stationary many $x \in \mathcal{P}_\kappa\lambda$ which is φ-s.i.a. of length $\mathrm{cf}(x \cap \kappa)$ by Lem.3.6 (1). Below we prove that if $I[\kappa]$ is proper, then there are stationary many $M \in \mathcal{P}_\kappa\mathcal{H}_\theta$ which is i.a. of length $M \cap \kappa$ but is not i.a. of length $\mathrm{cf}(M \cap \kappa)$. Then the similar holds for φ-s.i.a. by Lem.3.6:

Lemma 3.10. *Let μ and κ are regular cardinals with $\mu < \kappa$, and suppose that $\{\zeta < \kappa \mid \mathrm{cf}(\zeta) = \mu\} \notin I[\kappa]$.*

(1) Suppose that θ is a regular cardinal $\geq \kappa$. Then there are stationary many $M \in \mathcal{P}_\kappa \mathcal{H}_\theta$ such that $M \cap \kappa \in \kappa$, $\mathrm{cf}(M \cap \kappa) = \mu$ and M is i.a. of length $M \cap \kappa$ but is not i.a. of length μ.

(2) Suppose that λ is a $<\kappa$-strong limit cardinal with $\mathrm{cf}(\lambda) < \kappa < \lambda$, and let $\varphi : \bigcup_{\lambda' < \lambda} {}^{<\kappa}\mathcal{P}_\kappa \lambda' \to \lambda$ be a bijection. Then there are stationary many $x \in \mathcal{P}_\kappa \lambda$ such that $x \cap \kappa \in \kappa$, $\mathrm{cf}(x \cap \kappa) = \mu$ and x is φ-s.i.a. of length $x \cap \kappa$ but is not φ-s.i.a. of length μ.

Proof. Note that (2) follows from (1) and Lem.3.6. We prove (1).

Take an arbitrary function $F : [\mathcal{H}_\theta]^{<\omega} \to \mathcal{H}_\theta$. It suffices to find $M \in \mathcal{P}_\kappa \mathcal{H}_\theta$ such that

(i) M is closed under F,

(ii) $M \cap \kappa \in \kappa$ and $\mathrm{cf}(M \cap \kappa) = \mu$,

(iii) M is i.a. of length $M \cap \kappa$ but is not i.a. of length μ.

Take a well-ordering Δ of $\mathcal{H}_\theta$, and let $\mathcal{M}$ be the structure $\langle \mathcal{H}_\theta, \in, \Delta, F \rangle$. By induction on $\xi < \kappa$ define $M_\xi \in \mathcal{P}_\kappa \mathcal{H}_\theta$ as follows: If ξ is a limit ordinal, then let $M_\xi := \bigcup_{\eta < \xi} M_\eta$. If ξ is a successor ordinal, then let

$$M_\xi \ := \ \mathrm{skull}^{\mathcal{M}}(M_{\xi-1} \cup \xi \cup \{\langle M_\eta \mid \eta < \xi \rangle\}) \ .$$

Note that $M_\xi \prec \mathcal{M}$ for every $\xi < \kappa$ and thus that M_ξ is closed under F for every $\xi < \kappa$. Note also that if ζ is a limit ordinal $< \kappa$, then $\langle M_\xi \mid \xi < \zeta \rangle$ is an i.a. sequence to M_ζ.

Let $M_\kappa := \bigcup_{\xi < \kappa} M_\xi$, and take an enumeration $\langle b_\xi \mid \xi < \kappa \rangle$ of all bounded subsets of κ which belong to M_κ. Moreover let C be the set of all limit $\zeta < \kappa$ such that $M_\zeta \cap \kappa = \zeta$ and such that $\{b_\xi \mid \xi < \zeta\}$ is equal to the set of all bounded subsets of κ which are in M_ζ. It is easy to see that C is a club in κ.

By our assumption on $I[\kappa]$ we can take $\zeta \in C$ of cofinality μ which is not approachable with respect to $\langle b_\xi \mid \xi < \kappa \rangle$. Then $M := M_\zeta$ satisfies the properties (i) and (ii) above and is i.a. of length $\zeta = M \cap \kappa$. Hence it suffices to show that M is not i.a. of length μ.

Assume that M is i.a. of length μ, and let $\langle N_\eta \mid \eta < \mu \rangle$ is an i.a. sequence to M. Let $b := \{\sup(N_\eta \cap \kappa) \mid \eta < \mu\}$. Then b is an unbounded subset of $M \cap \kappa = \zeta$ of order-type μ. Moreover all proper initial segments of b belong to M because all proper initial segments of $\langle N_\eta \mid \eta < \mu \rangle$ belong to M. Then all proper initial segments of b belong to $\{b_\xi \mid \xi < \zeta\}$ because $\zeta \in C$. This contradicts that ζ is not approachable with respect to $\langle b_\xi \mid \xi < \kappa \rangle$. □

3.2 i.a, preservation and reflection

First we review the relationship between the notion of i.a. and the preservation of stationary subsets of $\mathcal{P}_\kappa\lambda$ under $<\kappa$-closed forcing extensions, due to [4] and [5]:

Lemma 3.11. *Let κ be a regular uncountable cardinal and λ be a cardinal $\geq \kappa$. The following are equivalent for a stationary $S \subseteq \mathcal{P}_\kappa\lambda$:*

(I) S remains stationary for all $<\kappa$-closed forcing extensions.

(II) S remains stationary in the forcing extension by $\mathrm{Col}(\kappa, \lambda)$.

(III) For every regular cardinal $\theta \geq \lambda$ the set of all $M \in \mathcal{P}_\kappa\mathcal{H}_\theta$ with the following properties is stationary in $\mathcal{P}_\kappa\mathcal{H}_\theta$:

(i) there exist a limit ordinal $\zeta < \kappa$ and a $\subseteq$-increasing sequence $\langle x_\xi \mid \xi < \zeta\rangle$ in $\mathcal{P}_\kappa\lambda$ such that $\bigcup_{\xi<\zeta} x_\xi = M \cap \lambda$ and such that $\langle x_\xi \mid \xi < \zeta'\rangle \in M$ for every $\zeta' < \zeta$.

(ii) $M \cap \lambda \in S$.

(IV) For every regular cardinal $\theta \geq \lambda$ the set

$$\{M \in \mathcal{P}_\kappa\mathcal{H}_\theta \mid M \text{ is i.a.} \wedge M \cap \lambda \in S\}$$

is stationary in $\mathcal{P}_\kappa\mathcal{H}_\theta$.

To prove the lemma above we need the following lemma:

Lemma 3.12. *Let κ be a regular uncountable cardinal and W be a set with $|W| = \kappa \subseteq W$. Then all stationary subset of $\mathcal{P}_\kappa W$ remain stationary in all $<\kappa$-closed forcing extensions.*

Proof. Roughly speaking, the lemma follows from the facts that stationary subsets of $\mathcal{P}_\kappa W$ can be identified with stationary subsets of κ and that all stationary subsets remain stationary in all $<\kappa$-closed forcing extensions. But here we give a direct proof of the lemma.

Suppose that $S \subseteq \mathcal{P}_\kappa W$ is stationary and that $\mathbb{P}$ is a $<\kappa$-closed forcing notion. We show that S remains stationary in $V^{\mathbb{P}}$. Take an arbitrary $\mathbb{P}$-name $\dot{F}$ for a function from $[W]^{<\omega}$ to W and an arbitrary $p \in \mathbb{P}$. It suffices to find $p^* \leq p$ and $x \in S$ such that $x \cap \kappa \in \kappa$ and $p^* \Vdash$ "x is closed under $\dot{F}$".

Using the facts that $|W| = \kappa$ and that $\mathbb{P}$ is $<\kappa$-closed, we can easily construct a descending sequence $\langle p_\xi \mid \xi < \kappa\rangle$ in $\mathbb{P}$ below p such that for every $a \in [W]^{<\omega}$ some p_ξ decides $\dot{F}(a)$. For each $a \in [W]^{<\omega}$ let ξ_a be the least $\xi < \kappa$ such that p_ξ decides $\dot{F}(a)$. Moreover let $F : [W]^{<\omega} \to W$ be such that $F(a)$ is the value of $\dot{F}(a)$ decided by p_{ξ_a}.

Then there exists $x \in S$ such that $x \cap \kappa \in \kappa$ and x is closed under F because S is stationary. Take $\xi^* < \kappa$ such that $\xi_a \leq \xi^*$ for every $a \in [x]^{<\omega}$. Then $p_{\xi^*} \leq p$. Moreover $p_{\xi^*} \Vdash$ "x is closed under $\dot{F}$" because x is closed under F and $p_{\xi^*} \Vdash$ "$\dot{F} \restriction [x]^{<\omega} = F \restriction [x]^{<\omega}$". Thus p_{ξ^*} and x are what we seek. ☐

Proof of Lem.3.11. We prove (IV)⇒(III)⇒(II)⇒(I)⇒(IV). Fix a stationary $S \subseteq \mathcal{P}_\kappa\lambda$.

(IV)⇒(III):

Assume (IV). Let θ be a regular cardinal $\geq \lambda$. Let $\bar{S}$ be the set of all $M \in \mathcal{P}_\kappa\mathcal{H}_\theta$ such that $M \prec \langle \mathcal{H}_\theta, \in, \lambda \rangle$, M is i.a. and $M \cap \lambda \in S$. Then $\bar{S}$ is stationary by (IV). Moreover if $M \in \bar{S}$ and $\langle M_\xi \mid \xi < \zeta \rangle$ is an i.a. sequence to M, then it is easy to see that $\langle M_\xi \cap \lambda \mid \xi < \zeta \rangle$ witnesses that M satisfies (i) in (III). Hence the set in (III) includes $\bar{S}$, and therefore it is also stationary.

(III)⇒(II):

Assume (III). To prove (II) take an arbitrary $\mathrm{Col}(\kappa, \lambda)$-name $\dot{F}$ of a function from $[\lambda]^{<\omega}$ to λ and an arbitrary $p \in \mathrm{Col}(\kappa, \lambda)$. It suffices to find $x \in S$ and $p^* \leq p$ such that $x \cap \kappa \in \kappa$ and $p^* \Vdash_{\mathrm{Col}(\kappa,\lambda)}$ "x is closed under $\dot{F}$".

Take a sufficiently large regular cardinal θ and a well-ordering Δ of $\mathcal{H}_\theta$. Let $\mathcal{M}$ be the structure $\langle \mathcal{H}_\theta, \in, \Delta, \kappa, \lambda, p, \dot{F} \rangle$. Then, by (III), we can take $M \in \mathcal{P}_\kappa\mathcal{H}_\theta$ such that $M \prec \mathcal{M}$, $M \cap \kappa \in \kappa$ and M satisfies (i) and (ii) in (III). It suffices to find $p^* \leq p$ which forces that $x := M \cap \lambda$ is closed under $\dot{F}$.

Suppose that a sequence $\langle x_\xi \mid \xi < \zeta \rangle$ witnesses (i) for M. By induction on $\xi < \zeta$ let $p_\xi \in \mathrm{Col}(\kappa, \lambda)$ be the Δ-least lower bound of $\{p\} \cup \{p_\eta \mid \eta < \xi\}$ such that p_ξ decides $\dot{F}(a)$ for all $a \in [x_\xi]^{<\omega}$. We can take such p_ξ by the $<\kappa$-closure of $\mathrm{Col}(\kappa, \lambda)$.

Here note that $p_\xi \in M$ for each $\xi < \zeta$ because p_ξ can be definable in $\mathcal{M}$ from a parameter $\langle x_\eta \mid \eta \leq \xi \rangle \in M$. Hence p_ξ decides $\dot{F}(a)$ to be in $M \cap \lambda = x$ for each $a \in [x_\xi]^{<\omega}$.

Now take a lower bound p^* of $\{p_\xi \mid \xi < \zeta\}$. Then $p^* \leq p$, and p^* forces that x is closed under $\dot{F}$.

(II)⇒(I):

Assume (II). Take an arbitrary $<\kappa$-closed forcing notion $\mathbb{P}$. We must show that S remains stationary in $V^{\mathbb{P}}$.

Let υ be a regular cardinal with $\lambda, 2^{|\mathbb{P}|} < \upsilon$. In $V^{\mathrm{Col}(\kappa,\upsilon)}$, for every $p \in \mathbb{P}$, there exists a $\mathbb{P}$-generic filter over V containing p. Hence it suffices to show that S remains stationary in $V^{\mathrm{Col}(\kappa,\upsilon)}$. But note that $\mathrm{Col}(\kappa, \upsilon) \cong \mathrm{Col}(\kappa, \lambda) \times \mathrm{Col}(\kappa, \upsilon)$. So it suffices to show that S remains stationary in $V^{\mathrm{Col}(\kappa,\lambda)\times\mathrm{Col}(\kappa,\upsilon)}$.

First, S remains stationary in $V^{\mathrm{Col}(\kappa,\lambda)}$ by (II). Here note that $|\lambda| = \kappa$ in $V^{\mathrm{Col}(\kappa,\lambda)}$. Moreover $\mathrm{Col}(\kappa, \upsilon)$ is absolute between V and $V^{\mathrm{Col}(\kappa,\lambda)}$, and so $V^{\mathrm{Col}(\kappa,\lambda)\times\mathrm{Col}(\kappa,\upsilon)}$ is a $<\kappa$-closed forcing extension of $V^{\mathrm{Col}(\kappa,\lambda)}$. Therefore S remains stationary in $V^{\mathrm{Col}(\kappa,\lambda)\times\mathrm{Col}(\kappa,\upsilon)}$ by Lem.3.12.

(I)⇒(IV):

Assume (I). Let θ be a regular cardinal $\geq \lambda$ in V, let $H := {\mathcal{H}_\theta}^V$, and let $|H|^V = \upsilon$. Here note that $\mathcal{P}_\kappa H$ and the internally approachability of sets are absolute between V and $V^{\mathrm{Col}(\kappa,\upsilon)}$. Moreover $V \subseteq V^{\mathrm{Col}(\kappa,\upsilon)}$. Hence it suffices to show that the set $\{M \in \mathcal{P}_\kappa H \mid M \text{ is i.a.} \wedge M \cap \lambda \in S\}$ is stationary in $V^{\mathrm{Col}(\kappa,\upsilon)}$. But S remains stationary in $V^{\mathrm{Col}(\kappa,\upsilon)}$ by (1). So it suffices to prove that the set $\{M \in \mathcal{P}_\kappa H \mid M \text{ is i.a.}\}$ contains a club in $V^{\mathrm{Col}(\kappa,\upsilon)}$. We work in $V^{\mathrm{Col}(\kappa,\upsilon)}$.

Let $F : \kappa \to H$ be a surjection. Note that $\langle F[\xi] \mid \xi < \zeta\rangle \in H$ for each $\zeta < \kappa$ by the $<\kappa$-closure of $\mathrm{Col}(\kappa, \upsilon)$. Let C be the set of all limit $\zeta < \kappa$ such that for every $\zeta' < \zeta$ there exists ζ'' with $F(\zeta'') = \langle F[\xi] \mid \xi < \zeta'\rangle$. Then C is a club in κ. Hence the set $\{F[\zeta] \mid \zeta \in C\}$ contains a club in $\mathcal{P}_\kappa H$. Moreover $\langle F[\xi] \mid \xi < \zeta\rangle$ is an i.a. sequence to $F[\zeta]$ for every $\zeta \in C$. Thus $F[\zeta]$ is i.a. for every $\zeta \in C$. □

Fact 1.2 and what we mentioned below Fact 1.2 easily follow from Lem.3.11:

Corollary 3.13. *Let κ be a regular uncountable cardinal and λ be a cardinal with $\lambda^{<\kappa} = \lambda$.*

(1) All stationary subsets of $S_{\kappa\lambda}$ remain stationary in all $<\kappa$-closed forcing extensions.

(2) $\mathcal{P}_\kappa\lambda \setminus (S_{\kappa\lambda})^V$ becomes nonstationary in $V^{\mathrm{Col}(\kappa,\lambda)}$.

Proof. Let $\psi : {}^{<\kappa}\mathcal{P}_\kappa\lambda \to \lambda$ be a bijection, and let $S_{\kappa\lambda}$ denote $S^\psi_{\kappa\lambda}$. Take an arbitrary regular cardinal $\theta \geq \lambda$. Let $\bar{C}$ be the set of all $M \in \mathcal{P}_\kappa\mathcal{H}_\theta$ with $M \prec \langle\mathcal{H}_\theta, \in, \kappa, \lambda, \psi\rangle$. Moreover for each $S \subseteq \mathcal{P}_\kappa\lambda$ let $\bar{S}$ be the set of all $M \in \mathcal{P}_\kappa\mathcal{H}_\theta$ satisfying the properties (i) and (ii) in (III) of Lem.3.11.

Note that for each $M \in \bar{C}$ and each sequence $\langle x_\xi \mid \xi < \zeta\rangle$ in $\mathcal{P}_\kappa\lambda$, $\langle x_\xi \mid \xi < \zeta\rangle$ is a ψ-i.a. sequence to $M \cap \lambda$ if and only if it witnesses that M satisfies (i) in (III) of Lem.3.11. Hence

$$\bar{C} \cap \bar{S} \;=\; \bar{C} \cap \{M \in \mathcal{P}_\kappa\mathcal{H}_\theta \mid M \cap \lambda \in S \cap S_{\kappa\lambda}\}\,.$$

Here note that $\bar{C}$ is a club. Thus $\bar{S}$ is stationary in $\mathcal{P}_\kappa\mathcal{H}_\theta$ if and only if the set $\{M \in \mathcal{P}_\kappa\mathcal{H}_\theta \mid M \cap \lambda \in S \cap S_{\kappa\lambda}\}$ is stationary in $\mathcal{P}_\kappa\mathcal{H}_\theta$. But the latter is equivalent to that $S \cap S_{\kappa\lambda}$ is stationary in $\mathcal{P}_\kappa\lambda$. Therefore $\bar{S}$ is stationary in $\mathcal{P}_\kappa\mathcal{H}_\theta$ if and only if $S \cap S_{\kappa\lambda}$ is stationary in $\mathcal{P}_\kappa\lambda$. Both (1) and (2) easily follow from this fact and Lem.3.11. □

The corollary below immediately follows from Lem.3.6 (1) and 3.11:

Corollary 3.14. *Let κ be a regular uncountable cardinal, and λ be a $<\kappa$-strong limit cardinal with $\mathrm{cf}(\lambda) < \kappa < \lambda$. Then $\mathcal{P}_\kappa\lambda \setminus (T^0_{\kappa\lambda} \cup T^1_{\kappa\lambda})^V$ becomes nonstationary in $V^{\mathrm{Col}(\kappa,\lambda)}$.*

Finally we review the relationship between the preservation and the reflection of stationary subsets of $\mathcal{P}_\kappa\lambda$. We give the proof of Fact 1.3:

Proposition 3.15 ([5])**.** *Let κ be a regular uncountable cardinal and λ be a cardinal $> \kappa$. Suppose that ν is a λ-supercompact cardinal with $\kappa < \nu < \lambda$. In $V^{\mathrm{Col}(\kappa,<\nu)}$ assume that T is a stationary subset of $\mathcal{P}_\kappa\lambda$ such that all stationary subsets of T remain stationary in all $<\kappa$-closed forcing extensions. Then $\mathsf{SR}_{\kappa\lambda}(T)$ holds in $V^{\mathrm{Col}(\kappa,<\nu)}$.*

Proof. We make a remark before starting the proof. Several models of ZFC will appear in the proof below. But ${}^{<\kappa}\mathrm{On}$ is absolute among all models, and thus notions such as $\mathrm{Col}(\kappa, *)$ and $\mathcal{P}_\kappa *$ are absolute among them.

Now we start the proof. In V let $j : V \to M$ be a λ-supercompact embedding with $\mathrm{crit}(j) = \nu$, where crit denotes the critical point. Let G be a $\mathrm{Col}(\kappa, <\nu)$-generic filter over V, and in $V[G]$ let T be a stationary subset of $\mathcal{P}_\kappa\lambda$ such that all stationary subsets of T remain stationary in all $<\kappa$-closed forcing extensions of $V[G]$. We show that $\mathsf{SR}_{\kappa\lambda}(T)$ holds in $V[G]$. For this take an arbitrary stationary $S \subseteq T$ in $V[G]$. We must show that the following holds in $V[G]$:

(I) There exists $W \subseteq \lambda$ such that $|W| = \kappa \subseteq W$ and such that $S \cap \mathcal{P}_\kappa W$ is stationary.

First note that

$$j(\mathrm{Col}(\kappa, <\nu)) \; = \; \mathrm{Col}(\kappa, <j(\nu)) \; \cong \; \mathrm{Col}(\kappa, <\nu) \times \mathrm{Col}(\kappa, [\nu, j(\nu))) \; ,$$

where $\mathrm{Col}(\kappa, [\nu, j(\nu)))$ is the $<\kappa$-support product of $\langle \mathrm{Col}(\kappa, \nu') \mid \nu \leq \nu' < j(\nu) \rangle$. Let H be a $\mathrm{Col}(\kappa, [\nu, j(\nu)))$-generic filter over $V[G]$. We work in $V[G*H]$ below.

Note that $G * H$ is $j(\mathrm{Col}(\kappa, <\nu))$-generic over M. Then by the standard argument, $j : V \to M$ can be extended to the elementary embedding $j^* : V[G] \to M[G*H]$. For proving (I) in $V[G]$, it suffices to show that the following holds in $M[G*H]$ by the elementarity of j^*:

(II) There exists $W \subseteq j^*(\lambda)$ such that $|W| = \kappa \subseteq W$ and $j^*(S) \cap \mathcal{P}_\kappa W$ is stationary.

(Here note that $\kappa < \nu = \mathrm{crit}(j^*)$ and thus that $j^*(\kappa) = \kappa$.)

We show that $W := j^*[\lambda] \in M[G*H]$ witnesses (II). Clearly $|j^*[\lambda]| = |\lambda| = \kappa \subseteq j^*[\lambda]$ in $M[G*H]$. We prove that $j^*(S) \cap \mathcal{P}_\kappa j^*[\lambda]$ is stationary in $M[G*H]$.

First note that S remains stationary in $\mathcal{P}_\kappa\lambda$ in $V[G*H]$ by the assumption on T. Hence the set $\{j^*[x] \mid x \in S\}$ is stationary in $\mathcal{P}_\kappa j^*[\lambda]$ in $V[G*H]$. But $j^*(x) = j^*[x]$ for each $x \in S$ because $|x| < \kappa < \nu = \mathrm{crit}(j^*)$. Thus $\{j^*[x] \mid x \in S\} \subseteq j^*(S) \cap \mathcal{P}_\kappa j^*[\lambda]$. So $j^*(S) \cap \mathcal{P}_\kappa j^*[\lambda]$ is stationary in $V[G*H]$. Then $j^*(S) \cap \mathcal{P}_\kappa j^*[\lambda]$ is stationary also in $M[G*H]$ because $M[G*H] \subseteq V[G*H]$.

This completes the proof. □

4 Preservation and reflection below $T^0_{\kappa\lambda}$ and $T^1_{\kappa\lambda}$

In this section we prove Thm.1.7 and 1.8. We prove Thm.1.7 in Subsection 4.1 and prove Thm.1.8 in Subsection 4.2.

4.1 $T^0_{\kappa\lambda}$

Here we prove Thm.1.7:

Theorem 1.7. *Let κ be a regular uncountable cardinal and λ be a $<\kappa$-strong limit cardinal with $\mathrm{cf}(\lambda) < \kappa < \lambda$.*

(1) All stationary subsets of $T^0_{\kappa\lambda}$ remain stationary in all $<\kappa$-closed forcing extensions.

(2) *Suppose that ν is a λ-supercompact cardinal with $\kappa < \nu < \lambda$. Then* $\mathsf{SR}_{\kappa\lambda}(T^0_{\kappa\lambda})$ *holds in* $V^{\mathrm{Col}(\kappa,<\nu)}$.

Note that Thm.1.7 (2) follows from Thm.1.7 (1) and Prop.3.15. We prove Thm.1.7 (1). The point is that elements of $T^0_{\kappa\lambda}$ is essentially internally approachable:

Proof of Thm.1.7 (1). Let κ be a regular uncountable cardinal and λ be a $<\kappa$-strong limit cardinal with $\mathrm{cf}(\lambda) < \kappa < \lambda$. Fix a bijection $\varphi : \bigcup_{\lambda'<\lambda} {}^{<\kappa}(\mathcal{P}_\kappa\lambda') \to \lambda$, and let $T^0_{\kappa\lambda}$ be $T^{\varphi,0}_{\kappa\lambda}$. Let S be an arbitrary stationary subset of $T^0_{\kappa\lambda}$. It suffices to show that S satisfies (III) in Lem.3.11.

Let θ be a regular cardinal $> \lambda$, and let $\bar{S}$ be the set of all $M \in \mathcal{P}_\kappa\mathcal{H}_\theta$ such that $M \prec \langle \mathcal{H}_\theta, \in, \kappa, \lambda, \varphi\rangle$, $M \cap \lambda \in S$ and $M \cap \kappa \in \kappa$. Then $\bar{S}$ is stationary because S is stationary. We prove that if $M \in \bar{S}$, then M satisfies the properties (i) and (ii) in (III) of Lem.3.11. Fix $M \in \bar{S}$. All we have to show is that M satisfies (i).

Because $M \cap \lambda \in S \subseteq T^0_{\kappa\lambda}$ there exists a φ-s.i.a. sequence $\langle y_\xi \mid \xi < \mathrm{cf}(\lambda)\rangle$ to $M\cap\lambda$. Moreover take an increasing sequence $\langle \lambda_\xi \mid \xi < \mathrm{cf}(\lambda)\rangle \in M$ converging to λ. Note that $\mathrm{cf}(\lambda) \subseteq M$ because $\mathrm{cf}(\lambda) \in M \cap \kappa \in \kappa$. Hence all initial segments of $\langle \lambda_\xi \mid \xi < \mathrm{cf}(\lambda)\rangle$ belong to M.

Let $x_\xi := y_\xi \cap \lambda_\xi$ for each $\xi < \mathrm{cf}(\lambda)$. Then $\langle x_\xi \mid \xi < \mathrm{cf}(\lambda)\rangle$ is a $\subseteq$-increasing sequence in $\mathcal{P}_\kappa\lambda$ with $\bigcup_{\xi<\mathrm{cf}(\lambda)} x_\xi = M \cap \lambda$. Moreover $\langle x_\xi \mid \xi < \zeta\rangle \in M$ for every $\zeta < \mathrm{cf}(\lambda)$ because $\langle x_\xi \mid \xi < \zeta\rangle$ can be recovered from $\langle y_\xi \cap \lambda_\zeta \mid \xi < \zeta\rangle$ and $\langle \lambda_\xi \mid \xi < \zeta\rangle$, both of which belong to M. Therefore $\langle x_\xi \mid \xi < \mathrm{cf}(\lambda)\rangle$ witnesses that M satisfies (i) in (III) of Lem.3.11. □

4.2 $T^1_{\kappa\lambda}$

Here we prove Thm.1.8:

Theorem 1.8. *Assume* GCH. *Let κ be a regular cardinal $\geq \omega_2$ and λ be a singular cardinal with* $\mathrm{cf}(\lambda) < \kappa < \lambda$.

(1) *$T^1_{\kappa\lambda}$ (defined in V) remains stationary in all $<\kappa$-closed forcing extensions. But for every stationary $T \subseteq T^1_{\kappa\lambda}$ there exists stationary $S \subseteq T$ which becomes nonstationary in $V^{\mathrm{Col}(\kappa,\lambda)}$.*

(2) $\mathsf{SR}_{\kappa\lambda}(T)$ *fails for every stationary $T \subseteq T^1_{\kappa\lambda}$.*

Note that the first statement of Thm.1.8 (1) follows from Lem.3.2, 3.6 and 3.11. In fact we do not need the assumption of GCH for the first statement of Thm.1.8 (1):

Proposition 4.1. *Let κ be a regular cardinal $\geq \omega_2$ and λ be a $<\kappa$-strong limit cardinal with $\mathrm{cf}(\lambda) < \kappa < \lambda$. Then $(T^1_{\kappa\lambda})^V$ remains stationary in $\mathcal{P}_\kappa\lambda$ in all $<\kappa$-closed forcing extensions.*

Below we prove the remaining part of Thm.1.8. In fact we prove it from a weaker assumption than GCH. We use the following assumption:

Definition 4.2. *For a singular cardinal λ let Ψ_λ be the following assertion:*

$\Psi_\lambda \equiv 2^\lambda$ *is a regular cardinal, and there exist an increasing sequence $\vec{\lambda} = \langle \lambda_\eta \mid \eta < \mathrm{cf}(\lambda) \rangle$ of regular cardinals converging to λ and a $\mathrm{cf}(\lambda)$-complete proper ideal J over $\mathrm{cf}(\lambda)$ such that $\mathrm{tcf}(\Pi\vec{\lambda}/J) = 2^\lambda$.*

Note that if $2^\lambda = \lambda^+$, then $\mathrm{tcf}(\Pi\vec{\lambda}/\mathrm{BDD}_{\mathrm{cf}(\lambda)}) = 2^\lambda$ for every increasing sequence $\vec{\lambda}$ of regular cardinals converging to λ. Here $\mathrm{BDD}_{\mathrm{cf}(\lambda)}$ denotes the bounded ideal over $\mathrm{cf}(\lambda)$, i.e. the ideal consisting of all bounded subsets of $\mathrm{cf}(\lambda)$. Hence if GCH holds, then Ψ_λ holds for every singular cardinal λ.

Ψ_λ holds in some other situation. Suppose that $\lambda = \omega_\omega$ is strong limit. Then

$$2^{\omega_\omega} = \mathrm{cf}(\mathcal{P}_{\omega_2}\omega_\omega, \subseteq) = \max \mathrm{pcf}(\{\omega_n \mid n \in \omega\}) .$$

The former equality follows from the fact that ω_ω is strong limit. The latter is due to Shelah [14], and its proof can be also found in Holz-Steffens-Weitz [7] §8.4. Hence there exists a maximal ideal J over ω such that $\mathrm{tcf}(\Pi\langle \omega_n \mid n \in \omega\rangle/J) = 2^{\omega_\omega}$. Here note that every ideal is ω-complete. Therefore Ψ_{ω_ω} holds if ω_ω is strong limit. See [14] for other situation in which Ψ_λ holds.

We prove the following proposition which implies Thm.1.8:

Proposition 4.3. *Let κ be a regular uncountable cardinal, and let λ be a $<\kappa$-strong limit cardinal such that $\mathrm{cf}(\lambda) < \kappa < \lambda$ and such that Ψ_λ holds. Then the following hold:*

(1) For every stationary $T \subseteq T^1_{\kappa\lambda}$ there exists a stationary $S \subseteq T$ such that S becomes nonstationary in $V^{\mathrm{Col}(\kappa,\lambda)}$.

(2) Assume also that $\kappa^{<\kappa} = \kappa$. Then $\mathsf{SR}_{\kappa\lambda}(T)$ fails for every stationary $T \subseteq T^1_{\kappa\lambda}$.

The rest of this subsection is devoted to the proof of Prop.4.3. We need some preliminaries. First we fix our notation in the rest of this subsection. Let

κ: regular uncountable cardinal,

λ: $<\kappa$-strong limit cardinal with $\mathrm{cf}(\lambda) < \kappa < \lambda$ and Ψ_λ.

Note that if $\vec{\lambda} = \langle \lambda_\eta \mid \eta < \mathrm{cf}(\lambda) \rangle$ and J witnesses Ψ_λ, then $J \supseteq \mathrm{BDD}_{\mathrm{cf}(\lambda)}$, i.e. $\rho \in J$ for every $\rho < \mathrm{cf}(\lambda)$. This is because if $J \not\supseteq \mathrm{BDD}_{\mathrm{cf}(\lambda)}$, then there exists $\eta^* < \mathrm{cf}(\lambda)$ with $\{\eta^*\} \notin J$ by the $\mathrm{cf}(\lambda)$-completeness of J, and then there are no $<_J$-increasing sequence of length $> \lambda_{\eta^*}$. Hence we can take a pair of witness $\vec{\lambda} = \langle \lambda_\eta \mid \eta < \mathrm{cf}(\lambda) \rangle$ and J of Ψ_λ so that $\lambda_0 > \kappa$. Let

$\vec{\lambda} = \langle \lambda_\eta \mid \eta < \mathrm{cf}(\lambda) \rangle$: increasing sequence of regular cardinals converging to λ with $\lambda_0 > \kappa$,

J: $\mathrm{cf}(\lambda)$-complete proper ideal over $\mathrm{cf}(\lambda)$,

$\vec{h} = \langle h_\alpha \mid \alpha < 2^\lambda \rangle$: $(\vec{\lambda}, J)$-scale.

For each $x \in \mathcal{P}_\kappa\lambda$ let

ch_x := the characteristic function for x with respect to $\vec{\lambda}$, i.e. the function on $\mathrm{cf}(\lambda)$ such that $\mathrm{ch}_x(\eta) = \sup(x \cap \lambda_\eta)$ for each $\eta < \mathrm{cf}(\lambda)$,

β_x := the least $\beta < 2^\lambda$ with $\mathrm{ch}_x \leq_J h_\beta$.

Finally let

$\varphi : \bigcup_{\lambda' < \lambda} {}^{<\kappa}(\mathcal{P}_\kappa \lambda') \to \lambda$: bijection, $T^1_{\kappa\lambda} := T^{\varphi,1}_{\kappa\lambda}$,

θ: regular cardinal $> 2^\lambda$, Δ: well-ordering of $\mathcal{H}_\theta$,

$\mathcal{M} := \langle \mathcal{H}_\theta, \in, \Delta, \kappa, \lambda, \vec{\lambda}, J, \vec{h}, \varphi \rangle$.

$\mathcal{T} := \{ x \in T^1_{\kappa\lambda} \mid x \cap \kappa \in \kappa \wedge \mathrm{skull}^{\mathcal{M}}(x) \cap \lambda = x \}$

Note that $T^1_{\kappa\lambda} \setminus \mathcal{T}$ is nonstationary. Note also that $\mathrm{cf}(\lambda), \{\lambda_\eta \mid \eta < \mathrm{cf}(\lambda)\} \subseteq x$ for every $x \in \mathcal{T}$ by Lem.2.3.

Next we present lemmata on elements of $\mathcal{T}$:

Lemma 4.4. *Let M be an element of $\mathcal{P}_\kappa \mathcal{H}_\theta$ such that $M \cap \lambda \in \mathcal{T}$ and $M \prec \mathcal{M}$. Suppose that $\langle x_\xi \mid \xi < \zeta \rangle$ is a φ-s.i.a. sequence to $M \cap \lambda$, and let $\gamma^* := \sup_{\xi < \zeta} \beta_{x_\xi} < 2^\lambda$. Moreover let g be an element of $\Pi\vec{\lambda}$ such that $g \restriction \rho \in M$ for every $\rho < \mathrm{cf}(\lambda)$. Then $g \leq_J h_{\gamma^*}$.*

Proof. First of all note that $g(\eta) \in M \cap \lambda$ for every $\eta < \mathrm{cf}(\lambda)$ by the assumption on g. Thus for every $\eta < \mathrm{cf}(\lambda)$ there exists $\xi < \zeta$ with $g(\eta) \in x_\xi$.

The proof of the lemma splits into two cases by the cofinality of ζ:

Case 1: $\mathrm{cf}(\zeta) < \mathrm{cf}(\lambda)$.

Take an increasing sequence $\langle \xi_k \mid k < \mathrm{cf}(\zeta) \rangle$ cofinal in ζ. For each $k < \mathrm{cf}(\zeta)$ let $A_k := \{ \eta < \mathrm{cf}(\lambda) \mid g(\eta) \in x_{\xi_k} \}$. Note that $\mathrm{cf}(\lambda) = \bigcup_{k < \mathrm{cf}(\zeta)} A_k$ by the remark at the beginning of the proof.

Moreover let $B_k := \{ \eta \in A_k \mid g(\eta) > h_{\gamma^*}(\eta) \}$ for each $k < \mathrm{cf}(\zeta)$. Note that $B_k \in J$ for every $k < \mathrm{cf}(\zeta)$. This follows from the facts that $g(\eta) \leq \mathrm{ch}_{x_{\xi_k}}(\eta)$ for every $\eta \in A_k$ and that $\mathrm{ch}_{x_{\xi_k}} \leq_J h_{\gamma^*}$. Thus $B := \bigcup_{k < \mathrm{cf}(\zeta)} B_k \in J$ by the $\mathrm{cf}(\lambda)$-completeness of J.

Here note that $B = \{ \eta < \mathrm{cf}(\eta) \mid g(\eta) > h_{\gamma^*}(\eta) \}$ because $\bigcup_{k < \mathrm{cf}(\zeta)} A_k = \mathrm{cf}(\lambda)$. Therefore $B \in J$ means that $g \leq_J h_{\gamma^*}$.

Case 2: $\mathrm{cf}(\zeta) \geq \mathrm{cf}(\lambda)$.

In this case we claim that there exists $\xi^* < \zeta$ with $g[\mathrm{cf}(\lambda)] \subseteq x_{\xi^*}$. Note that if such ξ^* exists, then $g \leq \mathrm{ch}_{x_{\xi^*}} \leq_J h_{\gamma^*}$, and so $g \leq_J h_{\gamma^*}$. Thus it suffices to show that such ξ^* exists.

First note that for each $\eta < \mathrm{cf}(\lambda)$ there exists $\xi < \zeta$ with $g[\eta] \subseteq x_\xi$. This follows from the remark at the beginning of this proof and the fact that $\mathrm{cf}(\zeta) \geq \mathrm{cf}(\lambda)$. For each $\eta < \mathrm{cf}(\lambda)$ let ξ_η be the least $\xi < \zeta$ with $g[\eta] \subseteq x_\xi$.

Clearly $\langle \xi_\eta \mid \eta < \mathrm{cf}(\lambda) \rangle$ is an increasing sequence below ζ. Moreover note that $\langle \xi_\eta \mid \eta < \rho \rangle \in M$ for every $\rho < \mathrm{cf}(\lambda)$ because $\langle \xi_\eta \mid \eta < \rho \rangle$ can be recovered from $g \restriction \rho$ and $\langle x_\xi \cap \lambda_\rho \mid \xi \leq \xi_\rho \rangle$ both of which belong to M. Hence $\langle x_{\xi_\eta} \cap \lambda' \mid \eta < \rho \rangle \in M$ for every $\rho < \mathrm{cf}(\lambda)$ and every $\lambda' < \lambda$.

Thus if $\sup_{\eta<\mathrm{cf}(\lambda)} \xi_\eta = \zeta$, then it is easy to see that $\langle x_{\xi_\eta} \mid \eta < \mathrm{cf}(\lambda)\rangle$ becomes a φ-s.i.a. sequence to x, which contradicts that $x \in T^1_{\kappa\lambda}$. Therefore $\xi^* := \sup_{\eta<\mathrm{cf}(\lambda)} \xi_\eta < \zeta$. Then $g[\mathrm{cf}(\lambda)] \subseteq x_{\xi^*}$. □

Lemma 4.5. *Suppose that $x \in \mathcal{T}$ and that both $\langle x_\xi \mid \xi < \zeta\rangle$ and $\langle y_\iota \mid \iota < \epsilon\rangle$ are φ-s.i.a. sequence to x. Then $\sup_{\xi<\zeta} \beta_{x_\xi} = \sup_{\iota<\epsilon} \beta_{y_\iota}$.*

Proof. It suffices to show that $\sup_{\xi<\zeta} \beta_{x_\xi} \geq \sup_{\iota<\epsilon} \beta_{y_\iota}$. Let $\gamma^* := \sup_{\xi<\zeta} \beta_{x_\xi}$, and let $M := \mathrm{skull}^{\mathcal{M}}(x)$. Note that $M \cap \lambda = x$ and thus that $\langle x_\xi \mid \xi < \zeta\rangle$ is a φ-s.i.a. sequence to $M \cap \lambda$. Note also that if $\iota < \epsilon$, then $\mathrm{ch}_{y_\iota} \restriction \rho \in M$ for every $\rho < \mathrm{cf}(\lambda)$ because $\langle \lambda_\eta \mid \eta < \rho\rangle,\ y_\iota \cap \lambda_\rho \in M$. Hence $\mathrm{ch}_{y_\iota} \leq_J h_{\gamma^*}$, i.e. $\beta_{y_\iota} \leq \gamma^*$, for every $\iota < \epsilon$ by Lem.4.4. Therefore $\sup_{\iota<\epsilon} \beta_{y_\iota} \leq \gamma^*$. □

For each $x \in \mathcal{T}$, taking a φ-s.i.a. sequence $\langle x_\xi \mid \xi < \zeta\rangle$ to x, let

$$\gamma_x \ := \ \sup_{\xi<\zeta} \beta_{x_\xi} \ < \ 2^\lambda \ .$$

γ_x is independent of the choice of a φ-s.i.a. sequence to x by the above lemma. Note that if $x \in \mathcal{T}$ and $\langle x_\xi \mid \xi < \zeta\rangle$ is a φ-s.i.a. sequence to x, then $\beta_{x_\xi} \leq \beta_x$ for each $\xi < \zeta$. Hence

$$\gamma_x \leq \beta_x$$

for every $x \in \mathcal{T}$.

Lemma 4.6. *Let M be an element of $\mathcal{P}_\kappa\mathcal{H}_\theta$ such that $M \cap \lambda \in \mathcal{T}$, $M \prec \mathcal{M}$ and M is i.a. Then $\sup(M \cap 2^\lambda) = \gamma_{M\cap\lambda}$.*

Proof. Suppose that $\langle M_\xi \mid \xi < \zeta\rangle$ is an i.a. sequence to M. Note that $\langle M_\xi \cap \lambda \mid \xi < \zeta\rangle$ is a φ-s.i.a. sequence to $M \cap \lambda$. Hence $\gamma_{M\cap\lambda} = \sup_{\xi<\zeta} \beta_{M_\xi\cap\lambda}$. Moreover $\beta_{M_\xi\cap\lambda} \in M \cap 2^\lambda$ for every $\xi < \zeta$ because $M_\xi \cap \lambda \in M \prec \mathcal{M}$. Therefore $\gamma_{M\cap\lambda} \leq \sup(M \cap 2^\lambda)$.

To see the converse inequality, take an arbitrary $\beta \in M \cap 2^\lambda$. It suffices to show that $\beta \leq \gamma_{M\cap\lambda}$.

First note that $h_\beta \in M$ because $\beta \in M \prec \mathcal{M}$. Hence $h_\beta \restriction \rho \in M$ for every $\rho < \mathrm{cf}(\lambda)$. Then $h_\beta \leq_J h_{\gamma_{M\cap\lambda}}$ by Lem.4.4. Therefore $\beta \leq \gamma_{M\cap\lambda}$. □

Lemma 4.7. *$\{\gamma_x \mid x \in T\}$ is unbounded in 2^λ for every stationary $T \subseteq \mathcal{T}$.*

Proof. Let T be a stationary subset of $\mathcal{T}$. Take an arbitrary $\alpha < 2^\lambda$. We must find $x \in T$ with $\gamma_x \geq \alpha$.

Because T is stationary, we can take $M \in \mathcal{P}_\kappa\mathcal{H}_\theta$ such that $M \cap \lambda \in T$, $M \prec \mathcal{M}$ and $\alpha \in M \cap 2^\lambda$. Then $h_\alpha \restriction \rho \in M$ for every $\rho < \mathrm{cf}(\lambda)$. Then $h_\alpha \leq_J h_{\gamma_{M\cap\lambda}}$ by Lem.4.4. Therefore $\alpha \leq \gamma_{M\cap\lambda}$. This completes the proof because $M \cap \lambda \in T$. □

To prove Prop.4.3 (1), for each stationary $T \subseteq \mathcal{T}$ we must construct a stationary $S \subseteq T$ which becomes nonstationary in $V^{\mathrm{Col}(\kappa,\lambda)}$. To prove (2) we must construct a nonreflecting stationary $S \subseteq T$ for each stationary $T \subseteq \mathcal{T}$. The constructions of such S are the same. Here we present the construction. For each stationary $T \subseteq \mathcal{T}$ define $S_T \subseteq T$ as follows:

First let $\langle F_\alpha \mid \alpha \in 2^\lambda \rangle$ be the Δ-least enumeration of all functions from $[\lambda]^{<\omega}$ to $\mathcal{P}_\kappa\lambda$. By induction on $\alpha < 2^\lambda$ let z^T_α be the Δ-least $z \in T$ such that

- z is closed under F_α, i.e. $F_\alpha(a) \subseteq z$ for every $a \in [z]^{<\omega}$,
- $\gamma_z > \sup_{\alpha'<\alpha} \beta_{z_{\alpha'}}$.

Note that the set $\{z \in T \mid z \text{ is closed under } F_\alpha\}$ is stationary. Hence we can take such $z^T_\alpha \in T$ by Lem.4.7. Then let

$$S_T \;:=\; \{z^T_\alpha \mid \alpha < 2^\lambda\}\;.$$

Here we present basic properties of S_T:

Lemma 4.8. *Let T be a stationary subset of $\mathcal{T}$.*

(1) S_T is a stationary subset of T.

(2) The set $\{\gamma_z \mid z \in S_T\}$ is nonstationary in 2^λ.

(3) Suppose that ζ is a limit ordinal $< \kappa$ and that $\langle y_\xi \mid \xi < \zeta\rangle$ is a strictly $\subseteq$-increasing sequence of elements of S_T. Then there are no $z \in S_T$ with $\gamma_z = \sup_{\xi<\zeta} \beta_{y_\xi}$.

Proof. Throughout this proof we let z_α denote z^T_α for each $\alpha < 2^\lambda$.

(1) For every function $F : [\lambda]^{<\omega} \to \mathcal{P}_\kappa\lambda$ there exists $z \in S_T$ closed under F by the first property of the choice of each z_α. This means that S_T is stationary. Clearly $S_T \subseteq T$.

(2) First of all recall that $\gamma_z \leq \beta_z$ for every $z \in \mathcal{T}$. Hence $\gamma_{z_\alpha} > \sup_{\alpha'<\alpha} \gamma_{z_{\alpha'}}$ for every $\alpha < 2^\lambda$ by the second property of the choice of z_α.

Thus the set $\{\gamma_{z_\alpha} \mid \alpha < 2^\lambda\} = \{\gamma_z \mid z \in S_T\}$ is nonstationary in 2^λ.

(3) First of all note that if $\alpha' < \alpha < 2^\lambda$, then $\gamma_{z_{\alpha'}} \leq \beta_{z_{\alpha'}} < \gamma_{z_\alpha} \leq \beta_{z_\alpha}$ by the choice of z_α and the fact that $\gamma_z \leq \beta_z$ for every $z \in \mathcal{T}$.

For each $\xi < \zeta$ let $\alpha_\xi < 2^\lambda$ be such that $y_\xi = z_{\alpha_\xi}$. Then $\langle \alpha_\xi \mid \xi < \zeta\rangle$ is strictly increasing because both $\langle y_\xi \mid \xi < \zeta\rangle$ and $\langle \beta_{z_\alpha} \mid \alpha < 2^\lambda\rangle$ are strictly increasing. Let $\alpha^* := \sup_{\xi<\zeta} \alpha_\xi$. If $\alpha \geq \alpha^*$, then $\gamma_{z_\alpha} > \sup_{\xi<\zeta} \beta_{y_\xi}$ by the second property of the choice of z_α. On the other hand if $\alpha < \alpha^*$, then $\gamma_{z_\alpha} < \sup_{\xi<\zeta} \beta_{y_\xi}$ by the remark above. Therefore there are no $z \in S_T$ with $\gamma_z = \sup_{\xi<\zeta} \beta_{y_\xi}$. □

Below we prove that S_T becomes nonstationary in $V^{\mathrm{Col}(\kappa,\lambda)}$ and is nonreflecting for each stationary $T \subseteq \mathcal{T}$. First we prove that S_T becomes nonstationary in $V^{\mathrm{Col}(\kappa,\lambda)}$. This implies Prop.4.3 (1):

Lemma 4.9. *S_T becomes nonstationary in $V^{\mathrm{Col}(\kappa,\lambda)}$ for all stationary $T \subseteq \mathcal{T}$.*

Proof. Fix a stationary $T \subseteq \mathcal{T}$. Let $\bar{S}$ be the set of all $M \in \mathcal{P}_\kappa \mathcal{H}_\theta$ such that $M \cap \lambda \in S_T$, $M \prec \mathcal{M}$ and M is i.a. By Lem.3.11 it suffices to show that $\bar{S}$ is nonstationary in $\mathcal{P}_\kappa \mathcal{H}_\theta$.

Note that $\{\sup(M \cap 2^\lambda) \mid M \in \bar{S}\} \subseteq \{\gamma_z \mid z \in S_T\}$ by Lem.4.6 and that $\{\gamma_z \mid z \in S_T\}$ is nonstationary in 2^λ by Lem.4.8 (2). Hence $\{\sup(M \cap 2^\lambda) \mid M \in \bar{S}\}$ is nonstationary. This implies that $\bar{S}$ is nonstationary in $\mathcal{P}_\kappa \mathcal{H}_\theta$. □

Now we have proved Prop.4.3 (1). For (2) we must prove that S_T is nonreflecting. We prove this by splitting $\mathcal{T}$ into the following two sets:

$\mathcal{T}_0$:= the set of all $x \in \mathcal{T}$ which is φ-s.i.a. of length ζ for some $\zeta < \kappa$ with $\mathrm{cf}(\zeta) = \mathrm{cf}(\lambda)$.

$\mathcal{T}_1$:= the set of all $x \in \mathcal{T}$ which is φ-s.i.a. of length ζ for some $\zeta < \kappa$ with $\mathrm{cf}(\zeta) \neq \mathrm{cf}(\lambda)$.

Note that $\mathcal{T}_1$ is stationary because $\kappa \geq \omega_2$. In fact we proved that $\mathcal{T}_1$ is stationary in the proof of Lem.3.7 (2). On the other hand, note that elements of $\mathcal{T}_0$ are φ-s.i.a. of length ζ for some ζ with $\mathrm{cf}(\zeta) = \mathrm{cf}(\lambda)$ but are not i.a. of length $\mathrm{cf}(\lambda)$. By Lem.3.10, $\mathcal{T}_0$ is stationary if the set $\{\zeta < \kappa \mid \mathrm{cf}(\zeta) = \mathrm{cf}(\lambda)\}$ does not belong to $I[\kappa]$. Note also that if $x \in \mathcal{T}_0$, then x is φ-s.i.a. of length $x \cap \kappa$ by Lem.3.9.

Clearly the following two lemmata imply Prop.4.3 (2):

Lemma 4.10. *Assume that $\kappa^{<\kappa} = \kappa$, and suppose that $T \subseteq \mathcal{T}_0$ is stationary. Then $S_T \cap \mathcal{P}_\kappa W$ is nonstationary for every $W \subseteq \lambda$ with $|W| = \kappa \subseteq W$.*

Lemma 4.11. *Suppose that $T \subseteq \mathcal{T}_1$ is stationary. Then $S_T \cap \mathcal{P}_\kappa W$ is nonstationary for every $W \subseteq \lambda$ with $|W| = \kappa \subseteq W$.*

First we prove Lem.4.11. We use the following lemma, due to Shelah:

Lemma 4.12 (Shelah)**.** *Suppose that ζ is a limit ordinal $< \kappa$ with $\mathrm{cf}(\zeta) \neq \mathrm{cf}(\lambda)$ and that $\langle x_\xi \mid \xi < \zeta \rangle$ is a $\subseteq$-increasing sequence. Let $x := \bigcup_{\xi<\zeta} x_\xi$. Then $\beta_x = \sup_{\xi<\zeta} \beta_{x_\xi}$.*

Proof. Clearly $\beta_x \geq \sup_{\xi<\zeta} \beta_{x_\xi}$. We prove that $\beta_x \leq \sup_{\xi<\zeta} \beta_{x_\xi}$. Let $\gamma^* := \sup_{\xi<\zeta} \beta_{x_\xi}$. All we have to show is that $\mathrm{ch}_x \leq_J h_{\gamma^*}$. The proof splits into two cases:

Case 1: $\mathrm{cf}(\zeta) < \mathrm{cf}(\lambda)$.

Take an increasing cofinal sequence $\langle \xi_k \mid k < \mathrm{cf}(\zeta) \rangle$ in ζ. Note that $B_k := \{\eta < \mathrm{cf}(\lambda) \mid \mathrm{ch}_{x_{\xi_k}}(\eta) > h_{\gamma^*}(\eta)\} \in J$ because $\beta_{x_{\xi_k}} \leq \gamma^*$. Then $B := \bigcup_{k<\mathrm{cf}(\zeta)} B_k \in J$ by the $\mathrm{cf}(\lambda)$-completeness of J.

Here note that $\mathrm{ch}_x(\eta) = \sup_{k<\mathrm{cf}(\zeta)} \mathrm{ch}_{x_{\xi_k}}(\eta)$ for every $\eta < \mathrm{cf}(\lambda)$. Thus $\{\eta < \mathrm{cf}(\lambda) \mid \mathrm{ch}_x(\eta) > h_{\gamma^*}(\eta)\} = B \in J$. Therefore $\mathrm{ch}_x \leq_J h_{\gamma^*}$.

Case 2: $\mathrm{cf}(\zeta) > \mathrm{cf}(\lambda)$.

Assume that $\mathrm{ch}_x \not\leq_J h_{\gamma^*}$. Then $A := \{\eta < \mathrm{cf}(\lambda) \mid \mathrm{ch}_x(\eta) > h_{\gamma^*}(\eta)\} \notin J$. For each $\eta \in A$ we can take $\xi_\eta < \zeta$ such that $\mathrm{ch}_{x_{\xi_\eta}}(\eta) > h_{\gamma^*}(\eta)$ because $\mathrm{ch}_x(\eta) = \sup_{\xi<\zeta} \mathrm{ch}_{x_\xi}(\eta)$.

Then $\xi^* := \sup_{\eta \in A} \xi_\eta < \zeta$ because $\mathrm{cf}(\zeta) > \mathrm{cf}(\lambda)$. Moreover $\{\eta < \mathrm{cf}(\lambda) \mid \mathrm{ch}_{x_{\xi^*}}(\eta) > h_{\gamma^*}(\eta)\} = A \notin J$. Hence $\mathrm{ch}_{x_{\xi^*}} \not\leq_J h_{\gamma^*}$. This contradicts that $\beta_{x_{\xi^*}} \leq \sup_{\xi<\zeta} \beta_{x_\xi} = \gamma^*$. □

The following is immediate from the above lemma:

Corollary 4.13. *$\beta_x = \gamma_x$ for every $x \in \mathcal{T}_1$.*

Now we prove Lem.4.11:

Proof of Lem.4.11. Suppose that $T \subseteq \mathcal{T}_1$ is stationary. We claim the following:

Claim. *Suppose that ζ is a limit ordinal $< \kappa$ and that $\langle y_\xi \mid \xi < \zeta \rangle$ is a $\subseteq$-increasing sequence in S_T with $\langle y_\xi \cap \kappa \mid \xi < \zeta \rangle$ strictly increasing. Then $\bigcup_{\xi<\zeta} y_\xi \notin S_T$.*

$\vdash$ Let $y := \bigcup_{\xi<\zeta} y_\xi$. The proof splits into two cases by the cofinality of ζ.

First suppose that $\mathrm{cf}(\zeta) = \mathrm{cf}(\lambda)$. Then $\mathrm{cf}(y \cap \kappa) = \mathrm{cf}(\lambda)$, and thus $y \notin \mathcal{T}_1$ by Lem.3.5. Therefore $y \notin S_T$ because $S_T \subseteq T \subseteq \mathcal{T}_1$.

Next suppose that $\mathrm{cf}(\zeta) \neq \mathrm{cf}(\lambda)$. Assume that $y \in S_T$. Then

$$\sup_{\xi<\zeta} \beta_{y_\xi} \;=\; \beta_y \;=\; \gamma_y$$

by Lem.4.12 and Cor.4.13. This contradicts Lem.4.8 (3) because $y \in S_T$ and $\langle y_\xi \mid \xi < \zeta \rangle$ is a strictly $\subseteq$-increasing sequence in S_T. Therefore $y \notin S_T$. $\dashv$

Take an arbitrary $W \subseteq \lambda$ with $|W| = \kappa \subseteq W$. We show that $S_T \cap \mathcal{P}_\kappa W$ is nonstationary.

We can take a $\subseteq$-increasing continuous cofinal sequence $\langle x_\xi \mid \xi < \kappa \rangle$ in $\mathcal{P}_\kappa W$ with $\langle x_\xi \cap \kappa \mid \xi < \kappa \rangle$ strictly increasing. Let $E := \{\xi < \kappa \mid x_\xi \in S_T\}$. By the claim above, if ζ is a limit ordinal $< \kappa$ with $\sup(E \cap \zeta) = \zeta$, then $\zeta \notin E$. This implies that E is nonstationary in κ. Therefore $S_T \cap \mathcal{P}_\kappa W$ is nonstationary. □

We turn our attention to Lem.4.10. First we extend the notion of the φ-semi-internally approachability to subsets of λ of cardinality κ:

Definition 4.14. *Suppose that W is a subset of λ with $|W| = \kappa$. Then W is said to be* φ-semi-internally approachable of length κ *if there exists a $\subseteq$-increasing sequence $\langle w_\xi \mid \xi < \kappa \rangle$ in $\mathcal{P}_\kappa\lambda$ such that $\bigcup_{\xi<\kappa} w_\xi = W$ and such that $\varphi(\langle w_\xi \cap \lambda' \mid \xi < \zeta \rangle) \in W$ for every $\zeta < \kappa$ and every $\lambda' < \lambda$. We call a sequence $\langle w_\xi \mid \xi < \kappa \rangle$ as above a* φ-semi-internally approaching sequence to W.

Clearly Lem.4.10 follows from the two lemmata below:

Lemma 4.15. *Suppose that $T \subseteq \mathcal{T}_0$ is stationary. If $|W| = \kappa \subseteq W \subseteq \lambda$ and W is φ-s.i.a. of length κ, then $S_T \cap \mathcal{P}_\kappa W$ is nonstationary.*

Lemma 4.16. *Assume that $\kappa^{<\kappa} = \kappa$. If $|W| = \kappa \subseteq W \subseteq \lambda$ and $\mathcal{T}_0 \cap \mathcal{P}_\kappa W$ is stationary, then W is φ-s.i.a. of length κ.*

First we prove Lem.4.15:

Proof of Lem.4.15. Suppose that $|W| = \kappa \subseteq W \subseteq \lambda$ and that W is φ-s.i.a. of length κ. Assume that $S_T \cap \mathcal{P}_\kappa W$ is stationary. We work for a contradiction.

First note that $\mathrm{skull}^{\mathcal{M}}(W) \cap \lambda = W$ because $\mathrm{skull}^{\mathcal{M}}(x) \cap \lambda = x$ for each $x \in S_T$ and $S_T \cap \mathcal{P}_\kappa W$ is stationary. Then we can take a φ-s.i.a. sequence $\langle w_\xi \mid \xi < \kappa \rangle$ to W which is strictly $\subseteq$-increasing continuous. (First take a φ-s.i.a. sequence $\langle w'_\xi \mid \xi < \kappa \rangle$ to W. Let B be the set of all $\xi < \kappa$ such that $w_\xi \cap \kappa \supsetneq w_\eta \cap \kappa$ for every $\eta < \xi$. Note that B is a club in κ. Let $\langle \rho_\xi \mid \xi < \kappa \rangle$ be the increasing enumeration of B, and let $w_\xi := \bigcup_{\eta < \rho_\xi} w'_\eta$ for each $\xi < \kappa$. Note that if $\zeta < \kappa$ and $\kappa \leq \lambda' < \lambda$, then $\langle w_\xi \cap \lambda' \mid \xi < \zeta \rangle \in \mathrm{skull}^{\mathcal{M}}(W)$ because this sequence can be recovered from $\langle w'_\xi \cap \lambda' \mid \xi \leq \rho_\zeta \rangle \in \mathrm{skull}^{\mathcal{M}}(W)$. Then it is easy to see that $\langle w_\xi \mid \xi < \kappa \rangle$ is a φ-s.i.a. sequence to W which is strictly $\subseteq$-increasing continuous.)

Next let $E := \{\xi < \kappa \mid w_\xi \in S_T\}$. Then E is stationary in κ. Moreover define a club $C \subseteq \kappa$ as follows:

First, for each $\eta < \mathrm{cf}(\lambda)$, let C_η be the set of all limit $\zeta < \kappa$ such that $\varphi(\langle w_\xi \cap \lambda_\eta \mid \xi < \zeta' \rangle) \in w_\zeta$ for every $\zeta' < \zeta$. Note that C_η is a club in κ for each $\eta < \mathrm{cf}(\lambda)$. Next let C' be the set of all $\zeta < \kappa$ such that $\mathrm{skull}^{\mathcal{M}}(w_\xi) \cap \lambda = w_\xi$. C' is also a club because $\mathrm{skull}^{\mathcal{M}}(W) \cap \lambda = W$. Finally let $C := C' \cap \bigcap_{\eta < \mathrm{cf}(\lambda)} C_\eta$. Then C is a club in κ. Here note that $\langle w_\xi \mid \xi < \zeta \rangle$ is a φ-s.i.a. sequence to w_ζ for every $\zeta \in C$ by Lem.3.4.

Because E is stationary we can take $\zeta \in E \cap C$ such that $\sup(E \cap \zeta) = \zeta$. Then $\langle w_\xi \mid \xi < \zeta \rangle$ is a φ-s.i.a. sequence to w_ζ, and so

$$\gamma_{w_\zeta} \;=\; \sup_{\xi < \zeta} \beta_{w_\xi} \;=\; \sup_{\xi \in E \cap \zeta} \beta_{w_\xi} \,.$$

But $w_\zeta \in S_T$, and $\langle w_\xi \mid \xi \in E \cap \zeta \rangle$ is a strictly $\subseteq$-increasing sequence in S_T. Hence this contradicts Lem.4.8 (3). □

For Lem.4.16 we introduce the notion of φ-semi-internally unboundedness:

Definition 4.17. *Let W be a subset of λ with $|W| = \kappa \subseteq W$. We say that W is* $\mathcal{M}$-semi-internally unbounded *($\mathcal{M}$-s.i.u.) if it satisfies the following:*

(i) $\mathrm{skull}^{\mathcal{M}}(W) \cap \lambda = W$.

(ii) For any $y \in \mathcal{P}_\kappa W$ there exists $x \in \mathcal{P}_\kappa W$ such that $y \subseteq x$ and such that $x \cap \lambda' \in \mathrm{skull}^{\mathcal{M}}(W)$ for every $\lambda' < \lambda$.

Lem.4.16 splits into the following two lemmata:

Lemma 4.18. *If $|W| = \kappa \subseteq W \subseteq \lambda$ and $\mathcal{T}_0 \cap \mathcal{P}_\kappa W$ is stationary, then W is $\mathcal{M}$-s.i.u.*

Lemma 4.19. *Assume that $\kappa^{<\kappa} = \kappa$. If $|W| = \kappa \subseteq W \subseteq \lambda$ and W is $\mathcal{M}$-s.i.u, then W is φ-s.i.a. of length κ.*

First we prove Lem.4.18. Lem.4.18 easily follows from the lemma below. Recall that if $x \in \mathcal{T}_0$, then x is φ-s.i.a. of length $x \cap \kappa$ by Lem.3.9:

Lemma 4.20. *Suppose that $x, y \in \mathcal{T}_0$, $y \subseteq x$ and $y \cap \kappa < x \cap \kappa$. Let $\langle x_\xi \mid \xi \in x \cap \kappa\rangle$ be a φ-s.i.a. sequence to x. Then $y \subseteq x_\xi$ for some $\xi < x \cap \kappa$.*

Proof. Let $\langle y_\iota \mid \iota < y \cap \kappa\rangle$ be a φ-s.i.a. sequence to y. Moreover let $M := \mathrm{skull}^{\mathcal{M}}(x)$. Note that $\langle y_\iota \cap \lambda' \mid \iota < \zeta\rangle \in M$ for every $\zeta < y \cap \kappa$ and $\lambda' < \lambda$ because $y \subseteq x$.

We use the following claim:

Claim. *For every $\lambda' < \lambda$ and every $\iota < y \cap \kappa$ there exists $\xi < x \cap \kappa$ with $y_\iota \cap \lambda' \subseteq x_\xi$.*

$\vdash$ Fix $\lambda' < \lambda$ and $\iota < y \cap \kappa$. Assume that $y_\iota \cap \lambda' \not\subseteq x_\xi$ for every $\xi < \kappa$. We work for a contradiction.

Note that $y_\iota \cap \lambda' \in M$. In M take a bijection $\pi : |y_\iota \cap \lambda'| \to y_\iota \cap \lambda'$. Moreover let ζ be the least ordinal $\leq |y_\iota \cap \lambda'|$ such that $\pi[\zeta] \not\subseteq x_\xi$ for every $\xi < x \cap \kappa$. Note that $\mathrm{cf}(\zeta) = \mathrm{cf}(x \cap \kappa) = \mathrm{cf}(\lambda)$. Moreover $\zeta \in M \cap \kappa$ because $\zeta \leq |y_\iota \cap \lambda'| \in M \cap \kappa \in \kappa$. Hence we can take an increasing sequence $\langle \zeta_\eta \mid \eta < \mathrm{cf}(\lambda)\rangle \in M$ converging to ζ. For each $\eta < \mathrm{cf}(\lambda)$ let ξ_η be the least $\xi < x \cap \kappa$ such that $\pi[\zeta_\eta] \subseteq x_\xi$.

Then $\langle \xi_\eta \mid \eta < \mathrm{cf}(\lambda)\rangle$ is an increasing cofinal sequence in $x \cap \kappa$. Moreover $\langle \xi_\eta \mid \eta < \rho\rangle \in M$ for every $\rho < \mathrm{cf}(\lambda)$ because $\langle \xi_\eta \mid \eta < \rho\rangle$ can be recovered from $\langle x_\xi \cap \lambda' \mid \xi \leq \xi_\rho\rangle$ and $\langle \pi[\zeta_\eta] \mid \eta < \rho\rangle$ both of which belong to M. From these it is easy to see that $\langle x_{\xi_\eta} \mid \eta < \mathrm{cf}(\lambda)\rangle$ is a φ-s.i.a. sequence to x. Hence x is φ-s.i.a. of length $\mathrm{cf}(\lambda)$. This contradicts that $x \in \mathcal{T}_0$. $\dashv$

We proceed to the proof of the lemma. Assume that there are no $\xi < x \cap \kappa$ with $y \subseteq x_\xi$. We work for a contradiction.

First note that $y \cap \kappa \in M \cap \kappa$ and that $\mathrm{cf}(y \cap \kappa) = \mathrm{cf}(\lambda)$. Take an increasing sequence $\langle \iota_\eta \mid \eta < \mathrm{cf}(\lambda)\rangle \in M$ converging to $y \cap \kappa$. By the claim above, for each $\eta < \mathrm{cf}(\lambda)$, let ξ_η be the least $\xi < x \cap \kappa$ such that $y_{\iota_\eta} \cap \lambda_\eta \subseteq x_\xi$.

Here note that $\langle y_{\iota_\eta} \cap \lambda_\eta \mid \eta < \mathrm{cf}(\lambda)\rangle$ is a $\subseteq$-increasing sequence with its union y. Note also that $\langle y_{\iota_\eta} \cap \lambda_\eta \mid \eta < \rho\rangle \in M$ for each $\rho < \mathrm{cf}(\lambda)$ because $\langle y_{\iota_\eta} \cap \lambda_\eta \mid \eta < \rho\rangle$ can be recovered from $\langle y_\iota \cap \lambda_\rho \mid \iota < \iota_\rho\rangle$, $\langle \iota_\eta \mid \eta < \rho\rangle$ and $\langle \lambda_\eta \mid \eta < \rho\rangle$, all of which belong to M.

Then $\langle \xi_\eta \mid \eta < \mathrm{cf}(\lambda)\rangle$ is an increasing cofinal sequence in $x \cap \kappa$ by the remark above and the fact that there are no $\xi < x \cap \kappa$ with $y \subseteq x_\xi$. Note also that $\langle \xi_\eta \mid \eta < \rho\rangle \in M$ for each $\rho < \mathrm{cf}(\lambda)$ because $\langle \xi_\eta \mid \eta < \rho\rangle$ can be recovered from $\langle x_\xi \cap \lambda_\rho \mid \xi \leq \xi_\rho\rangle$ and $\langle y_{\iota_\eta} \cap \lambda_\eta \mid \eta < \rho\rangle$ both of which belong to M.

Then it is easy to see that $\langle x_{\xi_\eta} \mid \eta < \mathrm{cf}(\lambda)\rangle$ is a φ-s.i.a. sequence to x. Thus x is φ-s.i.a. of length $\mathrm{cf}(\lambda)$. This contradicts that $x \in \mathcal{T}_0$. $\square$

Now we prove Lem.4.18:

Proof of Lem.4.18. Suppose that $|W| = \kappa \subseteq W \subseteq \lambda$ and that $\mathcal{T}_0 \cap \mathcal{P}_\kappa W$ is stationary. First note that $\mathrm{skull}^{\mathcal{M}}(W) \cap \lambda = W$ because $\mathrm{skull}^{\mathcal{M}}(x) \cap \lambda = x$ for each $x \in \mathcal{T}_0$.

To prove (ii) in Def.4.17 take an arbitrary $y \in \mathcal{P}_\kappa W$. We will find $x^* \in \mathcal{P}_\kappa W$ such that $y \subseteq x^*$ and such that $x^* \cap \lambda' \in \mathrm{skull}^{\mathcal{M}}(W)$ for every $\lambda' < \lambda$. Because $\mathcal{T}_0 \cap \mathcal{P}_\kappa W$ is stationary we may assume that $y \in \mathcal{T}_0$.

We can take $x \in \mathcal{T}_0 \cap \mathcal{P}_\kappa W$ with $y \subseteq x$ and $y \cap \kappa < x \cap \kappa$. Let $\langle x_\xi \mid \xi < x \cap \kappa \rangle$ be a φ-s.i.a. sequence to x. By Lem.4.20 there exists $\xi < x \cap \kappa$ with $y \subseteq x_\xi$. Note that $x_\xi \cap \lambda' \in \mathrm{skull}^{\mathcal{M}}(x) \subseteq \mathrm{skull}^{\mathcal{M}}(W)$ for every $\lambda' < \lambda$. Therefore $x^* := x_\xi$ is what we seek. □

Finally we prove Lem.4.19:

Proof of Lem.4.19. Assume that $\kappa^{<\kappa} = \kappa$. Suppose that $|W| = \kappa \subseteq W \subseteq \lambda$ and that W is $\mathcal{M}$-s.i.u. Take an arbitrary $\subseteq$-increasing cofinal sequence $\langle w_\xi \mid \xi < \kappa \rangle$ in $\mathcal{P}_\kappa W$. We show that $\langle w_\xi \mid \xi < \kappa \rangle$ is a φ-s.i.a. sequence to W. For this it suffices to show that $\langle w_\xi \cap \lambda' \mid \xi < \zeta \rangle \in \mathrm{skull}^{\mathcal{M}}(W)$ for every $\zeta < \kappa$ and every $\lambda' < \lambda$. Let $M := \mathrm{skull}^{\mathcal{M}}(W)$.

Fix $\zeta < \kappa$ and $\lambda' < \lambda$. Then there exists $x \in \mathcal{P}_\kappa \lambda'$ such that $\bigcup_{\xi<\zeta} w_\xi \cap \lambda' \subseteq x \in M$ by the φ-s.i.u. of W. Here note that $\langle w_\xi \cap \lambda' \mid \xi < \zeta \rangle \in {}^{<\kappa}\mathcal{P}(x)$. Note also that $|{}^{<\kappa}\mathcal{P}(x)| = \kappa$ because $\kappa^{<\kappa} = \kappa$. Moreover ${}^{<\kappa}\mathcal{P}(x) \in M$ and $\kappa \subseteq M$. Therefore ${}^{<\kappa}\mathcal{P}(x) \subseteq M$. Thus $\langle w_\xi \cap \lambda' \mid \xi < \zeta \rangle \in M$. □

This completes the proof of Prop.4.3.

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