On the existence of skinny stationary subsets

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Abstract

Matsubara–Usuba [13] introduced the notion of skinniness and its variants for subsets of $\mathcal{P}_\kappa\lambda$ and showed that the existence of skinny stationary subsets of $\mathcal{P}_\kappa\lambda$ is related to large cardinal properties of ideals over $\mathcal{P}_\kappa\lambda$ and to Jensen's diamond principle on λ. In this paper, we study the existence of skinny stationary sets in more detail.

Key Words: $\mathcal{P}_\kappa\lambda$, skinny stationary set, saturated ideal

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1 Introduction

In Matsubara–Usuba [13], we introduced the notion of skinniness and its variants for subsets of $\mathcal{P}_\kappa\lambda$, where $\mathcal{P}_\kappa\lambda$ denotes the set $\{x \subseteq \lambda \mid |x| < \kappa\}$ as usual. The existence of skinny stationary subsets of $\mathcal{P}_\kappa\lambda$ is related to large cardinal properties of ideals over $\mathcal{P}_\kappa\lambda$ and to Jensen's diamond principle on λ. In this paper, we study the existence of skinny stationary subsets of $\mathcal{P}_\kappa\lambda$ in more detail. Throughout this paper, we let κ denote an uncountable regular cardinal and λ denote a cardinal $\geq \kappa$.

In order to state the definition of skinniness and its variants, we introduce some notation:

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Notation. *For a set x of ordinals, we define* $\sup^*(x)$ *by* $\sup^*(x) = \sup(x)$ *if* $\sup(x) \notin x$. *Let* $\sup^*(x)$ *be undefined if* $\sup(x) \in x$. *For* $X \subseteq \mathcal{P}_\kappa\lambda$, *we let*

$$E_X := \{\sup^*(x) \mid x \in X\} .$$

For $X \subseteq \mathcal{P}_\kappa\lambda$ *and* $\alpha \leq \lambda$, *let*

$$X^\alpha := \{x \in X \mid \sup^*(x) = \alpha\} .$$

Note that $E_X \subseteq E^\lambda_{<\kappa} \cup \{\lambda\}$ for all $X \subseteq \mathcal{P}_\kappa\lambda$, where $E^\lambda_{<\kappa} = \{\alpha < \lambda \mid \mathrm{cf}(\alpha) < \kappa\}$.

Now we present the notion of skinniness and its variants:

Definition 1.1. *Let X be a subset of $\mathcal{P}_\kappa\lambda$ and μ be some cardinal.*

(1) X is said to be skinny *if* $|X^\alpha| < |\mathcal{P}_\kappa\alpha|$ *for every* $\alpha \leq \lambda$.

(2) X is said to be really skinny *if* $|X^\alpha| < \mathrm{cf}(\mathcal{P}_\kappa\alpha, \subseteq)$ *for every* $\alpha \leq \lambda$, *where* $\mathrm{cf}(\mathcal{P}_\kappa\alpha, \subseteq)$ *is the smallest size of a $\subseteq$-cofinal subset of* $\mathcal{P}_\kappa\alpha$.

(3) X is said to be skinnier *if* $|X^\alpha| \leq |\alpha|$ *for every* $\alpha \leq \lambda$.

(4) X is said to be skinniest *if* $|X^\alpha| \leq 1$ *for every* $\alpha \leq \lambda$.

(5) X is said to be μ-skinny *if* $|X^\alpha| < \mu$ *for every* $\alpha \leq \lambda$.

Note that X is skinniest if and only if X is 2-skinny. Moreover, if X is μ-skinny for some $\mu < \lambda$, then $\{x \in X \mid \sup(x) \geq \mu\}$ is skinnier. Note also that X is μ-skinny if and only if $\sup^* \restriction X$ is $< \mu$ to one. In particular, X is skinniest if and only if $\sup^* \restriction X$ is one to one. Also, if $\lambda = \mu^+$ and $\sup(x) \geq \mu$ for every $x \in X$, then X is skinnier exactly when $\sup^* \restriction X$ is $\leq \mu$ to one.

Before stating our results in this paper, we recall important facts on skinny stationary sets. For the case where λ is a singular cardinal, we presented the following result in [12] and [13]:

Theorem 1.2 ((1) Matsubara–Shelah [12], (2) Matsubara–Usuba [13])**.**

(1) If λ is a strong limit singular cardinal $> \kappa$, then there is no skinny stationary subset of $\mathcal{P}_\kappa\lambda$.

(2) If λ is a singular cardinal $> \kappa$, then there is no skinnier stationary subset of $\mathcal{P}_\kappa\lambda$.

For a regular λ, the existence of skinnier or skinniest stationary subsets of $\mathcal{P}_\kappa\lambda$ is related to large cardinal properties of ideals over $\mathcal{P}_\kappa\lambda$:

Theorem 1.3 (Solovay [20])**.** *Suppose λ is regular and κ is λ-supercompact. Let U be a normal fine ultrafilter over $\mathcal{P}_\kappa\lambda$. Then there is a skinniest $X \subseteq \mathcal{P}_\kappa\lambda$ with $X \in U$.*

Theorem 1.4 (Matsubara–Usuba [13])**.** *Suppose λ is a regular cardinal with $2^{<\kappa} < \lambda = 2^{<\lambda}$. Let X be a stationary subset of $\mathcal{P}_\kappa\lambda$. If $\mathrm{NS}_{\kappa\lambda} \restriction X$ is precipitous, then X contains a skinnier stationary subset.*

It is also known that the existence of skinnier or skinniest stationary sets is related to Jensen's diamond principle. We introduce some definitions to state this fact:

Definition 1.5. *Let S be a stationary subset of $E^\lambda_{<\kappa}$. We say that S* bears *a skinny (skinnier, skinniest, μ-skinny) stationary set if there is a skinny (skinnier, skinniest, μ-skinny, respectively) stationary $X \subseteq \mathcal{P}_\kappa\lambda$ with $E_X \subseteq S$.*

Theorem 1.6 (Shelah [16], Matsubara–Usuba [13])**.** *Let λ be a regular cardinal $> 2^{<\kappa}$. Then the following are equivalent for a stationary $S \subseteq E^\lambda_{<\kappa}$:*

(i) $\diamondsuit_\lambda(S)$.

(ii) S bears a skinniest stationary subset of $\mathcal{P}_\kappa\lambda$, and $2^{<\lambda} = \lambda$.

(iii) S bears a skinnier stationary subset of $\mathcal{P}_\kappa\lambda$, and $2^{<\lambda} = \lambda$.

In this paper, focusing on the case where λ is a regular cardinal, we study the existence of skinnier and μ-skinny stationary subsets of $\mathcal{P}_\kappa\lambda$ in more detail.

In §3, we will present basic facts on the existence of skinnier stationary sets. We show that, for many successor cardinals λ, there exists a skinnier stationary subset of $\mathcal{P}_\kappa\lambda$. Among other things, we prove the following:

Proposition 1.7. *Suppose $\lambda = \kappa^{+n}$ for some $n < \omega$. Then every stationary subset of $E^\lambda_{<\kappa}$ bears a skinnier stationary subset of $\mathcal{P}_\kappa\lambda$.*

In §3, we also observe the relationship between the existence of skinny stationary sets and the Singular Cardinal Hypothesis SCH. We prove that if there is a λ^+-skinny stationary subset of $\mathcal{P}_\kappa\lambda$ for every regular cardinal $\lambda \geq \kappa$, then SCH holds above κ. From this relationship, we also obtain the consistency of the non-existence of a skinnier stationary subset of $\mathcal{P}_\kappa\kappa^{+\omega+1}$.

In §4 of this paper, we relate the existence of skinniest stationary subsets of $\mathcal{P}_\kappa\lambda$ with combinatorial principles. Recall Theorem 1.4, which states that if $2^{<\kappa} < \lambda = 2^{<\lambda}$ and $\mathrm{NS}_{\kappa\lambda} \restriction X$ is precipitous, then X contains a skinnier stationary set. For a stationary $X \subseteq \mathcal{P}_\kappa\lambda$, we introduce a combinatorial principle $\diamondsuit^M_\lambda(X)$ which implies that X contains a skinniest stationary set. Moreover, using this combinatorial principle, we prove the following theorem, which is an unpublished result of Donder (See König-Larson-Yoshinobu [8], 25 Theorem):

Theorem 1.8 (Donder)**.** *Assume $V = L$. If λ is a regular cardinal, then every stationary subset of $\mathcal{P}_\kappa\lambda$ has a skinniest stationary subset.*

In §4, we also relate the existence of skinniest stationary sets with Jensen's $\square$-principle. We show that, under $\square$-principles, similar facts to those proved in §3 for skinnier sets hold for skinniest sets. In particular, we prove the following:

Theorem 1.9. *Suppose $\lambda = \kappa^{+n}$ for some $n < \omega$. If $\square_{\kappa^{+m}}$ holds for every $m < n$, then there exists a skinniest stationary subset of $\mathcal{P}_\kappa\lambda$.*

This theorem implies that the non-existence of such a skinniest set has a strong consistency strength. We do not know whether it is consistent.

In §5, we prove some variations of Theorem 1.3. More precisely, we show that the dual filters of normal saturated ideals over $\mathcal{P}_\kappa\lambda$ contain skinny stationary subsets of $\mathcal{P}_\kappa\lambda$ with various degrees of skinniness. The degree of skinniness depends on the degree of saturation of the ideal.

In §6 and §7, we discuss the existence of stationary subsets of $E^\lambda_{<\kappa}$ which bear no skinnier or μ-skinny stationary sets.

By Theorem 1.6, under the assumption of $2^{<\kappa} < \lambda = 2^{<\lambda}$, if S is a stationary subset of $E^\lambda_{<\kappa}$ such that $\mathrm{NS}_\lambda \restriction S$ is λ^+-saturated, then S cannot bear a skinnier stationary subset of $\mathcal{P}_\kappa\lambda$. In §6, we drop the cardinal arithmetical assumption from this fact. This is done by incorporating the combinatorial principle $\clubsuit^-_{\lambda,<\kappa}$.

Comparing Proposition 1.7 and Theorem 1.9, it is natural to ask whether, if $\lambda = \kappa^{+n}$ for some $n < \omega$ and $\square_{\kappa^{+m}}$ holds for all $m < n$, then every stationary subset of $E^\lambda_{<\kappa}$ bears a skinniest stationary set. In §7, we prove that this is not the case. In fact, we prove the following more general theorem. In the following theorem, notice that if $\square_\nu$ holds in V for a cardinal ν, then it holds in $V^{\mathbb{P}}$ since $\mathbb{P}$ preserves all cofinalities.

Theorem 1.10. *Let κ, μ and λ be uncountable regular cardinals with $\kappa \leq \mu < \lambda$. Suppose $2^{<\mu} = \mu$. Then there is a poset $\mathbb{P}$ satisfying the following:*

(i) $\mathbb{P}$ *has the* μ^+*-c.c. and adds no new sequence of ordinals of length* $< \mu$*. (In particular,* $\mathbb{P}$ *preserves all cofinalities.)*

(ii) In $V^{\mathbb{P}}$*, there is a sequence* $\langle S_\delta \mid \delta < \mu \rangle$ *of subsets of* $E^\lambda_{<\kappa}$ *such that* $\bigcup_{\delta<\mu} S_\delta = E^\lambda_{<\kappa}$ *and* S_δ *bears no* μ*-skinny stationary subsets of* $\mathcal{P}_\kappa\lambda$ *for any* $\delta < \mu$*.*

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2 Preliminaries

In this section, we present our notation and basic facts used in this paper.

First, we give some notation on sets of ordinals. For a regular cardinal μ and an ordinal $\nu > \mu$, let E^ν_μ ($E^\nu_{<\mu}$, $E^\nu_{\neq\mu}$) be the set of all $\alpha < \nu$ with $\mathrm{cf}(\alpha) = \mu$ ($\mathrm{cf}(\alpha) < \mu$, $\mathrm{cf}(\alpha) \neq \mu$, respectively). Suppose x is a set of ordinals. We let $\mathrm{o.t.}(x)$ denote the order type of x. Let $\mathrm{Lim}(x)$ be the set $\{\alpha \in x \mid \sup(x \cap \alpha) = \alpha\}$. Moreover, let $\mathrm{Cl}(x)$ denote the closure of x with respect to the order topology, i.e. $\mathrm{Cl}(x) := x \cup \{\alpha \in \mathrm{On} \mid \sup(x \cap \alpha) = \alpha\}$.

Next, we give our notation and basic facts on $\mathcal{P}_\kappa(\lambda)$. Let $\mathrm{cf}(\mathcal{P}_\kappa\lambda, \subseteq)$ denote the smallest size of a $\subseteq$-cofinal subset of $\mathcal{P}_\kappa\lambda$. In this paper, we adopt the notion of stationary subsets of $\mathcal{P}_\kappa\lambda$, introduced in Jech [6]. $Z \subseteq \mathcal{P}_\kappa\lambda$ is said to be *club* if Z is $\subseteq$-cofinal in $\mathcal{P}_\kappa\lambda$, and, for any $\zeta < \kappa$ and any $\subseteq$-increasing sequence $\langle z_\xi \mid \xi < \zeta \rangle$ of elements of Z, we have $\bigcup_{\xi<\zeta} z_\xi \in Z$. $X \subseteq \mathcal{P}_\kappa\lambda$ is said to be *stationary* if X intersects with every club subset of $\mathcal{P}_\kappa\lambda$. We often use the following fact (See Jech [7], Exercise 38.10):

Fact 2.1. *Suppose* $X \subseteq \mathcal{P}_\kappa\lambda$*. Then* X *is stationary in* $\mathcal{P}_\kappa\lambda$ *if and only if, for any function* $F : {}^{<\omega}\lambda \to \lambda$*, there is* $x \in X$ *such that* $x \cap \kappa \in \kappa$ *and* x *is closed under* F*, i.e.* $F[{}^{<\omega}x] \subseteq x$*.*

Next, we give notation on ideals. Let A be an infinite set. In this paper, an *ideal* over A means a non-principal proper ideal over A. Let I be an ideal over A. Then I^* denotes the dual filter of I, i.e. $I^* = \{A \setminus X \mid X \in I\}$. For $B \subseteq A$, let $I \restriction B$ be the ideal over A defined by $I \restriction B := \{X \subseteq A \mid X \cap B \in I\}$. Let NS_λ and $\mathrm{NS}_{\kappa\lambda}$ denote the non-stationary ideals over λ and $\mathcal{P}_\kappa\lambda$, respectively. For a cardinal μ, we say that I is *μ-saturated* if there is no $\mathcal{X} \subseteq \mathcal{P}(A) \setminus I$ of size μ such that $X_0 \cap X_1 \in I$ for any distinct $X_0, X_1 \in \mathcal{X}$.

Let I be an ideal over $\mathcal{P}_\kappa \lambda$. We say that I is *normal* if it is closed under taking diagonal unions, that is, for any sequence $\vec{X} = \langle X_\alpha \mid \alpha \in \lambda \rangle$, its diagonal union $\nabla \vec{X} = \{x \in \mathcal{P}_\kappa \lambda \mid \exists \alpha \in x,\ x \in X_\alpha\}$ belongs to I. We say that I is *fine* if the set $\{x \in \mathcal{P}_\kappa \lambda \mid y \not\subseteq x\}$ belongs to I for any $y \in \mathcal{P}_\kappa \lambda$. Note that if I is normal and fine, then I is κ-complete, that is, $\bigcup J \in I$ for all $J \subseteq I$ of size $< \kappa$.

Finally, we present our notation on forcing. Let $\mathbb{P}$ be a poset and μ be a regular uncountable cardinal. We say that $\mathbb{P}$ is *μ-closed* if every descending sequence in $\mathbb{P}$ of length $< \mu$ has a lower bound in $\mathbb{P}$. $\mathbb{P}$ has the μ-c.c. if every antichain in $\mathbb{P}$ has size $< \mu$. $\mathbb{P}$ is *μ-distributive* if $\bigcap \mathcal{D}$ is dense open in $\mathbb{P}$ for every family $\mathcal{D}$ of dense open subsets of $\mathbb{P}$ with $|\mathcal{D}| < \mu$. If $\mathbb{P}$ is μ-distributive, then forcing extensions by $\mathbb{P}$ add no new sequences of ordinals of length $< \mu$. For the proof, see Jech [7] (Chapter 15, Theorem 15.6).

Suppose I is an ideal over a set A. Then let $\mathbb{P}_I$ be the poset $\mathcal{P}(A) \setminus I$ ordered by inclusions. Note that I is μ-saturated if and only if $\mathbb{P}_I$ has the μ-c.c. If G is a $\mathbb{P}_I$-generic filter over V, then, in $V[G]$, we can construct an ultrapower of V by G. This ultrapower is called a generic ultrapower and denoted as $\mathrm{Ult}_G(V)$. See Jech [7] (Chapter 22) for more details on generic ultrapowers. We say that I is *precipitous* if $\mathrm{Ult}_G(V)$ is well-founded for any $\mathbb{P}_I$-generic filter G over V. Recall that for a regular uncountable cardinal λ, every λ-complete λ^+-saturated ideal is precipitous. The proof can be found in Jech [7] (Chapter 22).

Let $\mathbb{P}$ be a poset and G be a $\mathbb{P}$-generic filter over V. Then, for a set M in V, we let $M[G] = \{\dot{a}_G \mid \dot{a} \in M \wedge \dot{a} \text{ is a } \mathbb{P}\text{-name}\}$, where $\dot{a}_G$ is the evaluation of $\dot{a}$ by G.

Let γ be an ordinal and μ be a regular uncountable cardinal. We say that $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \gamma, \beta < \gamma \rangle$ is a $< \mu$-support iteration if each $\dot{\mathbb{Q}}_\beta$ is a $\mathbb{P}_\beta$-name for a poset, and each $\mathbb{P}_\alpha$ consists of all partial functions p on α such that $|\operatorname{dom}(p)| < \mu$ and $p(\beta) \in \dot{\mathbb{Q}}_\beta$ for all $\beta \in \operatorname{dom}(p)$. $p_0 \leq p_1$ in $\mathbb{P}_\alpha$ if $\operatorname{dom}(p_0) \supseteq \operatorname{dom}(p_1)$, and $p_0 \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$p_0(\beta) \leq p_1(\beta)$" for every $\beta \in \operatorname{dom}(p_1)$. For each $\alpha \leq \gamma$, $\Vdash_{\mathbb{P}_\alpha}$ is simply denoted as $\Vdash_\alpha$.

3 Basic facts

In this section, we present basic facts on the existence of skinnier and μ-skinny stationary sets.

We begin with a lemma, which tells us that, for many successor cardinals λ, there is a skinnier stationary subset of $\mathcal{P}_\kappa\lambda$.

Lemma 3.1. *Suppose* $\mathrm{cf}(\mathcal{P}_\kappa\lambda, \subseteq) = \lambda$. *Then every stationary subset of* $E^{\lambda^+}_{<\kappa}$ *bears a skinnier stationary subset of* $\mathcal{P}_\kappa\lambda^+$.

We can easily prove this lemma using the following well-known theorem:

Theorem 3.2 (Shelah [17]). *The smallest size of stationary subsets of* $\mathcal{P}_\kappa\lambda$ *is* $\mathrm{cf}(\mathcal{P}_\kappa\lambda, \subseteq)$.

Proof. Suppose S is a stationary subset of $E^{\lambda^+}_{<\kappa}$. We will construct a skinnier stationary $X \subseteq \mathcal{P}_\kappa\lambda^+$ with $E_X \subseteq S$.

By Theorem 3.2 and the assumption of the lemma, for each $\alpha \in S \setminus \lambda$, there is a stationary $X_\alpha \subseteq \mathcal{P}_\kappa\alpha$ of size λ. Notice that, for each $\alpha \in S \setminus \lambda$, there are club many $x \in \mathcal{P}_\kappa\alpha$ with $\sup(x) = \alpha$. So we may assume that $\sup(x) = \alpha$ for all $x \in X_\alpha$. Let $X := \bigcup\{X_\alpha \mid \alpha \in S \setminus \lambda\}$. Then X is skinnier, and $E_X \subseteq S$.

In order to prove that X is stationary, take an arbitrary $F : {}^{<\omega}\lambda^+ \to \lambda^+$. It suffices to find $x \in X$ such that $x \cap \kappa \in \kappa$ and x is closed under F. Since S is stationary in λ^+, there is $\alpha \in S$ which is closed under F. Then, since X_α is stationary in $\mathcal{P}_\kappa\alpha$, there is $x \in X_\alpha$ such that $x \cap \kappa \in \kappa$ and x is closed under $F \restriction {}^{<\omega}\alpha$. This x is as desired. $\square$

By induction on $n < \omega$, it can be easily prove that $\mathrm{cf}(\mathcal{P}_\kappa\kappa^{+n}, \subseteq) = \kappa^{+n}$ for all $n < \omega$. So Proposition 1.7 follows from Lemma 3.1.

The size of stationary subsets of $\mathcal{P}_\kappa\kappa^{+n}$ is studied in Baumgartner [2] in details. Let X be the set of all $x \in \mathcal{P}_\kappa\kappa^{+n}$ such that $\mathrm{cf}(\sup(x \cap \kappa^{+i})) > \omega$ for all $i \leq n$. It is easy to see that X is stationary if $\kappa \geq \omega_2$. In [2], it was proved that there is a club $Z \subseteq \mathcal{P}_\kappa\kappa^{+n}$ such that for any $x, y \in X \cap Z$, if $\sup(x \cap \kappa^{+i}) = \sup(y \cap \kappa^{+i})$ for all $i \leq n$, then $x = y$. So $X \cap Z$ is a skinnier stationary set for such a club Z. In [2], it was also proved that $|Z| \supseteq (\kappa^{+n})^\omega = \max\{\kappa^{+n}, \kappa^\omega\}$ for any club $Z \subseteq \mathcal{P}_\kappa\kappa^{+n}$. So if $\kappa^\omega > \kappa^{+n}$, then any club subset of $\mathcal{P}_\kappa\kappa^{+n}$ is not κ^ω-skinny.

With some cardinal arithmetical assumptions, we have the existence of skinniest stationary sets. Shelah [18] proved that, for a cardinal λ, if $2^\lambda = \lambda^+$, then

$\diamondsuit_{\lambda^+}(S)$ holds for every stationary $S \subseteq E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$. Note that, if $\max\{\kappa, \mathrm{cf}(\lambda)^+\} \geq \omega_2$, then $E^{\lambda^+}_{<\kappa} \cap E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$ is stationary. Hence the following proposition is an immediate corollary of Theorem 1.6:

Proposition 3.3. *Let λ be a cardinal with $2^\lambda = \lambda^+$. If $\max\{\kappa, \mathrm{cf}(\lambda)^+\} \geq \omega_2$, then there is a skinniest stationary subset of $\mathcal{P}_\kappa \lambda^+$.*

Next, we turn our attention to the non-existence of skinny stationary sets. We first observe the relationship between the existence of skinny stationary sets and the Singular Cardinal Hypothesis SCH, which asserts that $2^\delta = \delta^+$ for every singular strong limit cardinal δ. For this, we prove the next lemma:

Lemma 3.4. *Suppose λ is a regular cardinal. Then there exists a λ^+-skinny stationary subset of $\mathcal{P}_\kappa \lambda$ if and only if $\mathrm{cf}(\mathcal{P}_\kappa \lambda, \subseteq) = \lambda$.*

Proof. Note that there is a λ^+-skinny stationary subset of $\mathcal{P}_\kappa \lambda$ if and only if there is a stationary subset of $\mathcal{P}_\kappa \lambda$ of size λ. The latter is equivalent to that $\mathrm{cf}(\mathcal{P}_\kappa \lambda, \subseteq) = \lambda$ by Theorem 3.2. □

Note that if λ is a singular strong limit cardinal with $\mathrm{cf}(\lambda) < \kappa < \lambda$, then

$$2^\lambda = \lambda^{\mathrm{cf}(\lambda)} \leq \mathrm{cf}(\mathcal{P}_\kappa \lambda, \subseteq) \cdot \kappa^{\mathrm{cf}(\lambda)} = \mathrm{cf}(\mathcal{P}_\kappa \lambda, \subseteq) \leq \mathrm{cf}(\mathcal{P}_\kappa \lambda^+, \subseteq)\,.$$

Thus, if λ is a singular strong limit cardinal with $\mathrm{cf}(\lambda) < \kappa < \lambda$ and $2^\lambda > \lambda^+$, then $\mathrm{cf}(\mathcal{P}_\kappa \lambda^+, \subseteq) > \lambda^+$. So, by Lemma 3.4, there is no λ^{++}-skinny stationary subset of $\mathcal{P}_\kappa \lambda^+$ for such λ. Note also that, by Silver's Theorem [19], if SCH fails above κ, then there is such λ. Hence we have the following:

Corollary 3.5. *Suppose there exists a λ^+-skinny stationary subset of $\mathcal{P}_\kappa \lambda$ for every regular $\lambda \geq \kappa$. Then* SCH *holds above κ.*

Recall that Magidor [10] constructed a model in which SCH fails at $\aleph_\omega$. In this model, $\mathrm{cf}(\mathcal{P}_\kappa \aleph_{\omega+1}, \subseteq) \geq 2^{\aleph_\omega} > \aleph_{\omega+1}$ for all $\kappa < \aleph_\omega$ by the same calculation as above. By Lemma 3.4, in this model, there is no skinnier stationary subset of $\mathcal{P}_\kappa \kappa^{+\omega+1}$ for every $\kappa < \aleph_\omega$. In this sense, Proposition 1.7 is optimal.

It is known that there may be a regular cardinal λ with $2^{<\lambda} = \lambda$ for which $\mathcal{P}_\kappa \lambda$ contains no skinnier stationary subsets. By Proposition 3.3, if λ is a regular cardinal with $2^{<\lambda} = \lambda$ and there is no skinnier stationary subset of $\mathcal{P}_\kappa \lambda$, then either λ is an inaccessible cardinal, or $\kappa = \omega_1$ and λ is a successor of a singular cardinal of countable cofinality.

First we mention the case where λ is inaccessible. Using Radin forcing, Woodin [3] built a model in which $\diamondsuit_\lambda$ fails for an inaccessible cardinal λ. By Theorem 1.6, in this model of Woodin, there is no skinnier stationary subset of $\mathcal{P}_\kappa\lambda$ for every regular uncountable cardinal $\kappa < \lambda$.

As for the case where $\kappa = \omega_1$ and λ is a successor of a singular cardinal of countable cofinality with $2^{<\lambda} = \lambda$, we do not know whether the non-existence of a skinnier stationary subset of $\mathcal{P}_\kappa\lambda$ is consistent:

Question 3.6. *Is it consistent that there is no skinnier stationary subset of $\mathcal{P}_{\omega_1}\lambda$ for some successor cardinal λ with $2^{<\lambda} = \lambda$?*

Gitik–Rinot [4] obtained a partial result on this question. They built a model of $\neg\diamondsuit_{\aleph_{\omega+1}}(S)$ for some stationary $S \subseteq E^{\aleph_{\omega+1}}_\omega$ together with GCH. So, in this model, S bears no skinnier stationary subset of $\mathcal{P}_{\omega_1}\aleph_{\omega+1}$.

4 Combinatorial principles

By Theorem 1.4, if λ is a regular cardinal $> \kappa$ with $2^{<\kappa} < \lambda = 2^{<\lambda}$ and $\mathrm{NS}_{\kappa\lambda}$ is precipitous, then every stationary subset of $\mathcal{P}_\kappa\lambda$ contains a skinnier stationary subset, that is, skinnier stationary sets are dense in the family of stationary subsets of $\mathcal{P}_\kappa\lambda$. We consider when skinnier stationary sets are dense. For this, we introduce the following variant of Jensen's diamond principle.

Definition 4.1. *Let X be a stationary subset of $\mathcal{P}_\kappa\lambda$. Then $\diamondsuit^M_\lambda(X)$ is the following assertion:*

> *There is a sequence $\langle b_\alpha \mid \alpha \in E_X\rangle$ such that*
>
> *(i) $b_\alpha \subseteq \alpha$ for every $\alpha \in E_X$,*
>
> *(ii) for every $B \subseteq \lambda$, the set $\{x \in X \mid B \cap \sup(x) = b_{\sup(x)}\}$ is stationary in $\mathcal{P}_\kappa\lambda$.*

A sequence $\langle b_\alpha \mid \alpha \in E_X\rangle$ satisfying (i) and (ii) is called a $\diamondsuit^M_\lambda(X)$-sequence.

This variant of Jensen's diamond principle produces a skinniest stationary subset of X as follows:

Proposition 4.2. *Let λ be a regular cardinal and X be a stationary subset of $\mathcal{P}_\kappa\lambda$. If $\diamondsuit^M_\lambda(X)$ holds, then X has a skinniest stationary subset.*

Proof. We may assume that $x \cap \kappa \in \kappa$ and $\sup(x) \notin x$ for every $x \in X$, since there are club many such $x \in \mathcal{P}_\kappa\lambda$. Using $\diamondsuit_\lambda^M(X)$, it is not difficult to build a sequence $\langle f_\alpha \mid \alpha \in E_X \rangle$ such that

(i) $f_\alpha : {}^{<\omega}\alpha \to \alpha$ for every $\alpha \in E_X$,

(ii) for every $F : {}^{<\omega}\lambda \to \lambda$, the set $\{x \in X \mid F \restriction {}^{<\omega}\sup(x) = f_{\sup(x)}\}$ is stationary in $\mathcal{P}_\kappa\lambda$.

Now, for each $\alpha \in E_X$, we consider the set of all $x \in X^\alpha$ closed under f_α. If this set is not empty, then choose an element and call it x_α. Otherwise, let x_α be any element of X^α. Clearly $Y := \{x_\alpha \mid \alpha \in E_X\}$ is a skinniest subset of X. So it suffices to prove that Y is stationary in $\mathcal{P}_\kappa\lambda$.

Take an arbitrary function $F : {}^{<\omega}\lambda \to \lambda$. Since $x_\alpha \cap \kappa \in \kappa$ for each $\alpha \in E_X$, it is enough to find $\alpha^* \in E_X$ such that x_{α^*} is closed under F. By property (ii) of $\langle f_\alpha \mid \alpha \in E_X \rangle$, we can find some $x^* \in X$ such that $F \restriction {}^{<\omega}\sup(x^*) = f_{\sup(x^*)}$, and x^* is closed under F. Let $\alpha^* := \sup(x^*) \in E_X$. Note that x_{α^*} must be closed under f_{α^*} since $x^* \in X^{\alpha^*}$, and x^* is closed under f_{α^*}. Then x_{α^*} is closed under F since $x_{\alpha^*} \subseteq \alpha^*$, and $F \restriction {}^{<\omega}\alpha^* = f_{\alpha^*}$. □

Next, we consider when $\diamondsuit_\lambda^M(X)$ holds. First, we show that it holds in L. Note that Theorem 1.8 is an immediate consequence of the next proposition together with Proposition 4.2:

Proposition 4.3. *Assume $V = L$. Suppose λ is a regular cardinal $> \kappa$, and X is a stationary subset of $\mathcal{P}_\kappa\lambda$. Then $\diamondsuit_\lambda^M(X)$ holds.*

Proof. By shrinking X if necessary, we may assume that $x \cap \kappa \in \kappa$ and $\sup(x) \notin x$ for every $x \in X$. We will define a sequence $\vec{b} = \langle b_\alpha \mid \alpha \in E_X \rangle$, where $b_\alpha \subseteq \alpha$ for each $\alpha \in E_X$, by induction on α as follows:

Assume that $\alpha \in E_X$ and $\langle b_\beta \mid \beta \in E_X \cap \alpha \rangle$ has been defined. Let $\langle b, f \rangle$ be the $<_L$-least pair with the following properties, if such a pair exists:

(i) $b \subseteq \alpha$, and $f : {}^{<\omega}\alpha \to \alpha$.

(ii) There are no $x \in L_\alpha \cap X$ such that $b \cap \sup(x) = b_{\sup(x)}$ and x is closed under f.

Then let $b_\alpha := b$. If such a pair $\langle b, f \rangle$ does not exist, then let $b_\alpha := \emptyset$.

We have defined $\vec{b}$. We prove that $\vec{b}$ is a $\diamondsuit^M_\lambda(X)$-sequence. For a contradiction, assume not. Then, for some $B \subseteq \lambda$ and some $F : {}^{<\omega}\lambda \to \lambda$, there are no $x \in X$ such that $B \cap \sup(x) = b_{\sup(x)}$ and x is closed under F. Let $\langle B, F\rangle$ be the $<_L$-least such pair. Moreover, let $\mathcal{M} := \langle L_{\lambda^+}, \in, \lambda, X, \vec{b}, B, F\rangle$, and, for each $\alpha < \lambda$, let M_α be the Skolem hull of α in $\mathcal{M}$. Note that there are club many $\alpha < \lambda$ with $M_\alpha \cap \lambda = \alpha$. Since X is stationary in $\mathcal{P}_\kappa\lambda$, there exists $x \in X$ such that $M_{\sup(x)} \cap \lambda = \sup(x)$ and x is closed under F. Let $\alpha := \sup(x)$, $b := B \cap \alpha$, and $f := F \restriction {}^{<\omega}\alpha$. By the standard argument using the transitive collapse of M_α, it is easy to see that $\langle b, f\rangle$ is the $<_L$-least pair satisfying (i) and (ii) above. So $b_\alpha = b$, and hence $b_{\sup(x)} = B \cap \sup(x)$. Since x is closed under F, this contradicts the choice of B and F. □

As we mentioned before, if $2^{<\kappa} < \lambda = 2^{<\lambda}$, and $\mathsf{NS}_{\kappa\lambda}$ is precipitous, then skinnier stationary subsets of $\mathcal{P}_\kappa\lambda$ are dense. Recall the fact, due to Goldring [5], that $\mathsf{NS}_{\kappa\lambda}$ is precipitous if a Woodin cardinal is Lévy collapsed to λ^+. The next proposition, together with Proposition 4.2, tells us that the Lévy collapse of an inaccessible cardinal always provides the denseness of skinniest stationary subsets:

Proposition 4.4. *Let λ be a regular cardinal $> \kappa$, and suppose δ is an inaccessible cardinal $> \lambda$. Then $\mathrm{Col}(\lambda, < \delta)$ forces that $\diamondsuit^M_\lambda(X)$ holds for every stationary $X \subseteq \mathcal{P}_\kappa\lambda$, where $\mathrm{Col}(\lambda, < \delta)$ is the Levy collapse making $\delta = \lambda^+$.*

To prove this proposition, we use the following lemma:

Lemma 4.5. *Let λ be a regular cardinal $> \kappa$, and suppose X is a stationary subset of $\mathcal{P}_\kappa\lambda$.*

(1) If $2^{<\lambda} = \lambda$, then $\mathrm{Add}(\lambda)$ forces $\diamondsuit^M_\lambda(X)$, where $\mathrm{Add}(\lambda)$ denotes the poset ${}^{<\lambda}2$ ordered by reverse inclusions.

(2) If $\diamondsuit^M_\lambda(X)$ holds, then every λ-closed forcing preserves $\diamondsuit^M_\lambda(X)$.

Proof. Without loss of generality, we may assume that $x \cap \kappa \in \kappa$ and $\sup(x) \notin x$ for every $x \in X$.

(1) Let $\mathbb{P}$ be the poset of all functions p such that $\mathrm{dom}(p) \in \lambda$ and $p(\alpha) \subseteq \alpha$ for every $\alpha \in \mathrm{dom}(p)$. $\mathbb{P}$ is ordered by reverse inclusions. It is easy to see that $\mathbb{P}$ is forcing equivalent to $\mathrm{Add}(\lambda)$. We will show that if G is a $\mathbb{P}$-generic filter over V, then $\langle \bigcup G(\alpha) \mid \alpha \in E_X\rangle$ is a $\diamondsuit^M_\lambda(X)$-sequence in $V[G]$.

Take a $\mathbb{P}$-name $\dot{B}$ for a subset of λ, a $\mathbb{P}$-name $\dot{F}$ for a function from ${}^{<\omega}\lambda$ to λ, and an element p of $\mathbb{P}$. It suffices to find $p^* \leq p$ and $x^* \in X$ such that

$$p^* \Vdash_{\mathbb{P}} \text{“}\dot{B} \cap \sup(x^*) = \bigcup \dot{G}(\sup(x^*)) \wedge x^* \text{ is closed under } \dot{F}\text{”} .$$

Using the λ-closedness of $\mathbb{P}$, by induction on $\alpha < \lambda$, we can build a descending sequence $\langle p_\alpha \mid \alpha < \lambda \rangle$ below p such that p_α decides $\dot{B} \cap \alpha$ and $\dot{F} \restriction {}^{<\omega}\alpha$ for each $\alpha < \lambda$. Let $B \subseteq \lambda$ and $F : {}^{<\omega}\lambda \to \lambda$ be the evaluations of $\dot{B}$ and $\dot{F}$ by $\langle p_\alpha \mid \alpha < \lambda \rangle$, that is,

- $B = \{\beta \in \lambda \mid (\exists \alpha < \lambda)\, p_\alpha \Vdash_{\mathbb{P}} \text{“}\beta \in \dot{B}\text{”}\}$,
- $\dot{F}(a) = \beta$ if and only if $p_\alpha \Vdash_{\mathbb{P}} \text{“}\dot{F}(a) = \beta\text{”}$ for some $\alpha < \lambda$.

Since X is stationary, we can find some $x^* \in X$ such that x^* is closed under F and $\mathrm{dom}(p_\alpha) < \sup(x^*)$ for every $\alpha < \sup(x^*)$. Let $q := \bigcup_{\alpha < \sup(x^*)} p_\alpha$. Then q forces that $\dot{F} \restriction {}^{<\omega} \sup(x^*) = F \restriction {}^{<\omega} \sup(x^*)$ and $\dot{B} \cap \sup(x^*) = B \cap \sup(x^*)$. Note also that $\mathrm{dom}(q) \leq \sup(x^*)$. Now take $p^* \leq q$ such that $p^*(\sup(x^*)) = B \cap \sup(x^*)$. Then p^* and x^* are clearly as desired.

(2) Let $\vec{b} = \langle b_\alpha \mid \alpha \in E_X \rangle$ be a $\diamondsuit^M_\lambda(X)$-sequence. Take an arbitrary λ-closed poset $\mathbb{P}$. We show that $\vec{b}$ will remain a $\diamondsuit^M_\lambda(X)$-sequence in any $\mathbb{P}$-generic extension. Let $\dot{B}$ be a $\mathbb{P}$-name for a subset of λ, $\dot{F}$ be a $\mathbb{P}$-name for a function from ${}^{<\omega}\lambda$ to λ, and p be a condition in $\mathbb{P}$. We will find $p^* \leq p$ and $x^* \in X$ such that

$$p^* \Vdash_{\mathbb{P}} \text{“}\dot{B} \cap \sup(x^*) = b_{\sup(x^*)} \wedge x^* \text{ is closed under } \dot{F}\text{”} .$$

As in the proof of (1), we can build a descending sequence $\langle p_\alpha \mid \alpha < \lambda \rangle$ below p such that p_α decides $\dot{B} \cap \alpha$ and $\dot{F} \restriction {}^{<\omega}\alpha$. Let B and F be the evaluations of $\dot{B}$ and $\dot{F}$ by $\langle p_\alpha \mid \alpha < \lambda \rangle$. Since $\vec{b}$ is a $\diamondsuit^M_\lambda(X)$-sequence in V, there is some $x^* \in X$ such that $B \cap \sup(x^*) = b_{\sup(x^*)}$ and x^* is closed under F. Let $p^* := p_{\sup(x^*)}$. Then p^* and x^* are clearly as desired. □

Proof of Proposition 4.4. Suppose G is a $\mathrm{Col}(\lambda, <\delta)$-generic filter over V. In $V[G]$, let X be a stationary subset of $\mathcal{P}_\kappa\lambda$. Then $X \in V[G \cap \mathrm{Col}(\lambda, <\gamma)]$ for some $\gamma < \delta$. Let G_0 denote $G \cap \mathrm{Col}(\lambda, <\gamma)$. Note that, in V, $\mathrm{Col}(\lambda, <\delta)$ is forcing equivalent to $\mathrm{Col}(\lambda, <\gamma) \times \mathrm{Add}(\lambda) \times \mathrm{Col}(\lambda, <\delta)$. Furthermore, both $\mathrm{Add}(\lambda)$ and $\mathrm{Col}(\lambda, <\delta)$ are absolute among all models in consideration between V and $V[G]$. There are an $\mathrm{Add}(\lambda)$-generic filter G_1 over $V[G_0]$ and a $\mathrm{Col}(\lambda, <\delta)$-generic filter G_2 over $V[G_0][G_1]$ such that $V[G] = V[G_0][G_1][G_2]$. We know that $\diamondsuit_\lambda^M(X)$ holds in $V[G_0][G_1]$ by Lemma 4.5 (1), and then it also holds in $V[G_0][G_1][G_2] = V[G]$ by Lemma 4.5 (2). $\square$

Next, we relate the existence of skinniest stationary sets with Jensen's square principle. Recall Jensen's square principle:

Definition 4.6. $\square_\lambda$ *asserts the following:*

There exists $\langle c_\alpha \mid \alpha \in \mathrm{Lim}(\lambda^+)\rangle$ *such that*

(i) c_α *is club in* α *of order type* $\leq \lambda$ *for each* $\alpha \in \mathrm{Lim}(\lambda^+)$,

(ii) if $\alpha \in \mathrm{Lim}(\lambda^+)$, *and* $\beta \in \mathrm{Lim}(c_\alpha)$, *then* $c_\beta = c_\alpha \cap \beta$.

A sequence $\langle c_\alpha \mid \alpha \in \mathrm{Lim}(\lambda^+)\rangle$ *satisfying (i) and (ii) is called a* $\square_\lambda$*-sequence.*

We prove that, under $\square$-principle, some variation of Lemma 3.1 holds for skinniest stationary sets. Note that Theorem 1.9 is an immediate consequence of the following theorem since κ is a skinniest club subset of $\mathcal{P}_\kappa\kappa$.

Theorem 4.7. *Suppose* λ *is regular. If* $\square_\lambda$ *holds, and there exists a skinniest (skinny, really skinny,* μ*-skinny) stationary subset of* $\mathcal{P}_\kappa\lambda$*, then there exists a skinniest (skinny, really skinny,* μ*-skinny, respectively) stationary subset of* $\mathcal{P}_\kappa\lambda^+$*, too.*

Proof. Let Y be a skinniest (skinny, really skinny, μ-skinny) stationary subset of $\mathcal{P}_\kappa\lambda$ and $\vec{c} = \langle c_\alpha \mid \alpha \in \mathrm{Lim}(\lambda^+)\rangle$ be a $\square_\lambda$-sequence. For each $\alpha \in \mathrm{Lim}(\lambda^+)$, let $\langle \beta_\eta^\alpha \mid \eta < \mathrm{o.t.}(c_\alpha)\rangle$ be an increasing enumeration of c_α. Take a sequence $\vec{\pi} = \langle \pi_\beta \mid \beta < \lambda^+\rangle$ such that π_β is a surjection from λ to β for every $\beta < \lambda^+$. Then let X be the set of all $x \in \mathcal{P}_\kappa\lambda^+$ with $\alpha := \sup(x) \notin x$ such that

(i) $x \cap \lambda \in Y$,

(ii) $\sup(x \cap \lambda) \notin x$, and $\sup(x \cap \lambda) = \mathrm{o.t.}(c_\alpha)$,

(iii) $x = \bigcup\{\pi_{\beta_\eta^\alpha}[x \cap \lambda] \mid \eta \in x \cap \lambda\}$.

We claim that X is a skinniest (skinny, really skinny, μ-skinny) stationary subset of $\mathcal{P}_\kappa\lambda^+$.

First, we prove that X is skinniest (skinny, really skinny, μ-skinny). Take an arbitrary $\alpha \in E^{\lambda^+}_{<\kappa}$. It suffices to prove that $|X^\alpha| \leq |Y^{\mathrm{o.t.}(c_\alpha)}|$. By (ii), for every $x \in X^\alpha$, we have $x \cap \lambda \in Y^{\mathrm{o.t.}(c_\alpha)}$. Moreover, by (iii), the mapping $x \mapsto x \cap \lambda$ is an injection from X^α to $Y^{\mathrm{o.t.}(c_\alpha)}$. So $|X^\alpha| \leq |Y^{\mathrm{o.t.}(c_\alpha)}|$.

Next, we prove that X is stationary in $\mathcal{P}_\kappa\lambda^+$. For this, it is enough to prove that $X\cap\mathcal{P}_\kappa\gamma$ is stationary in $\mathcal{P}_\kappa\gamma$ for every $\gamma<\lambda^+$ of cofinality λ. Take $\gamma<\lambda^+$ of cofinality λ and a function $F:{}^{<\omega}\gamma\to\gamma$ arbitrarily. We will find $x\in X\cap\mathcal{P}_\kappa\gamma$ such that $x\cap\kappa\in\kappa$ and x is closed under F.

Take a sufficiently large regular cardinal θ, and let $\mathcal{M}$ be the structure $\langle\mathcal{H}_\theta,\in,\kappa,\lambda,\gamma,\vec{c},\vec{\pi},F\rangle$. Since Y is stationary, we can take $M\prec\mathcal{M}$ such that $|M|<\kappa$, $M\cap\kappa\in\kappa$ and $M\cap\lambda\in Y$. Let $x:=M\cap\gamma\in\mathcal{P}_\kappa\gamma$. Then $x\cap\kappa\in\kappa$, and x is closed under F. So it suffices to show that $x\in X$.

Let $\alpha:=\sup(x)$. Clearly $\alpha\notin x$, and x satisfies (i). We will check (ii) and (iii). For this, note that $c_\gamma\in M$ and c_γ is a club subset of γ of order type λ. Then the following are easily seen using the elementarity of M:

(iv) $\alpha\in\mathrm{Lim}(c_\gamma)$, and so $c_\alpha=c_\gamma\cap\alpha$.

(v) $\sup(c_\gamma\cap x)=\sup(x)=\alpha$.

(vi) $c_\gamma\cap x=\{\beta^\gamma_\eta\mid\eta\in x\cap\lambda\}$.

First, we check (ii). Clearly $\xi:=\sup(x\cap\lambda)\notin x$. By (v) and (vi), we have $\beta^\gamma_\xi=\alpha$. Then $\xi=\mathrm{o.t.}(c_\gamma\cap\alpha)=\mathrm{o.t.}(c_\alpha)$ by (iv). Next, we check (iii). From (v) and the elementarity of M, it easily follows that $x=\bigcup\{\pi_\beta[x\cap\lambda]\mid\beta\in c_\gamma\cap x\}$. Moreover $c_\gamma\cap x=\{\beta^\alpha_\eta\mid\eta\in x\cap\lambda\}$ by (iv) and (vi). So x satisfies (iii). □

We end this section with the following question:

Question 4.8. *Is the non-existence of skinniest stationary subsets of $\mathcal{P}_\kappa\kappa^{+n}$ consistent with* ZFC*?*

5 Saturated ideals

In this section, we prove some variants of Theorem 1.3. More precisely, we show that, for every regular cardinal λ, the dual filters of saturated ideals over $\mathcal{P}_\kappa\lambda$ contain skinny stationary subsets of $\mathcal{P}_\kappa\lambda$ with various degrees of skinniness. The degree of skinniness depends on the degree of saturation of ideals.

In Matsubara [11], it is proved that if there is an ω-strategically closed normal ideal over $\mathcal{P}_\kappa\lambda$, then SCH holds between κ and λ. The proofs we present in this section are based upon the idea of the proof of this result.

Our first theorem tells us that if $(2^{<\kappa})^+ < \lambda$, and λ is regular, then the existence of a λ-saturated normal fine ideal over $\mathcal{P}_\kappa\lambda$ implies the existence of a skinnier stationary subset of $\mathcal{P}_\kappa\lambda$.

Theorem 5.1. *Let λ be a regular cardinal $> \kappa$. Suppose I is a λ-saturated normal fine ideal over $\mathcal{P}_\kappa\lambda$. Then there is a $(2^{<\kappa})^+$-skinny subset of $\mathcal{P}_\kappa\lambda$ which is in I^*.*

The following lemma will be used to prove the above theorem and the next theorem as well:

Lemma 5.2. *Suppose λ and I are as in Theorem 5.1. Let S be a stationary subset of $E^\lambda_{<\kappa}$. Then*

$$X := \{x \in \mathcal{P}_\kappa\lambda \mid \mathrm{cf}(\sup(x)) > \omega \wedge S \cap \sup(x) \textit{ is stationary in } \sup(x)\} \in I^* .$$

Proof. Take an arbitrary $\mathbb{P}_I$-generic filter G over V. In $V[G]$, note that the ultrapower of V by G is well-founded since I is λ-saturated. Let M be its transitive collapse and $j : V \to M$ be the ultrapower map. It suffices to prove that $j[\lambda] \in j(X)$. Below we work in $V[G]$.

First note that λ is regular in $V[G]$ since $\mathbb{P}_I$ has the λ-c.c. Thus $\mathrm{cf}(j[\lambda]) = \lambda > \omega$ in $V[G]$. Then this also holds in M since $M \subseteq V[G]$. We must show that $j(S) \cap \sup(j[\lambda])$ is stationary in $\sup(j[\lambda])$ in M. For this, note that S remains stationary in λ in $V[G]$, since $\mathbb{P}_I$ has the λ-c.c. Moreover, $S \subseteq (E^\lambda_{<\kappa})^V$, and $j \restriction (E^\lambda_{<\kappa})^V$ is continuous since the critical point of j is κ. Then $j[S]$ is stationary in $\sup(j[\lambda])$ in $V[G]$. But $j[S] \subseteq j(S) \cap \sup(j[\lambda])$. Thus $j(S) \cap \sup(j[\lambda])$ is stationary in $\sup(j[\lambda])$ in $V[G]$. This also holds in M since $M \subseteq V[G]$. □

Proof of Theorem 5.1. Take a partition $\langle S_\beta \mid \beta < \lambda \rangle$ of E^λ_ω into stationary sets. Let X be the set of all $x \in \mathcal{P}_\kappa\lambda$ such that $\mathrm{cf}(\sup(x)) > \omega$ and $S_\beta \cap \sup(x)$ is stationary in $\sup(x)$ for all $\beta \in x$. Then $X \in I^*$ by Lemma 5.2 and the normality of I. So it suffices to prove that X is $(2^{<\kappa})^+$-skinny.

Take an arbitrary $\alpha \in E^\lambda_{<\kappa}$. We prove that $|X^\alpha| \leq 2^{<\kappa}$. Let B be the set of all $\beta < \alpha$ such that $S_\beta \cap \alpha$ is stationary in α. Take a club $c \subseteq \alpha$ of size $\mathrm{cf}(\alpha)$. Then $\langle S_\beta \cap c \mid \beta \in B \rangle$ is a pairwise disjoint sequence of non-empty subsets of c. So $|B| \leq |c| = \mathrm{cf}(\alpha) < \kappa$. Moreover $X^\alpha \subseteq \mathcal{P}(B)$. So $|X^\alpha| \leq 2^{|B|} \leq 2^{<\kappa}$. □

Next, we show that more stringent requirements on saturation of our ideal I guarantee the existence of a skinniest subset of $\mathcal{P}_\kappa\lambda$ which is in I^*:

Theorem 5.3. *Let λ be a regular cardinal $> \kappa$ and δ be a cardinal with $\delta < \kappa^{+\kappa}$ and $\delta \leq \lambda$. Suppose I is a δ-saturated normal fine ideal over $\mathcal{P}_\kappa\lambda$. Then there exists a skinniest subset of $\mathcal{P}_\kappa\lambda$ which is in I^*.*

Proof. To define a skinniest $X \in I^*$, we do some preliminaries. First, note that κ is weakly inaccessible by the assumptions of our theorem. (If $\kappa = \mu^+$, then forcing with $\mathbb{P}_I$ collapses the cardinality of λ to be μ, violating λ-saturation of I.) Since $\delta < \kappa^{+\kappa}$, there is some $\nu < \kappa$ such that $|\mathrm{REG} \cap [\kappa, \delta)| < |\mathrm{REG} \cap \nu|$, where REG denotes the class of all regular cardinals. For each $\mu \in \mathrm{REG} \cap \nu$, fix a partition $\langle T^\mu_\beta \mid \beta < \lambda \rangle$ of E^λ_μ into stationary sets. Let h be the function on $\mathcal{P}_\kappa\lambda$ defined by

$$h(x) := \{\mathrm{cf}(\mathrm{o.t.}(x \cap \gamma)) \mid \gamma \in \mathrm{REG} \cap [\kappa, \delta)\}\,.$$

Note that $\mathrm{REG} \cap \nu \setminus h(x) \neq \emptyset$ for every $x \in \mathcal{P}_\kappa\lambda$.

Now let X be the set of all $x \in \mathcal{P}_\kappa\lambda$ such that $\mathrm{cf}(\sup(x)) > \omega$, and

$$x = \{\beta < \lambda \mid T^\mu_\beta \text{ is stationary in } \sup(x)\}$$

for every $\mu \in \mathrm{REG} \cap \nu \setminus h(x)$. We will prove that $X \in I^*$ and X is skinniest.

First, we prove that X is skinniest. Suppose that $x, y \in X$ and $\sup(x) = \sup(y) = \alpha$. We show that $x = y$. Since $|h(x)|, |h(y)| < \mathrm{REG} \cap \nu$, we can pick $\mu \in \mathrm{REG} \cap \nu \setminus (h(x) \cup h(y))$. Then $x = \{\beta < \lambda \mid T^\mu_\beta \text{ is stationary in } \alpha\} = y$.

Next, we prove that $X \in I^*$. Let X_0 be the set of all $x \in \mathcal{P}_\kappa\lambda$ such that $\mathrm{cf}(\sup(x)) > \omega$ and $T^\mu_\beta \cap \sup(x)$ is stationary in $\sup(x)$ for every $\mu \in \mathrm{REG} \cap \nu$ and every $\beta \in x$. Moreover, let X_1 be the set of all $x \in \mathcal{P}_\kappa\lambda$ such that $T^\mu_\beta \cap \sup(x)$ is non-stationary in $\sup(x)$ for every $\mu \in \mathrm{REG} \cap \nu \setminus h(x)$ and every $\beta \in \lambda \setminus x$. Then $X = X_0 \cap X_1$. Moreover, $X_0 \in I^*$ by Lemma 5.2 together with the normality and the κ-completeness of I. So it suffices to show that $X_1 \in I^*$.

Take an arbitrary $\mathbb{P}_I$-generic filter G over V. In $V[G]$, let M be the transitive collapse of the ultrapower of V by G, and let $j : V \to M$ be the ultrapower map. It suffices to prove that $j[\lambda] \in j(X_1)$. Let $\alpha := \sup(j[\lambda])$. Take an arbitrary $\mu \in \mathrm{REG}^M \cap \nu \setminus j(h)(j[\lambda])$, and let $\langle T_\beta \mid \beta < j(\lambda) \rangle$ be $j(\langle T^\mu_\beta \mid \beta < \lambda \rangle)$. We must prove that, in M, $T_\beta \cap \alpha$ is non-stationary in α for any $\beta \in j(\lambda) \setminus j[\lambda]$.

For this, first we prove that $\mathrm{cf}^M(\gamma) \neq \mu$ for any $\gamma \in \mathrm{REG}^V \setminus \{\mu\}$. Suppose $\gamma \in \mathrm{REG}^V \setminus \{\mu\}$. If $\gamma < \kappa$, then $\gamma = j(\gamma)$ is regular in M, and so $\mathrm{cf}^M(\gamma) = \gamma \neq \mu$. Next, suppose $\gamma \geq \delta$. Then γ is regular in $V[G]$ by the δ-c.c. of $\mathbb{P}_I$, and hence this also holds in M. Thus $\mathrm{cf}^M(\gamma) = \gamma \neq \mu$. Finally, suppose that $\gamma \in [\kappa, \delta)$. Then

$$\mathrm{cf}^M(\gamma) = \mathrm{cf}^M(\mathrm{o.t.}(j[\gamma])) = \mathrm{cf}^M(\mathrm{o.t.}(j(\gamma) \cap j[\lambda])) \in j(h)(j[\lambda])\,.$$

Since $\mu \notin j(h)(j[\lambda])$, we have $\mathrm{cf}^M(\gamma) \neq \mu$.

Thus $S := (E^\lambda_\mu)^V = (E^\lambda_\mu)^M$. Note that $j \restriction S$ is continuous since $\mu < \kappa$. Hence $(E^\alpha_\mu)^M \setminus j[S]$ is non-stationary in α in M. But $j[S] \subseteq \bigcup_{\beta \in j[\lambda]} T_\beta$, and $\langle T_\beta \mid \beta < j(\lambda)\rangle$ is a pairwise disjoint sequence of subsets of $(E^{j(\lambda)}_\mu)^M$ since $\langle T^\mu_\beta \mid \beta < \lambda\rangle$ is a partition of S. Therefore $T_\beta \cap \alpha$ is non-stationary in α in M for any $\beta \in j(\lambda) \setminus j[\lambda]$. □

6 Saturation of $\mathrm{NS}_\lambda \restriction S$

Recall that the λ^+-saturation of $\mathrm{NS}_\lambda \restriction S$ implies the failure of $\diamondsuit_\lambda(S)$. So, by Theorem 1.6, assuming $2^{<\kappa} < \lambda = 2^{<\lambda}$, if $\mathrm{NS}_\lambda \restriction S$ is λ^+-saturated, then S does not bear a skinnier stationary subset of $\mathcal{P}_\kappa\lambda$. It turns out that we can drop the assumption on cardinal arithmetic from this proposition:

Theorem 6.1. *Suppose λ is a regular cardinal $> \kappa$ and S is a stationary subset of $E^\lambda_{<\kappa}$. If $\mathrm{NS}_\lambda \restriction S$ is λ^+-saturated, then S cannot bear a skinnier stationary subset of $\mathcal{P}_\kappa\lambda$.*

To present our proof, we use the following combinatorial principle:

Definition 6.2. *Suppose that λ is a regular cardinal $> \kappa$ and S is a stationary subset of $E^\lambda_{<\kappa}$. Let $\clubsuit^-_{\lambda,<\kappa}(S)$ be the following assertion:*

There is a sequence $\langle \mathfrak{b}_\alpha \mid \alpha \in S\rangle$ such that

- *(i) $\mathfrak{b}_\alpha \subseteq \mathcal{P}_\kappa\alpha$ and $|\mathfrak{b}_\alpha| \leq \alpha$ for each $\alpha \in S$,*
- *(ii) for any cofinal $B \subseteq \lambda$, the set $\{\alpha \in S \mid \exists b \in \mathfrak{b}_\alpha\, (\sup(B \cap b) = \alpha)\}$ is stationary in λ.*

A sequence $\langle \mathfrak{b}_\alpha \mid \alpha \in S\rangle$ satisfying (i) and (ii) is called a $\clubsuit^-_{\lambda,<\kappa}(S)$-sequence.

First, we prove the following lemma. Gitik–Rinot [4] proved the same lemma for the case λ is a successor cardinal. We state our proof for the sake of completeness.

Lemma 6.3. *Suppose that λ is a regular cardinal $> \kappa$ and X is a skinnier stationary subset of $\mathcal{P}_\kappa\lambda$. Then $\clubsuit^-_{\lambda,<\kappa}(E_X)$ holds.*

Proof. We prove that $\langle X^\alpha \mid \alpha \in E_X\rangle$ is a $\clubsuit^-_{\lambda,<\kappa}(E_X)$-sequence. Clearly it satisfies property (i) of $\clubsuit^-_{\lambda,<\kappa}$. We will prove that it also satisfies (ii).

Take an arbitrary cofinal $B \subseteq \lambda$. Define a function $F : \lambda \to \lambda$ by $F(\beta) = \min(B\backslash\beta)$ for each $\beta < \lambda$. Let Y be the set of all $x \in X$ such that $\sup(x) \notin x$ and x is closed under F. Then Y is stationary in $\mathcal{P}_\kappa\lambda$, and so E_Y is a stationary subset of E_X. Moreover, for each $\alpha \in E_Y$, if we take $x \in Y^\alpha \subseteq X^\alpha$, then $\sup(B \cap x) = \alpha$. Hence $\langle X^\alpha \mid \alpha \in E_X\rangle$ satisfies (ii) for B. □

The next theorem, together with the last lemma, immediately provides a proof for Theorem 6.1. Rinot [14] proved the next theorem for the case when λ is a successor cardinal.

Theorem 6.4. *Suppose that λ is a regular cardinal $> \kappa$ and S is a stationary subset of $E^\lambda_{<\kappa}$. If $\mathrm{NS}_\lambda \restriction S$ is λ^+-saturated, then $\clubsuit^-_{\lambda,<\kappa}(S)$ fails.*

Our proof of this theorem is modeled after the proof of the well-known fact that if $\lambda = \mu^+$, then there is no λ^+-saturated normal ideal over λ concentrating on $E^\lambda_{<\mathrm{cf}(\mu)}$, which follows from the theorem by Shelah [15] (Chapter XIII, 4.9 Lemma) stating that if λ is a regular cardinal in V, then $(\lambda^+)^V$ is not a cardinal in any outer model satisfying $\mathrm{cf}(\lambda) < \mathrm{cf}(|\lambda|)$. As in the proof of this fact, the following notion of strongly pairwise almost disjoint families plays a central role in our proof:

Definition 6.5 (Shelah [15])**.** *Let ρ be a limit ordinal and $\mathcal{B}$ be a family of cofinal subsets of ρ. $\mathcal{B}$ is said to be* strongly pairwise almost disjoint *if, for every $\mathcal{B}' \subseteq \mathcal{B}$ of size $\leq \rho$, there is a function $\sigma : \mathcal{B}' \to \rho$ such that $B_0 \cap B_1 \subseteq \max\{\sigma(B_0), \sigma(B_1)\}$ for any distinct $B_0, B_1 \in \mathcal{B}$.*

In the proof of the above mentioned theorem by Shelah [15] (Chapter XIII, 4.9 Lemma), it is mentioned that if λ is a regular cardinal, then there is a strongly pairwise almost disjoint family $\mathcal{B}$ of cofinal subsets of λ with $|\mathcal{B}| = \lambda^+$: Suppose λ is regular. It is well-known that there is a pairwise almost disjoint family $\mathcal{B}$ of cofinal subsets of λ with $|\mathcal{B}| = \lambda^+$. We claim that $\mathcal{B}$ is strongly pairwise almost disjoint. Suppose $\{B_\alpha \mid \alpha < \lambda\} \subseteq \mathcal{B}$. For each $\beta < \lambda$, let $\sigma(B_\beta) := \sup\{\sup(B_\beta \cap B_\alpha)+1 \mid \alpha < \beta\}$. Then $\sigma(B_\beta) < \lambda$ for all $\beta < \lambda$ since λ is regular. Moreover, if $\alpha < \beta < \lambda$, then $B_\alpha \cap B_\beta \subseteq \sigma(B_\beta) \leq \max\{\sigma(B_\alpha), \sigma(B_\beta)\}$.

To present our proof of Theorem 6.4, we need the next lemma concerning strongly pairwise almost disjoint families:

Lemma 6.6. *Suppose that ρ is a limit ordinal and $\mathcal{B}$ is a strongly pairwise almost disjoint family of cofinal subsets of ρ. Let b be a subset of ρ with $|b|^+ \leq \rho$. Then $|\{B \in \mathcal{B} \mid \sup(B \cap b) = \rho\}| \leq |b|$.*

Proof. Let $\mathcal{B}' := \{B \in \mathcal{B} \mid \sup(B \cap b) = \rho\}$. For a contradiction, assume $|\mathcal{B}'| > |b|$. Note that b is cofinal in ρ. In particular, $\mathrm{cf}(\rho) \leq |b|$.

Take $\mathcal{B}'' \subseteq \mathcal{B}'$ with $|\mathcal{B}''| = |b|^+ \leq \rho$. Since $\mathcal{B}$ is strongly pairwise almost disjoint, there is $\sigma : \mathcal{B}'' \to \rho$ such that $B_0 \cap B_1 \subseteq \max\{\sigma(B_0), \sigma(B_1)\}$ for all distinct $B_0, B_1 \in \mathcal{B}''$. Since $\mathrm{cf}(\rho) \leq |b|$, we can take $\alpha^* < \rho$ such that $\mathcal{B}^* = \{B \in \mathcal{B}'' \mid \sigma(B) < \alpha^*\}$ has size $|b|^+$. Then $\{B \setminus \alpha^* \mid B \in \mathcal{B}^*\}$ is a pairwise disjoint family of size $|b|^+$, whose members all intersect with b. This is a contradiction. $\square$

Now we present a proof of Theorem 6.4:

Proof of Theorem 6.4. Towards a contradiction, assume that $\clubsuit^-_{\lambda,<\kappa}(S)$ holds, where $\mathrm{NS}_\lambda \restriction S$ is λ^+-saturated. Let $\langle \mathfrak{b}_\alpha \mid \alpha \in S \rangle$ be a $\clubsuit^-_{\lambda,<\kappa}(S)$-sequence. Take a strongly pairwise almost disjoint family $\mathcal{B}$ of cofinal subsets of λ with $|\mathcal{B}| = \lambda^+$.

Let $\mathbb{P}$ be $\mathbb{P}_{\mathrm{NS}_\lambda \restriction S}$ and $\dot{G}$ be the canonical name for a $\mathbb{P}$-generic filter. In $V^{\mathbb{P}}$, let M be the transitive collapse of the ultrapower of V by $\dot{G}$, and let $j : V \to M$ be the ultrapower map. Here note that $\Vdash_{\mathbb{P}}$ "$\lambda \in j(S)$". Let $\dot{\mathfrak{b}}_\lambda$ be a $\mathbb{P}$-name for the λ-th element of $j(\langle \mathfrak{b}_\alpha \mid \alpha \in S \rangle)$, and let $\dot{\mathcal{A}}$ be a $\mathbb{P}$-name for the set $\{B \in \mathcal{B} \mid \exists b \in \dot{\mathfrak{b}}_\lambda \, (\sup(B \cap b) = \lambda)\}$.

Now note that $\mathbb{P}$ forces the following:

(i) $\mathcal{B}$ remains a strongly pairwise disjoint family.

(ii) $|b|^+ \leq \lambda$ for every $b \in \dot{\mathfrak{b}}_\lambda$.

(iii) $|\dot{\mathfrak{b}}_\lambda| \leq \lambda$.

(i) is because, by the λ^+-c.c. of $\mathbb{P}$, every subset of $\mathcal{B}$ of size $\leq \lambda$ in the extension can be covered by a subset of $\mathcal{B}$ of size $\leq \lambda$ in the ground model. For (ii), note that, in M, $|b|^+ \leq \kappa < \lambda$ for every $b \in \dot{\mathfrak{b}}_\lambda$, since $j(\langle \mathfrak{b}_\alpha \mid \alpha \in S \rangle)$ is a $\clubsuit^-_{j(\lambda),<\kappa}(j(S))$-sequence. Then this also holds in $V^{\mathbb{P}}$ since ${}^\lambda M \cap V^{\mathbb{P}} \subseteq M$. Finally, (iii) follows from the fact that $j(\langle \mathfrak{b}_\alpha \mid \alpha \in S \rangle)$ is a $\clubsuit^-_{j(\lambda),<\kappa}(j(S))$-sequence in M.

By Lemma 6.6 and (i)–(iii) above, we know that $\Vdash_{\mathbb{P}}$ "$|\dot{\mathcal{A}}| \leq \lambda$". By the λ^+-c.c. of $\mathbb{P}$ in V, we can take $\mathcal{A}^* \subseteq \mathcal{B}$ of size $\leq \lambda$ such that $\Vdash_{\mathbb{P}}$ "$\dot{\mathcal{A}} \subseteq \mathcal{A}^*$". Take $B \in \mathcal{B} \setminus \mathcal{A}^*$. Then $\mathbb{P}$ forces that there is no $b \in \dot{\mathfrak{b}}_\lambda$ with $\sup(B \cap b) = \lambda$. Here note that $\mathbb{P}$ forces $j(B) \cap \lambda = B$. Thus $\mathbb{P}$ forces that there is no $b \in \dot{\mathfrak{b}}_\lambda$ with $\sup(j(B) \cap b) = \lambda$. On the other hand, the set $\{\alpha \in S \mid \exists b \in \mathfrak{b}_\alpha\, (\sup(B \cap b) = \alpha)\}$ is stationary since $\langle \mathfrak{b}_\alpha \mid \alpha \in S \rangle$ is a $\clubsuit^-_{\lambda, <\kappa}(S)$-sequence, and this set forces that there is $b \in \dot{\mathfrak{b}}_\lambda$ with $\sup(j(B) \cap b) = \lambda$. This is a contradiction. $\Box$

7 Bearing no μ-skinny stationary sets

In this section, we prove Theorem 1.10. Throughout this section, let μ be a regular cardinal with $\kappa \leq \mu < \lambda$.

Before we present the details of our proof, we sketch its outline. First, we force our sequence $\langle S_\delta \mid \delta < \mu \rangle$ of subsets of $E^\lambda_{<\kappa}$ with certain desirable properties described in the beginning of §7.2. That forcing is presented in Lemma 7.8. Then we perform a $< \mu$-support iteration of some "club shooting" posets of length 2^λ, making all of S_δ ($\delta < \mu$) of our sequence bear no μ-skinny stationary subsets of $\mathcal{P}_\kappa \lambda$. In order to guarantee the μ-distributivity and the μ^+-c.c. of our iteration, we rely on the notion of Z-closedness which is described in §7.1.

7.1 Z-closed forcing

In this subsection, we introduce the notion of Z-closed forcing for a stationary subset Z of $\mathcal{P}_\mu \lambda$. This notion is a generalization of that of E-complete forcing, which was introduced by Shelah [15] (Chapter V, §1). We also present basic properties of Z-closed forcings, which will be used in the proof of Theorem 1.10. More precisely, we show that if Z is fat, then the Z-closedness implies the μ-distributivity and is preserved by $< \mu$-support iteration. Here the notion of fat subsets of $\mathcal{P}_\mu \lambda$ is the one introduced in Krueger [9], which generalizes the notion of fat subsets of a regular cardinal introduced by Abraham-Shelah [1].

First, we define the notion of Z-closed forcing:

Notation. *Let $\mathbb{P}$ be a poset and M be a set. An $(M, \mathbb{P})$*-generic sequence *is a descending sequence $\langle p_\xi \mid \xi < \zeta \rangle$ in $\mathbb{P} \cap M$ for some ordinal ζ such that for every dense open $D \subseteq \mathbb{P}$ with $D \in M$, there is $\xi < \zeta$ with $p_\xi \in D$. We say that $\mathbb{P}$ is* M-closed *if every $(M, \mathbb{P})$-generic sequence has a lower bound in $\mathbb{P}$.*

Definition 7.1. *Let Z be a subset of $\mathcal{P}_\mu\lambda$. We say that a poset $\mathbb{P}$ is Z-*closed *if, for every sufficiently large regular cardinal θ and every $M \prec \langle \mathcal{H}_\theta, \in, \mu, \lambda, Z, \mathbb{P}\rangle$ of size $< \mu$ with $\bigcup(\mathcal{P}_\mu\lambda \cap M) \in Z$, we have that $\mathbb{P}$ is M-closed.*

In the above definition, note that $\mathbb{P}$ is Z-closed if there are a sufficiently large regular cardinal θ and a countable expansion $\mathcal{M}$ of $\langle \mathcal{H}_\theta, \in, \mu, \lambda, Z, \mathbb{P}\rangle$ such that $\mathbb{P}$ is M-closed for every $M \prec \mathcal{M}$ of size $< \mu$ with $\bigcup(\mathcal{P}_\mu\lambda \cap M) \in Z$. This can be proved by the same argument as a well-known similar fact on proper forcing. Note also that, if θ is a sufficiently large regular cardinal, M is an elementary submodel of $\langle \mathcal{H}_\theta, \in, \mu, \lambda\rangle$ of size $< \mu$, and $M \cap \mu \in \mu$, then $\bigcup(\mathcal{P}_\mu\lambda \cap M) = M \cap \lambda$. We will use these facts without any notice.

Next, we recall the notion of fat subsets of $\mathcal{P}_\mu\lambda$ introduced by Krueger [9], which generalizes the notion of fat subsets of a regular uncountable cardinal due to Abraham-Shelah [1]. The following definition is slightly different from the one in [9]. But it is easy to see that they are equivalent if μ is an inaccessible cardinal or $\mu = \nu^+$ for a regular cardinal ν with $\nu^{<\nu} = \nu$.

Notation. *A sequence $\langle M_\xi \mid \xi < \zeta\rangle$ for some ordinal ζ is called a* nice $\in$-chain *if it is $\subseteq$-increasing continuous and $\langle M_\eta \mid \eta \leq \xi\rangle \in M_{\xi+1}$ for every ξ with $\xi + 1 < \zeta$.*

Definition 7.2 (Krueger [9])**.** *Let Z be a subset of $\mathcal{P}_\mu\lambda$. We say that Z is* fat *if, for every regular cardinal $\theta > \lambda$, every countable expansion $\mathcal{M}$ of the structure $\langle \mathcal{H}_\theta, \in\rangle$, and every regular cardinal $\nu < \mu$, there is a nice $\in$-chain $\langle M_\xi \mid \xi \leq \nu\rangle$ of elementary submodels of $\mathcal{M}$ of size $< \mu$ such that $M_\xi \cap \mu \in \mu$ and $M_\xi \cap \lambda \in Z$ for every limit $\xi \leq \nu$.*

Krueger [9] proved that if μ is an inaccessible cardinal or $\mu = \nu^+$ for a regular cardinal ν with $\nu^{<\nu} = \nu$, then $Z \subseteq \mathcal{P}_\mu\lambda$ is fat if and only if there is a μ-distributive forcing notion shooting a $\subseteq$-increasing continuous sequence in Z of length μ, which is $\subseteq$-cofinal in $\mathcal{P}_\mu\lambda$. Here, as we mentioned at the beginning of this subsection, we prove that if Z is fat, then the Z-closedness implies the μ-distributivity:

Lemma 7.3. *Suppose that Z is a fat subset of $\mathcal{P}_\mu\lambda$ and $\mathbb{P}$ is a Z-closed poset. Then $\mathbb{P}$ is μ-distributive.*

Proof. By induction on cardinals $\nu \leq \mu$, we prove that $\mathbb{P}$ is ν-distributive. Clearly $\mathbb{P}$ is ω-distributive. Suppose ν is a cardinal $\leq \mu$ and $\mathbb{P}$ is ν'-distributive for every cardinal $\nu' < \nu$. We prove that $\mathbb{P}$ is ν-distributive. If ν is a limit cardinal or a successor of a singular cardinal, then this is clear. So assume ν is a successor of a regular cardinal, and let ν^- denote the predecessor of ν. Suppose $p \in \mathbb{P}$ and $\mathcal{D}$ is a family of dense open subsets of $\mathbb{P}$ with $|\mathcal{D}| \leq \nu^-$. We will find $p^* \leq p$ with $p^* \in \bigcap \mathcal{D}$.

Take a sufficiently large regular cardinal θ and a well-ordering Δ of $\mathcal{H}_\theta$, and let $\mathcal{M} := \langle \mathcal{H}_\theta, \in, \Delta, \mu, \lambda, Z, \mathbb{P}, p, \mathcal{D} \rangle$. First we claim the following:

Claim. *There is a nice $\in$-chain $\langle M_\xi \mid \xi \leq \nu^- \rangle$ of elementary submodels of $\mathcal{M}$ such that $|M_\xi| < \nu^-$ for every $\xi < \nu^-$ and $\bigcup(\mathcal{P}_\mu \lambda \cap M_\xi) \in Z$ for every limit $\xi \leq \nu^-$.*

Proof of Claim. By fatness of Z, there is a nice $\in$-chain $\langle N_\xi \mid \xi \leq \nu^- \rangle$ of elementary submodels of $\mathcal{M}$ of size $< \mu$ such that $N_\xi \cap \mu \in \mu$ and $N_\xi \cap \lambda \in Z$ for every limit $\xi \leq \nu^-$. For each $\xi \leq \nu^-$, let M_ξ be the Skolem hull of the set $\{\langle N_\eta \mid \eta \leq \xi' \rangle \mid \xi' < \xi\}$ in $\mathcal{M}$. We prove that $\langle M_\xi \mid \xi \leq \nu^- \rangle$ is as desired.

Clearly, $|M_\xi| < \nu^-$ for every $\xi < \nu^-$. It is also clear that $\langle M_\xi \mid \xi \leq \nu^- \rangle$ is $\subseteq$-increasing continuous. Moreover, $\langle M_\xi \mid \xi \leq \zeta \rangle \in M_{\zeta+1}$ for every $\zeta < \nu^-$, since $\langle N_\eta \mid \eta \leq \zeta \rangle \in M_{\zeta+1}$, and, for each $\xi \leq \zeta$, M_ξ is the Skolem hull of $\{\langle N_\eta \mid \eta \leq \xi' \rangle \mid \xi' < \xi\}$ in the restriction of $\mathcal{M}$ to N_ζ. So $\langle M_\xi \mid \xi \leq \nu^- \rangle$ is a nice $\in$-chain.

It remains to check that $\bigcup(\mathcal{P}_\mu \lambda \cap M_\xi) \in Z$ for every limit $\xi \leq \nu^-$. Fix a limit ordinal $\xi \leq \nu^-$. First, note that $\bigcup(\mathcal{P}_\mu \mathcal{H}_\theta \cap M_\xi) \subseteq N_\xi$, since $M_\xi \subseteq N_\xi \prec \mathcal{M}$, and $N_\xi \cap \mu \in \mu$. Moreover, $N_\xi \subseteq \bigcup(\mathcal{P}_\mu \mathcal{H}_\theta \cap M_\xi)$ since $N_\xi = \bigcup_{\eta<\xi} N_\eta$, and $N_\eta \in \mathcal{P}_\mu \mathcal{H}_\theta \cap M_\xi$ for every $\eta < \xi$. So $\bigcup(\mathcal{P}_\mu \mathcal{H}_\theta \cap M_\xi) = N_\xi$. Then $\bigcup(\mathcal{P}_\mu \lambda \cap M_\xi) = N_\xi \cap \lambda \in Z$. □(Claim)

Let $\langle M_\xi \mid \xi \leq \nu^- \rangle$ be as in Claim. Note that $\mathcal{D} \subseteq M_{\nu^-}$ since $\nu^- \subseteq M_\nu \prec \mathcal{M}$. By induction on ξ, we will construct a descending sequence $\langle p_\xi \mid \xi \leq \nu^- \rangle$ in $\mathbb{P}$ below p. We will construct it so that, for each $\zeta < \nu^-$, $\langle p_\xi \mid \xi \leq \zeta \rangle$ is definable in $\mathcal{M}$ from $\langle M_\xi \mid \xi \leq \zeta \rangle$. So p_ξ will belong to $M_{\xi+1}$ for all $\xi < \nu^-$. Let $p_0 := p$. Suppose $\xi < \nu^-$ and p_ξ has been defined. Let $\mathcal{D}_{\xi+1}$ be the family of all dense open subsets of $\mathbb{P}$ which belong to $M_{\xi+1}$. Then $\bigcap \mathcal{D}_{\xi+1}$ is dense open since $|\mathcal{D}_{\xi+1}| < \nu^-$, and $\mathbb{P}$ is ν^--distributive. Let $p_{\xi+1}$ be the Δ-least element of $\bigcap \mathcal{D}_{\xi+1}$ below p_ξ. Suppose ξ is a limit ordinal $\leq \nu^-$, and $\langle p_\eta \mid \eta < \xi \rangle$ has been defined. By the construction at successor steps and the fact that $M_\xi = \bigcup_{\eta<\xi} M_\eta$, we know that $\langle p_\eta \mid \eta < \xi \rangle$ is $(M_\xi, \mathbb{P})$-generic. Then $\langle p_\eta \mid \eta < \xi \rangle$

has a lower bound since $\mathbb{P}$ is Z-closed, and $\bigcup(\mathcal{P}_\mu\lambda \cap M_\xi) \in Z$. Let p_ξ be the Δ-least lower bound of $\langle p_\eta \mid \eta < \xi\rangle$ in $\mathbb{P}$.

Let $p^* := p_{\nu^-}$. Then $p^* \leq p$. Moreover, $p^* \in D$ for every dense open $D \subseteq \mathbb{P}$ with $D \in M_{\nu^-}$. Then $p^* \in \bigcap \mathcal{D}$ since $\mathcal{D} \subseteq M_{\nu^-}$. □

In the rest of this subsection, we prove that if Z is fat, then Z-closedness is preserved by $< \mu$-support iterations.

Proposition 7.4. *Let Z be a fat subset of $\mathcal{P}_\mu\lambda$ and $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \gamma, \beta < \gamma\rangle$ be a $< \mu$-support iteration of Z-closed posets for some ordinal γ. Then $\mathbb{P}_\gamma$ is Z-closed.*

For this, we will need the following lemmata:

Lemma 7.5. *Let Z and $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \gamma, \beta < \gamma\rangle$ be as in Proposition 7.4, and suppose γ is a limit ordinal. Let θ be a sufficiently large regular cardinal and M be an elementary submodel of $\langle \mathcal{H}_\theta, \in, \mu, \lambda, Z, \vec{\mathbb{P}}\rangle$ such that $|M| < \mu$, $M \cap \mu \in \mu$ and $M \cap \lambda \in Z$. Let $\vec{p} = \langle p_\xi \mid \xi < \zeta\rangle$ be a descending sequence in $\mathbb{P}_\gamma \cap M$ such that $\langle p_\xi \restriction \alpha \mid \xi < \zeta\rangle$ is $(M, \mathbb{P}_\alpha)$-generic for every $\alpha \in M \cap \gamma$. Then $\vec{p}$ has a lower bound in $\mathbb{P}_\gamma$.*

Proof. First, note that $|M \cap \gamma| < \mu$ and $\mathrm{dom}(p_\xi) \subseteq M \cap \gamma$ for every $\xi < \zeta$. We will construct a lower bound p^* of $\vec{p}$ such that $\mathrm{dom}(p^*) = M \cap \gamma$. It suffices to define $p^*(\beta)$ by induction on $\beta \in M \cap \gamma$ so that $p^* \restriction \beta + 1$ is a lower bound of $\langle p_\xi \restriction \beta + 1 \mid \xi < \zeta\rangle$.

Suppose that $\beta \in M \cap \gamma$ and $p^* \restriction \beta$ has been defined. Note that $p^* \restriction \beta$ is a lower bound of $\langle p_\xi \restriction \beta \mid \xi < \zeta\rangle$, which is an $(M, \mathbb{P}_\beta)$-generic sequence. Let $\dot{G}_\beta$ be the canonical name for a $\mathbb{P}_\beta$-generic filter. Then $p^* \restriction \beta$ forces that $M[\dot{G}_\beta] \prec \langle {\mathcal{H}_\theta}^{V^{\mathbb{P}_\beta}}, \in, \dot{\mathbb{Q}}_\beta\rangle$, $M[\dot{G}_\beta] \cap \mu = M \cap \mu \in \mu$ and $M[\dot{G}_\beta] \cap \lambda = M \cap \lambda \in Z$. Moreover, $p^* \restriction \beta$ forces that $\langle p_\xi(\beta) \mid \xi < \zeta\rangle$ is $(M[\dot{G}_\beta], \dot{\mathbb{Q}}_\beta)$-generic. So $p^* \restriction \beta$ forces that $\langle p_\xi(\beta) \mid \xi < \zeta\rangle$ has a lower bound in $\dot{\mathbb{Q}}_\beta$ by Z-closedness of $\dot{\mathbb{Q}}_\beta$. Let $p^*(\beta)$ be a $\mathbb{P}_\beta$-name which $p^* \restriction \beta$ forces to be a lower bound of $\langle p_\xi(\beta) \mid \xi < \zeta\rangle$. Then $p^*(\beta)$ is as desired. □

Lemma 7.6. *Let Z and $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \gamma, \beta < \gamma\rangle$ be as in Proposition 7.4, and suppose that $\mathbb{P}_\alpha$ is μ-distributive for every $\alpha < \gamma$. Let $\mathcal{D}$ be a set of size $< \mu$ such that each $D \in \mathcal{D}$ is a dense open subset of $\mathbb{P}_{\alpha_D}$ for some $\alpha_D < \gamma$. Then $D^* = \{p \in \mathbb{P}_\gamma \mid \forall D \in \mathcal{D}\, (p \restriction \alpha_D \in D)\}$ is dense open in $\mathbb{P}_\gamma$.*

Proof. Clearly D^* is open. We will prove that D^* is dense in $\mathbb{P}_\gamma$. If γ is a successor ordinal or a limit ordinal of cofinality $\geq \mu$, then there is $\alpha < \gamma$ with $\alpha_D \leq \alpha$ for all $D \in \mathcal{D}$. So, in this case, it easily follows from the μ-distributivity of $\mathbb{P}_\alpha$ that D^* is dense in $\mathbb{P}_\gamma$. Thus suppose γ is a limit ordinal with $\nu := \mathrm{cf}(\gamma) < \mu$.

Suppose $p \in \mathbb{P}_\gamma$. We will find $p^* \in D^*$ below p. Take an increasing continuous sequence $\vec{\alpha} = \langle \alpha_\xi \mid \xi < \nu \rangle$ cofinal in γ. Let $\alpha_\nu := \gamma$. Let θ be a sufficiently large regular cardinal, and let $\mathcal{M} := \langle \mathcal{H}_\theta, \in, \mu, \lambda, Z, \vec{\mathbb{P}}, \mathcal{D}, p, \vec{\alpha} \rangle$. By fatness of Z, take a nice $\in$-chain $\langle M_\xi \mid \xi \leq \nu \rangle$ of elementary submodels of $\mathcal{M}$ of size $< \mu$ such that $M_\xi \cap \mu \in \mu$ and $M_\xi \cap \lambda \in Z$ for every limit $\xi \leq \nu$.

By induction on $\xi \leq \nu$, we will construct a descending sequence $\langle p_\xi \mid \xi \leq \nu \rangle$ in $\mathbb{P}_\gamma$ such that $p_\xi \in \mathbb{P}_{\alpha_\xi}$ and $p_\xi \leq p \restriction \alpha_\xi$ for each $\xi < \nu$. Each p_ξ will be definable in $\mathcal{M}$ from a parameter $\langle M_\eta \mid \eta \leq \xi \rangle$, and so p_ξ will be in $M_{\xi+1}$. Let $p_0 := p \restriction \alpha_0$. Suppose $\xi < \nu$ and p_ξ has been taken. First, let $q_{\xi+1} := p_\xi \cup p \restriction [\alpha_\xi, \alpha_{\xi+1})$. Then $q_{\xi+1} \in \mathbb{P}_{\alpha_{\xi+1}} \cap M_{\xi+1}$. Let $\mathcal{D}_{\xi+1}$ be the set of all dense open subsets of $\mathbb{P}_{\alpha_{\xi+1}}$ in $M_{\xi+1}$. Then $\bigcap \mathcal{D}_{\xi+1}$ is dense open in $\mathbb{P}_{\alpha_{\xi+1}}$ since $|\mathcal{D}_{\xi+1}| < \mu$ and $\mathbb{P}_{\alpha_{\xi+1}}$ is μ-distributive. Let $p_{\xi+1}$ be the Δ-least element of $\bigcap \mathcal{D}_{\xi+1}$ below $q_{\xi+1}$. Next, suppose that ξ is a limit ordinal $\leq \nu$. By the construction at successor steps, we know that $\langle p_\eta \restriction \alpha \mid \eta < \xi \rangle$ is $(M_\xi, \mathbb{P}_\alpha)$-generic for every $\alpha \in M_\xi \cap \alpha_\xi$. By Lemma 7.5, let p_ξ be the Δ-least lower bound of $\langle p_\eta \mid \eta < \xi \rangle$ in $\mathbb{P}_{\alpha_\xi}$.

Let $p^* := p_\nu$. Then $p^* \leq p$ by the construction of $\langle p_\xi \mid \xi \leq \nu \rangle$. Moreover, note that $\mathcal{D} \subseteq M_\nu$ and that, if $\alpha \in M_\nu \cap \gamma$, then $p^* \restriction \alpha$ belongs to all dense open subsets of $\mathbb{P}_\alpha$ in M_ν. So $p^* \in D^*$. □

Now we prove Proposition 7.4:

Proof of Proposition 7.4. We prove the proposition by induction on γ. Assume it is true for all $\gamma' < \gamma$. We prove it for γ.

Let θ be a sufficiently large regular cardinal, Δ be a well-ordering of $\mathcal{H}_\theta$ and $\mathcal{M}$ be the structure $\langle \mathcal{H}_\theta, \in, \Delta, \mu, \lambda, Z, \vec{\mathbb{P}} \rangle$. Suppose that M is an elementary submodel of $\mathcal{M}$ of size $< \mu$ with $\bigcup(\mathcal{P}_\mu \lambda \cap M) \in Z$, and $\vec{p} = \langle p_\xi \mid \xi < \zeta \rangle$ is an $(M, \mathbb{P}_\gamma)$-generic sequence. It is enough to find a lower bound of $\vec{p}$ in $\mathbb{P}_\gamma$.

First, suppose γ is a successor ordinal. Let $\beta := \gamma - 1$. Then $\langle p_\xi \restriction \beta \mid \xi < \zeta \rangle$ is an $(M, \mathbb{P}_\beta)$-generic sequence. So, by the induction hypothesis, it has a lower bound $p' \in \mathbb{P}_\beta$. Here note that $\mathcal{P}_\mu \lambda$ is absolute between V and $V^{\mathbb{P}_\beta}$ since $\mathbb{P}_\beta$ is μ-distributive. Hence p' forces that $\bigcup(\mathcal{P}_\mu \lambda \cap M[\dot{G}_\beta]) = \bigcup(\mathcal{P}_\mu \lambda \cap M) \in Z$, where $\dot{G}_\beta$ is the canonical name for a $\mathbb{P}_\beta$-generic filter. Moreover, p' forces that $\langle p_\xi(\beta) \mid \xi < \zeta \rangle$ is an $(M[\dot{G}_\beta], \dot{\mathbb{Q}}_\beta)$-generic sequence. So there is a $\mathbb{P}_\beta$-name $\dot{q}$ which p' forces to be a lower bound of $\langle p_\xi(\beta) \mid \xi < \zeta \rangle$. Then $p' \cup \{(\beta, \dot{q})\}$ is a lower bound of $\vec{p}$ in $\mathbb{P}_\gamma$.

Next, suppose that γ is a limit ordinal. Let $N := \bigcup(\mathcal{P}_\mu \mathcal{H}_\theta \cap M)$. Using the elementarity of M and the Tarski-Vaught criterion, we can easily prove that $N \prec \mathcal{M}$. It is also easy to see that $N \cap \mu \in \mu$ and $N \cap \lambda \in Z$. Then, by Lemma 7.5, it suffices to prove that $\langle p_\xi \restriction \alpha \mid \xi < \zeta \rangle$ is $(N, \mathbb{P}_\alpha)$-generic for every $\alpha \in N \cap \gamma$. Fix $\alpha \in N \cap \gamma$, and take an arbitrary dense open $D \subseteq \mathbb{P}_\alpha$ with $D \in N$. We must find $\xi < \zeta$ with $p_\xi \restriction \alpha \in D$. By the construction of N, there is $\mathcal{D} \in M$ of size $< \mu$ such that $D \in \mathcal{D}$. By shrinking $\mathcal{D}$ if necessary, we may assume that each $D' \in \mathcal{D}$ is a dense open subset of $\mathbb{P}_{\alpha_{D'}}$ for some $\alpha_{D'} < \gamma$. Then, by Lemma 7.6, the set $D^* = \{p \in \mathbb{P}_\gamma \mid \forall D' \in \mathcal{D}\, (p \restriction \alpha_{D'} \in D')\}$ is dense open in $\mathbb{P}_\gamma$. Moreover $D^* \in M$. So there is $\xi < \zeta$ with $p_\xi \in D^*$. Then $p_\xi \restriction \alpha = p_\xi \restriction \alpha_D \in D$. □

7.2 Proof of Theorem 1.10

In this subsection, we complete our proof of Theorem 1.10. First we describe a certain property we impose on our sequence $\langle S_\delta \mid \delta < \mu \rangle$:

Definition 7.7. *Let $\vec{S} = \langle S_\delta \mid \delta < \mu \rangle$ be a sequence of subsets of $E^\lambda_{<\kappa}$. Then let $Z_{\vec{S}}$ be the set of all $z \in \mathcal{P}_\mu \lambda$ such that*

(i) $z \cap \mu \in \mu$,

(ii) $(\mathrm{Cl}(z) \setminus z) \cap \bigcup_{\delta \in z \cap \mu} S_\delta = \emptyset$.

We say that $\vec{S}$ is nice *if $\bigcup_{\delta < \mu} S_\delta = E^\lambda_{<\kappa}$ and $Z_{\vec{S}}$ is fat in $\mathcal{P}_\mu \lambda$.*

We can easily add a nice $\vec{S} = \langle S_\delta \mid \delta < \mu \rangle$ by forcing:

Lemma 7.8. *Suppose $2^{<\mu} = \mu$ and λ is regular. Then there is a μ-closed poset with the μ^+-c.c. which adds a nice sequence $\langle S_\delta \mid \delta < \mu \rangle$ of subsets of $E^\lambda_{<\kappa}$.*

Proof. Let $\mathbb{P}$ be the set of all partial functions $p : \mu \times E^\lambda_{<\kappa} \to 2$ of size $< \mu$, ordered by reverse inclusions. Then $\mathbb{P}$ is μ-closed and has the μ^+-c.c.

Let $\dot{G}$ be the canonical name for a $\mathbb{P}$-generic filter, and, for each $\delta < \mu$, let $\dot{S}_\delta$ be a $\mathbb{P}$-name for the set $\{\alpha \in E^\lambda_{<\kappa} \mid \bigcup \dot{G}(\delta, \alpha) = 1\}$. Moreover, let $\dot{Z}$ be a $\mathbb{P}$-name for $Z_{\langle \dot{S}_\delta \mid \delta < \mu \rangle}$. By a standard density argument, $\bigcup_{\delta < \mu} \dot{S}_\delta = E^\lambda_{<\kappa}$ in $V^{\mathbb{P}}$.

We will prove that $\dot{Z}$ is fat in $V^{\mathbb{P}}$. Let θ be a sufficiently large regular cardinal and $\dot{\mathcal{M}}$ be a $\mathbb{P}$-name for a countable expansion of $\langle {\mathcal{H}_\theta}^{V^{\mathbb{P}}}, \in \rangle$. Suppose ν is a regular cardinal $< \mu$ and $p \in \mathbb{P}$. We will find $p^* \leq p$ and a sequence $\langle \dot{M}_\xi \mid \xi \leq \nu \rangle$ of $\mathbb{P}$-names such that p^* forces $\langle \dot{M}_\xi \mid \xi \leq \nu \rangle$ to witness fatness of $\dot{Z}$ for $\dot{\mathcal{M}}$ and ν. We work in V.

Take a regular cardinal $\chi > 2^\theta$ and a well-ordering Δ of $\mathcal{H}_\chi$, and let $\mathcal{N} := \langle \mathcal{H}_\chi, \in, \Delta, \mathbb{P}, \dot{\mathcal{M}}, p \rangle$. Then we can take a nice $\in$-chain $\langle N_\xi \mid \xi \leq \nu \rangle$ such that $N_\xi \prec \mathcal{N}$, $|N_\xi| < \mu$ and $N_\xi \cap \mu \in \mu$ for every $\xi \leq \nu$. By induction on $\xi \leq \nu$, we will define a decreasing sequence $\langle p_\xi \mid \xi \leq \nu \rangle$ in $\mathbb{P}$ below p. For each $\zeta < \nu$, $\langle p_\xi \mid \xi \leq \zeta \rangle$ will be definable in $\mathcal{N}$ from $\langle N_\xi \mid \xi \leq \zeta \rangle$. So p_ξ will belong to $N_{\xi+1}$ for all $\xi < \nu$.

Let $p_0 := p$. If p_ξ is defined, then let $p_{\xi+1}$ be the Δ-least condition below p_ξ which belongs to all dense open subsets of $\mathbb{P}$ in $N_{\xi+1}$. Note that there exists such a condition, since $\mathbb{P}$ is μ-closed and $|N_{\xi+1}| < \mu$. Suppose ξ is a limit ordinal and $\langle p_\eta \mid \eta < \xi \rangle$ has been defined. Then $\langle p_\eta \mid \eta < \xi \rangle$ is $(N_\xi, \mathbb{P})$-generic by the construction at successor steps. Let $p'_\xi := \bigcup_{\eta < \xi} p_\eta$. Then $\mathrm{dom}(p'_\xi) = (N_\xi \cap \mu) \times (N_\xi \cap \lambda)$. Let p_ξ be an extension of p'_ξ such that $\mathrm{dom}(p_\xi) = (N_\xi \cap \mu) \times \mathrm{Cl}(N_\xi \cap \lambda)$, and $p_\xi(\delta, \alpha) = 0$ for all $\delta \in N_\xi \cap \mu$ and all $\alpha \in \mathrm{Cl}(N_\xi \cap \lambda) \setminus (N_\xi \cap \lambda)$.

Now we have defined $\langle p_\xi \mid \xi \leq \nu \rangle$. Note that, for every limit $\xi \leq \nu$, p_ξ forces that $N_\xi[\dot{G}] \cap \mu = N_\xi \cap \mu \in \mu$ and $N_\xi[\dot{G}] \cap \lambda = N_\xi \cap \lambda \in \dot{Z}$. Let $p^* := p_\nu$, and, for each $\xi \leq \nu$, let $\dot{M}_\xi$ be a $\mathbb{P}$-name for $N_\xi[\dot{G}] \cap {\mathcal{H}_\theta}^{V^{\mathbb{P}}}$. Then p^* and $\langle \dot{M}_\xi \mid \xi \leq \nu \rangle$ is as desired. □

After obtaining a nice sequence $\vec{S} = \langle S_\delta \mid \delta < \mu \rangle$ of subsets of $E^\lambda_{<\kappa}$, by an iteration of club shootings, we will force each S_δ to bear no μ-skinny stationary sets. Next, we present our poset for club shooting:

Definition 7.9. *For a subset X of $\mathcal{P}_\kappa \lambda$, let $\mathbb{C}(X)$ be the poset of all p such that*

(i) $p : (d_p)^2 \to \mu$ *for some* $d_p \in \mathcal{P}_\mu \lambda$,

(ii) every $x \in X$ with $x \subseteq d_p$ is not closed under p.

$\mathbb{C}(X)$ *is ordered by reverse inclusions.*

First, we present an easy lemma on $\mathbb{C}(X)$. Note that, by (ii) of Definition 7.9 and (2) of the following lemma, $\mathbb{C}(X)$ forces X to be non-stationary.

Lemma 7.10. *Let X be a subset of $\mathcal{P}_\kappa \lambda$.*

(1) Suppose $p, q \in \mathbb{C}(X)$ and $p \restriction (d_p \cap d_q)^2 = q \restriction (d_p \cap d_q)^2$. Then p and q are compatible in $\mathbb{C}(X)$.

(2) For any $d \in \mathcal{P}_\mu\lambda$, the set $\{p \in \mathbb{C}(X) \mid d \subseteq d_p\}$ is dense in $\mathbb{C}(X)$.

Proof. (1) Take $\gamma \in \mu \setminus (d_p \cup d_q)$, and let $r : (d_p \cup d_q)^2 \to \mu$ be such that $p, q \subseteq r$ and $r(\alpha, \beta) = \gamma$ for all $(\alpha, \beta) \in (d_p \cup d_q)^2 \setminus ((d_p)^2 \cup (d_q)^2)$. We claim that $r \le p, q$. For this it suffices to check that $r \in \mathbb{C}(X)$.

Clearly, r satisfies property (i) of conditions of $\mathbb{C}(X)$. To check (ii), suppose $x \in X$ and $x \subseteq d_p \cup d_q$. If $x \subseteq d_p$ or $x \subseteq d_q$, then x is not closed under r, since $p, q \subseteq r$ and $p, q \in \mathbb{C}(X)$. So suppose $x \not\subseteq d_p$ and $x \not\subseteq d_q$. Take $\alpha \in x \setminus d_p$ and $\beta \in x \setminus d_q$. Then $r(\alpha, \beta) = \gamma \notin d_p \cup d_q \supseteq x$. Thus x is not closed under r.

(2) Suppose $p \in \mathbb{C}(X)$ and $d \in \mathcal{P}_\mu\lambda$. We will find $q \le p$ with $d \subseteq d_q$. Take $\gamma \in \mu \setminus (d_p \cup d)$, and let $q : (d_p \cup d)^2 \to \mu$ be such that $p \subseteq q$ and $q(\alpha, \beta) = \gamma$ for every $(\alpha, \beta) \in (d_p \cup d)^2 \setminus (d_p)^2$. Then $q \in \mathbb{C}(X)$ by the same argument as in the proof of (1). Moreover, clearly $q \le p$ and $d \subseteq d_q$. □

Lemma 7.11. *Suppose $\vec{S} = \langle S_\delta \mid \delta < \mu \rangle$ is a sequence of subsets of $E^\lambda_{<\kappa}$. Let $\delta < \mu$, and let X be a μ-skinny subset of $\mathcal{P}_\kappa\lambda$ such that $E_X \subseteq S_\delta$ and $\sup(x) \notin x$ for all $x \in X$. Then $\mathbb{C}(X)$ is $Z_{\vec{S}}$-closed.*

Proof. Let θ be a sufficiently large regular cardinal, Δ be a well-ordering of $\mathcal{H}_\theta$ and $\mathcal{M}$ be the structure $\langle \mathcal{H}_\theta, \in, \Delta, X, \vec{S}, \delta \rangle$. Suppose $M \prec \mathcal{M}$, $|M| < \mu$ and $\bigcup(\mathcal{P}_\mu\lambda \cap M) \in Z_{\vec{S}}$. Suppose also that $\vec{p} = \langle p_\xi \mid \xi < \zeta \rangle$ is an $(M, \mathbb{C}(X))$-generic sequence. We show that $p^* := \bigcup_{\xi<\zeta} p_\xi$ is a lower bound of p^* in $\mathbb{C}(X)$. For this, it suffices to prove that $p^* \in \mathbb{C}(X)$.

Clearly, d_{p^*} satisfies property (i) of conditions of $\mathbb{C}(X)$. To check (ii), take an arbitrary $x \in X$ with $x \subseteq d_{p^*}$. We show that x is not closed under d_{p^*}. Let $z := \bigcup(\mathcal{P}_\mu\lambda \cap M) \in Z_{\vec{S}}$. Then it easily follows from Lemma 7.10 (2) that $d_{p^*} = z$. So $\sup(x) \in \mathrm{Cl}(z) \cap S_\delta$. But $(\mathrm{Cl}(z) \setminus z) \cap S_\delta = \emptyset$ since $\delta \in M \cap \mu \subseteq z \cap \mu$. Hence $\sup(x) \in z$. Take $y_0 \in \mathcal{P}_\mu\lambda \cap M$ with $\sup(x) \in y_0$. Then we have $y_1 := \bigcup\{x' \in X \mid \sup(x') \in y_0\} \in \mathcal{P}_\mu\lambda \cap M$ by the μ-skinniness of X. Again, by Lemma 7.10 (2), there is $\xi < \zeta$ with $d_{p_\xi} \supseteq y_1$. Then x is not closed under p_ξ since $x \subseteq y_1$ and $p_\xi \in \mathbb{C}(X)$. But $p_\xi \subseteq p^*$. So x is not closed under p^*, too. □

Note that $Z_{\vec{S}}$ in a μ-distributive forcing extension is the same as in the ground model. Then, by Proposition 7.4 and Lemmata 7.3 and 7.11, if $\vec{S}$ is a nice sequence of subsets of $E^\lambda_{<\kappa}$, then a $<\mu$-support iteration of $\mathbb{C}(X)$'s as in Lemma 7.11 is $Z_{\vec{S}}$-closed. Next, we show that such an iteration has the μ^+-c.c. if $2^{<\mu} = \mu$.

Lemma 7.12. *Suppose $2^{<\mu} = \mu$ and $\vec{S} = \langle S_\delta \mid \delta < \mu\rangle$ is a nice sequence of subsets of $E^\lambda_{<\kappa}$. Let γ be an ordinal and $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \mathbb{C}(\dot{X}_\beta) \mid \alpha \le \gamma, \beta < \gamma\rangle$ be a $<\mu$-support iteration, where each $\dot{X}_\beta$ is a $\mathbb{P}_\beta$-name for a μ-skinny subset of $\mathcal{P}_\kappa\lambda$ such that $E_{\dot{X}_\beta} \subseteq S_\delta$ for some $\delta < \mu$ and $\sup(x) \notin x$ for all $x \in \dot{X}_\beta$. Then $\mathbb{P}_\gamma$ has the μ^+-c.c.*

Proof. Let D be the set of all $p \in \mathbb{P}_\gamma$ such that, for every $\beta \in \mathrm{dom}(p)$, $p(\beta) = \check{q}$ for some function $q \in V$.

First, we prove that D is dense in $\mathbb{P}_\gamma$. Take an arbitrary $p \in \mathbb{P}_\gamma$. We will find $p^* \le p$ which is in D. Let θ be a sufficiently large regular cardinal, and let $\mathcal{M} := \langle \mathcal{H}_\theta, \in, \kappa, \mu, \lambda, \vec{S}, \vec{\mathbb{P}}, p\rangle$. Since $Z_{\vec{S}}$ is fat, we can take a nice $\in$-chain $\langle M_n \mid n \le \omega\rangle$ of elementary submodels of $\mathcal{M}$ of size $<\mu$ such that $M_\omega \cap \mu \in \mu$ and $M_\omega \cap \lambda \in Z_{\vec{S}}$. Using the μ-distributivity of $\mathbb{P}_\gamma$, we can easily construct an $(M_\omega, \mathbb{P}_\gamma)$-generic sequence $\langle p_n \mid n < \omega\rangle$ below p. Note that $\mathrm{dom}(p_n) \subseteq M_\omega \cap \gamma$ for every $n < \omega$.

p^* will be a lower bound of $\langle p_n \mid n < \omega\rangle$ whose domain is $M_\omega \cap \gamma$. By induction on $\beta \in M_\omega \cap \gamma$, we will define $p^*(\beta)$ so that $p^* \restriction \beta + 1 \le p_n \restriction \beta + 1$ for every $n < \omega$. Suppose $\beta \in M_\omega \cap \gamma$ and $p^*(\beta')$ has been defined for all $\beta' \in M_\omega \cap \beta$. By the genericity, $p^* \restriction \beta$ forces that $M_\omega[\dot{G}_\beta] \cap \mu = M_\omega \cap \mu \in \mu$ and $M_\omega[\dot{G}_\beta] \cap \lambda = M_\omega \cap \lambda \in Z_{\vec{S}}$, where $\dot{G}_\beta$ is the canonical name for a $\mathbb{P}_\beta$-generic filter. p^* also forces that $\langle p_n(\beta) \mid n < \omega\rangle$ is $(M_\omega[\dot{G}_\beta], \mathbb{C}(\dot{X}_\beta))$-generic. Then, by the $Z_{\vec{S}}$-closedness of $\mathbb{C}(\dot{X}_\beta)$, p^* forces that $\bigcup_{n<\omega} p_n(\beta)$ is a lower bound of $\langle p_n(\beta) \mid n < \omega\rangle$ in $\mathbb{C}(\dot{X}_\beta)$. Moreover, by the genericity of $p^* \restriction \beta$ and the μ-distributivity of $\mathbb{P}_\beta$, for each $n < \omega$, there is a function $q_n \in V$ such that $p^* \restriction \beta \Vdash_\beta$ "$p_n(\beta) = \check{q_n}$". Let $q := \bigcup_{n<\omega} q_n$ and $p^*(\beta) := \check{q}$. Then $p^* \restriction \beta + 1 \in \mathbb{P}_{\beta+1}$ and $p^* \restriction \beta + 1 \le p_n \restriction \beta + 1$ for every $n < \omega$. Now we have defined $p^* \in \mathbb{P}_\gamma$. Clearly $p^* \le p$ and $p^* \in D$.

We have proved that D is dense in $\mathbb{P}_\gamma$. For each $p \in D$ and $\beta \in \mathrm{dom}(p)$, if $p(\beta) = \check{q}$, then we let $p(\beta)$ denote q. It suffices to show that if $A \subseteq D$ and $|A| \ge \mu^+$, then there are distinct $p, p' \in A$ which are compatible in $\mathbb{P}_\gamma$. Suppose $A \subseteq D$ and $|A| \ge \mu^+$. By the standard argument using the Δ-system lemma, there are distinct $p, p' \in A$ such that $p(\beta) \restriction (d_{p(\beta)} \cap d_{p'(\beta)})^2 = p'(\beta) \restriction (d_{p(\beta)} \cap d_{p'(\beta)})^2$ for every $\beta \in \mathrm{dom}(p) \cap \mathrm{dom}(p')$. Then, using Lemma 7.10 (1), it is easy to see that p and p' are compatible in $\mathbb{P}_\gamma$. □

Now we can easily prove Theorem 1.10:

Proof of Theorem 1.10. By Lemma 7.8, we may assume that there is a nice sequence $\vec{S} = \langle S_\delta \mid \delta < \mu \rangle$ of subsets of $E^\lambda_{<\kappa}$ in the ground model. Then, by Lemma 7.12, using the bookkeeping method, we can construct a $< \mu$-support iteration $\langle \mathbb{P}_\alpha, \mathbb{C}(\dot{X}_\beta) \mid \alpha \leq 2^\lambda, \beta < 2^\lambda \rangle$ with the following properties:

(i) Each $\dot{X}_\beta$ is a $\mathbb{P}_\beta$-name for a μ-skinny subset of $\mathcal{P}_\kappa\lambda$ such that $E_{\dot{X}_\beta} \subseteq S_\delta$ for some $\delta < \mu$ and $\sup(x) \notin x$ for all $x \in \dot{X}_\beta$.

(ii) If $\dot{X}$ is a $\mathbb{P}_{2^\lambda}$-name for a μ-skinny subset of $\mathcal{P}_\kappa\lambda$ such that $E_{\dot{X}} \subseteq S_\delta$ for some $\delta < \mu$ and $\sup(x) \notin x$ for all $x \in \dot{X}_\beta$, then there is $\beta < 2^\lambda$ such that $\Vdash_{2^\lambda}$ "$\dot{X}_\beta = \dot{X}$".

Let $\mathbb{P} := \mathbb{P}_{2^\lambda}$. Then $\mathbb{P}$ preserves all cofinalities, since $\mathbb{P}$ is μ-distributive and has the μ^+-c.c. by Proposition 7.4 and Lemmata 7.3, 7.11, 7.12. Moreover, in $V^{\mathbb{P}}$, S_δ bears no μ-skinny stationary sets for any $\delta < \mu$ by the construction of the iteration. □

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