On Katětov and Katětov–Blass orders on analytic P-ideals and Borel ideals

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Abstract

Minami–Sakai [10] investigated the cofinal types of the Katětov and the Katětov–Blass orders on the family of all F_σ ideals. In this paper we discuss these orders on analytic P-ideals and Borel ideals. We prove the following:

- The family of all analytic P-ideals has the largest element with respect to the Katětov and the Katětov–Blass orders.
- The family of all Borel ideals is countably upward directed with respect to the Katětov and the Katětov–Blass orders.

In the course of the proof of the latter result, we also prove that for any analytic ideal $\mathcal{I}$ there is a Borel ideal $\mathcal{J}$ with $\mathcal{I} \subseteq \mathcal{J}$.

1 Introduction

In this paper we continue the study of the Katětov and the Katětov–Blass orders on Borel ideals from Minami–Sakai [10]. To state a background and our results, first we present our notation on ideals and recall the Katětov and the Katětov–Blass orders.

Throughout this paper an *ideal* over a set X means a proper ideal over X including $[X]^{<\omega}$. We are interested in ideals over countable infinite sets. If X is countable infinite, then $\mathcal{P}(X)$ can be naturally seen as a topological space, which is homeomorphic to the Cantor space. An ideal $\mathcal{I}$ over a countable infinite set X is said to be *Borel*, *analytic* or F_σ if it is Borel, analytic or F_σ as a subset of $\mathcal{P}(X)$, respectively. Let **Bo**, **An** and **Fs** denote the families of all Borel, analytic and F_σ ideals over ω, respectively. Moreover let **AP** denote the family of all analytic P-ideals over ω, where an ideal $\mathcal{I}$ is called a *P-ideal* if for any $\{A_n \mid n < \omega\} \subseteq \mathcal{I}$ there is $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for every $n < \omega$. Recall the fact, due to Solecki [11], that every analytic P-ideal is $F_{\sigma\delta}$.

The Katětov order $\leq_{\mathrm{K}}$ and the Katětov–Blass order $\leq_{\mathrm{KB}}$ are preorders on ideals over countable infinite sets, which refine the inclusion relation $\subseteq$. They are defined as follows: For an ideal $\mathcal{I}$ over X and an ideal $\mathcal{J}$ over Y,

$$\mathcal{I} \leq_{\mathrm{K}} \mathcal{J} \overset{\text{def}}{\Leftrightarrow} \text{there is a function } \tau : Y \to X \text{ such that } \tau^{-1}[A] \in \mathcal{J} \text{ for any } A \in \mathcal{I},$$

$\mathcal{I} \leq_{\mathrm{KB}} \mathcal{J} \overset{\text{def}}{\Leftrightarrow}$ there is a finite to one function $\tau : Y \to X$ such that $\tau^{-1}[A] \in \mathcal{J}$ for any $A \in \mathcal{I}$.

Note that if $X = Y$, and $\mathcal{I} \subseteq \mathcal{J}$, then the identity function witnesses that $\mathcal{I} \leq_{\mathrm{KB}} \mathcal{J}$. Note also that if $\mathcal{I} \leq_{\mathrm{KB}} \mathcal{J}$, then $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$.

The Katětov order was introduced by Katětov [5] to discuss limit constructions of continuous functions with Borel ideals. Recently it has turned out that many properties of ideals can be characterized in terms of the Katětov and the Katětov–Blass orders. More precisely, for many properties of ideals we can find some Borel ideals which are critical with respect to $\leq_{\mathrm{K}}$ or $\leq_{\mathrm{KB}}$. For example, it is not hard to see that an ultrafilter $\mathcal{U}$ over ω is a P-point if and only if $\mathsf{fin} \times \mathsf{fin} \not\leq_{\mathrm{K}} \mathcal{U}^*$, where fin is the ideal consisting all finite subsets of ω. It is also known, due to Hrušák–Meza–Minami [3], that an ultrafilter $\mathcal{U}$ is a Q-point if and only if $\mathcal{ED}_{\mathrm{fin}} \not\leq_{\mathrm{KB}} \mathcal{U}^*$, where $\mathcal{ED}_{\mathrm{fin}}$ is the eventually different ideal restricted to the set $\{(m, n) \mid n \leq m\}$. See Hrušák [2] for details and other examples.

Motivated by these facts, some structural analyses have been made for the Katětov and the Katětov–Blass orders on Borel ideals. Meza [9] proved that $\mathcal{P}(\omega)/\mathsf{fin}$ is order-embeddable into $(\mathbf{Fs}, \leq_{\mathrm{K}})$. Minami–Sakai [10] proved that both $(\mathbf{Fs}, \leq_{\mathrm{K}})$ and $(\mathbf{Fs}, \leq_{\mathrm{KB}})$ are upward directed and that their cofinal types are the same as $(\omega^\omega, \leq^*)$. In this paper we continue [10] to investigate the structures of $\mathbf{AP}$ and $\mathbf{Bo}$ with respect to $\leq_{\mathrm{K}}$ and $\leq_{\mathrm{KB}}$.

In [10] it was proved that $(\mathbf{AP}, \leq_{\mathrm{KB}})$ is countably upward directed, that is, every countable subset of $\mathbf{AP}$ has an upper bound in $(\mathbf{AP}, \leq_{\mathrm{KB}})$. (Hence so is $(\mathbf{AP}, \leq_{\mathrm{K}})$.) But the case of the cofinal types of $(\mathbf{AP}, \leq_{\mathrm{K}})$ and $(\mathbf{AP}, \leq_{\mathrm{KB}})$ were left open. In this paper we prove the following:

Theorem 1.1. $(\mathbf{AP}, \leq_{\mathrm{KB}})$ *has the largest element.*

It is easy to see that for any $\mathcal{I} \in \mathbf{Fs}$ there is $\mathcal{J} \in \mathbf{AP}$ with $\mathcal{I} \subseteq \mathcal{J}$ (See Lemma 2.6). So we have the following corollary:

Corollary 1.2. *There is* $\mathcal{J} \in \mathbf{AP}$ *such that* $\mathcal{I} \leq_{\mathrm{KB}} \mathcal{J}$ *for all* $\mathcal{I} \in \mathbf{Fs}$.

Here note that Theorem 1.1 and Corollary 1.2 also hold for $\leq_{\mathrm{K}}$ because $\leq_{\mathrm{K}} \supseteq \leq_{\mathrm{KB}}$.

Next we turn our attention to $\mathbf{Bo}$. It is easy to see that $(\mathbf{Bo}, \leq_{\mathrm{K}})$ is upward directed. If $\mathcal{I}_0, \mathcal{I}_1 \in \mathbf{Bo}$, then the Fubini product $\mathcal{I}_0 \times \mathcal{I}_1$ is a Borel ideal over $\omega \times \omega$, and the projections from $\omega \times \omega$ to ω witness that $\mathcal{I}_0, \mathcal{I}_1 \leq_{\mathrm{K}} \mathcal{I}_0 \times \mathcal{I}_1$. Here note that the projections are not finite to one, and this argument does not prove the upward directedness of $(\mathbf{Bo}, \leq_{\mathrm{KB}})$. The upward directedness of $(\mathbf{Bo}, \leq_{\mathrm{KB}})$ was asked in [10]. In this paper we give a positive answer:

Theorem 1.3. $(\mathbf{Bo}, \leq_{\mathrm{KB}})$ *is countably upward directed.*

In the course of the proof of this theorem, we also prove that for any analytic ideal $\mathcal{I}$ there is a Borel ideal $\mathcal{J}$ with $\mathcal{I} \subseteq \mathcal{J}$. See Theorem 3.1.

We do not know the cofinal types of $(\mathbf{Bo}, \leq_{\mathrm{K}})$ and $(\mathbf{Bo}, \leq_{\mathrm{KB}})$. But it follows from the results in Katětov [6] that $(\mathbf{Bo}, \leq_{\mathrm{K}})$ does not have the largest element.

(Hence neither does $(\mathbf{Bo}, \leq_{\mathrm{KB}})$.) We will also give a proof of this fact which is slightly simpler than the arguments in [6].

The construction of this paper is as follows: In §2 we will prove Theorem 1.1 and Corollary 1.2. In §3 we will prove Theorem 1.3 and give a proof of the fact that $(\mathbf{Bo}, \leq_{\mathrm{K}})$ does not have the largest element. Finally in §4 we will present several questions.

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2 $\leq_{\mathrm{KB}}$-largest analytic P-ideal

In this section we prove Theorem 1.1 and Corollary 1.2. For this we use the characterizations of F_σ-ideals and analytic P-ideals using submeasures, which were given by Mazur [8] and Solecki [11]. First we recall these characterizations.

A *submeasure* on a set X is a function $\varphi : [X]^{<\omega} \to \mathbb{R}_{\geq 0}$ with the following properties, where $\mathbb{R}_{\geq 0}$ denotes the set of all non-negative real numbers:

(i) $\varphi(A) \leq \varphi(B)$ if $A \subseteq B$. (Monotonicity)

(ii) $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$. (Subadditivity)

(iii) $\varphi(\emptyset) = 0$.

If φ is a submeasure on X, then X is denoted by $d(\varphi)$. Let φ be a submeasure on ω. For $A \subseteq \omega$ let

$$\hat{\varphi}(A) := \lim_{n\to\omega} \varphi(A \cap n) \in \mathbb{R}_{\geq 0} \cup \{\infty\} ,$$

If $\hat{\varphi}(\omega) = \infty$, then

$$\mathsf{Fin}(\varphi) := \{A \subseteq \omega \mid \hat{\varphi}(A) < \infty\}$$

is an ideal over ω. On the other hand, if $\lim_{n\to\omega} \hat{\varphi}(\omega \setminus n) > 0$, then

$$\mathsf{Exh}(\varphi) := \{A \subseteq \omega \mid \lim_{n\to\omega} \hat{\varphi}(A \setminus n) = 0\}$$

is an ideal over ω. $\mathsf{Exh}(\varphi)$ is called *the exhaustive ideal* of φ. The following are the characterizations of F_σ-ideals and analytic P-ideals mentioned above:

Fact 2.1 (Mazur [8])**.** *$\mathcal{I} \in \mathbf{Fs}$ if and only if $\mathcal{I} = \mathsf{Fin}(\varphi)$ for some submeasure φ on ω such that $\hat{\varphi}(\omega) = \infty$.*

Fact 2.2 (Solecki [11])**.** *$\mathcal{I} \in \mathbf{AP}$ if and only if $\mathcal{I} = \mathsf{Exh}(\varphi)$ for some submeasure φ on ω such that $\lim_{n\to\omega} \hat{\varphi}(\omega \setminus n) > 0$.*

Here we make a remark on Fact 2.2. For $\mathcal{I} \in \mathbf{AP}$ Solecki [11] indeed constructed a submeasure φ as in Fact 2.2 with the property that $\mathrm{range}(\varphi) \subseteq \mathbb{Q}$. See the proof of Fact 2.2 in [11].

Now we proceed to the proof of Theorem 1.1.

Theorem 1.1. $(\mathbf{AP}, \leq_{\mathrm{KB}})$ *has the largest element.*

Our proof of Theorem 1.1 uses similar arguments as Hrušák–Meza [4], which proved the existence of a universal analytic P-ideal. Here a universal analytic P-ideal is $\mathcal{J} \in \mathbf{AP}$ such that for any $\mathcal{I} \in \mathbf{AP}$ there is $A \subseteq \omega$ with $\mathcal{J} \restriction A \cong \mathcal{I}$. To construct such an ideal, they constructed a universal $\mathbb{Q}$-valued submeasure on ω as a Fraïssé limit of $\mathbb{Q}$-valued submeasures on finite sets. They also constructed a universal $\mathbb{Z}$-valued submeasure in the same way to prove the existence of a universal F_σ-ideal. For Theorem 1.1 we will construct a submeasure on ω by combining $\mathbb{Q}$-valued submeasures on finite sets in a different way from [4].

To prove Theorem 1.1 we need some preliminaries. First we slightly modify Fact 2.2. We say that a submeasure φ on ω is *nice* if

(i) $\mathrm{range}(\varphi) \subseteq \mathbb{Q}$,

(ii) there is an increasing sequence $\langle k_n \mid n < \omega \rangle$ in ω such that $k_0 = 0$, $k_1 = 1$, and $\varphi(k_{n+1} \setminus k_n) = 1$ for all n.

We will use the following:

Lemma 2.3. *For any $\mathcal{I} \in \mathbf{AP}$ there is a nice submeasure φ on ω such that $\mathcal{I} = \mathsf{Exh}(\varphi)$.*

Proof. Suppose that $\mathcal{I} \in \mathbf{AP}$. By the proof in [11] of Fact 2.2 there is a submeasure ψ on ω such that $\mathcal{I} = \mathsf{Exh}(\psi)$, $\alpha := \lim_{n\to\omega} \hat{\psi}(\omega \setminus n) > 0$, and $\mathrm{range}(\psi) \subseteq \mathbb{Q}$. We may assume that $\psi(\{0\}) > 0$ by modifying ψ if necessary. (Since $\hat{\psi}(\omega) \geq \alpha > 0$, there is $k < \omega$ with $\psi(\{k\}) > 0$. Let $\sigma : \omega \to \omega$ be the transposition of 0 and k. Replace ψ with the submeasure ψ' defined by $\psi'(A) := \psi(\sigma[A])$.) We may also assume that $\alpha > 1$ and $\psi(\{0\}) \geq 1$. (Replace ψ with $m \cdot \psi$ for some natural number m if necessary.)

Define a function $\varphi : [\omega]^{<\omega} \to \mathbb{Q}$ by $\varphi(A) := \min\{\psi(A), 1\}$. It is easy to check that φ is a submeasure on ω. Moreover, $\varphi(\{0\}) = 1$, and for each $k < \omega$ there is $l > k$ with $\varphi(l \setminus k) = 1$ because $\alpha > 1$. From this it easily follows that φ satisfies the property (ii) of nice submeasures. So φ is nice. Finally, $\mathsf{Exh}(\varphi) = \mathsf{Exh}(\psi) = \mathcal{I}$ because $\hat{\varphi}(A) = \min\{\hat{\psi}(A), 1\}$ for each $A \subseteq \omega$. Thus φ is as desired. □

Next we give a sufficient condition for that $\mathsf{Exh}(\varphi) \leq_{\mathrm{KB}} \mathsf{Exh}(\psi)$:

Lemma 2.4. *Let φ and ψ be submeasures on ω such that $\lim_{n\to\omega} \hat{\varphi}(\omega \setminus n) > 0$ and $\lim_{n\to\omega} \hat{\psi}(\omega \setminus n) > 0$. Suppose that there is a finite to one $\tau : \omega \to \omega$ with $\psi(\tau^{-1}[A]) \leq \varphi(A)$ for all $A \in [\omega]^{<\omega}$. Then $\mathsf{Exh}(\varphi) \leq_{\mathrm{KB}} \mathsf{Exh}(\psi)$.*

Proof. We show that τ witnesses that $\mathsf{Exh}(\varphi) \leq_{\mathrm{KB}} \mathsf{Exh}(\psi)$. Suppose that $A \in \mathsf{Exh}(\varphi)$. We show that $B := \tau^{-1}[A] \in \mathsf{Exh}(\psi)$, that is, $\lim_{n\to\omega} \hat{\psi}(B \setminus n) = 0$. Because τ is finite to one, for each $n < \omega$ there are $m, \bar{n} < \omega$ such that $B \setminus n \supseteq \tau^{-1}[A \setminus m] \supseteq B \setminus \bar{n}$. Then, by the monotonicity of ψ,

$$\lim_{n\to\omega} \hat{\psi}(B \setminus n) \;=\; \lim_{m\to\omega} \hat{\psi}(\tau^{-1}[A \setminus m]) \;=:\; \alpha\,.$$

Moreover $\hat{\psi}(\tau^{-1}[C]) \leq \hat{\varphi}(C)$ for all $C \subseteq \omega$ by the assumption of the lemma. Hence $\alpha \leq \lim_{m\to\omega} \hat{\varphi}(A \setminus m) = 0$. So $\lim_{n\to\omega} \hat{\psi}(B \setminus n) = 0$. □

To prove Theorem 1.1 it suffices to construct a nice submeasure ψ on ω satisfying the properties of Lemma 2.4 for every nice submeasure φ on ω. As we mentioned before, we will construct such a submeasure ψ by combining submeasures on finite sets. The last preliminary for Theorem 1.1 is an investigation of submeasures on finite sets, which is similar as the one in [10].

Let Φ be the set of all submeasures φ such that

(i) $d(\varphi)$ is a non-empty finite subset of ω,

(ii) $\mathrm{range}(\varphi) \subseteq \mathbb{Q}$,

(iii) $\varphi(d(\varphi)) = 1$.

Moreover define a preorder $\trianglelefteq$ on Φ as follows: For $\varphi, \psi \in \Phi$ let

$$\varphi \trianglelefteq \psi \overset{\text{def}}{\Leftrightarrow} \text{there is } \tau : d(\psi) \to d(\varphi) \text{ such that } \psi(\tau^{-1}[A]) \leq \varphi(A) \text{ for all } A \subseteq d(\varphi).$$

Note that a submeasure φ on $\{0\}$ defined by $\varphi(\{0\}) = 1$ and $\varphi(\emptyset) = 0$ is the smallest element of $(\Phi, \trianglelefteq)$. We will use the following:

Lemma 2.5. $(\Phi, \trianglelefteq)$ *is upward directed.*

Proof. Take $\varphi_0, \varphi_1 \in \Phi$ arbitrarily, and let $X_i := d(\varphi_i)$ for $i = 0, 1$. It suffices to find a submeasure ψ on some non-empty finite set Y and functions $\tau_i : Y \to X_i$ for $i = 0, 1$ such that $\mathrm{range}(\psi) \subseteq \mathbb{Q}$, $\psi(Y) = 1$, and $\psi(\tau_i^{-1}[A]) \leq \varphi_i(A)$ for each $i = 0, 1$ and each $A \subseteq X_i$.

Let $Y := X_0 \times X_1$, and for each $i = 0, 1$ let $\tau_i : Y \to X_i$ be the i-th projection, that is, $\tau_i(k_0, k_1) = k_i$. Define a function ψ on $\mathcal{P}(Y)$ by

$$\psi(B) := \min\{\varphi_0(A_0) + \varphi_1(A_1) \mid A_0 \subseteq X_0 \wedge A_1 \subseteq X_1 \wedge B \subseteq \tau_0^{-1}[A_0] \cup \tau_1^{-1}[A_1]\}$$

Then it is easy to check that ψ is a submeasure on Y, $\mathrm{range}(\psi) \subseteq \mathbb{Q}$, and $\psi(\tau_i^{-1}[A]) \leq \varphi_i(A)$ for each $i = 0, 1$ and each $A \subseteq X_i$. Hence it suffices to show that $\psi(Y) = 1$.

Note that $\psi(Y) \leq \varphi_0(X_0) + \varphi_1(\emptyset) = 1$ because $Y \subseteq \tau_0^{-1}[X_0] \cup \tau_1^{-1}[\emptyset]$. On the other hand, note that if $A_i \subseteq X_i$ for $i = 0, 1$, and $Y \subseteq \tau_0^{-1}[A_0] \cup \tau_1^{-1}[A_1]$, then either $A_0 = X_0$, or $A_1 = X_1$. This is because if $A_0 \neq X_0$, and $A_1 \neq X_1$, then, taking $k_i \in X_i \setminus A_i$, we have that $(k_0, k_1) \in Y \setminus \tau_0^{-1}[A_0] \cup \tau_1^{-1}[A_1]$. Then, because $\varphi_i(X_i) = 1$, it follows that $\psi(Y) \geq 1$. Thus $\psi(Y) = 1$. □

Now we prove Theorem 1.1:

Proof of Theorem 1.1. First we construct $\mathcal{J} \in \mathbf{AP}$, which will turn out to be the $\leq_{\mathrm{KB}}$-largest. Since Φ is countable and upward directed, we can take an increasing cofinal sequence $\langle \psi_n \mid n < \omega \rangle$ in $(\Phi, \trianglelefteq)$. We may assume that $d(\psi_n) =$

$l_{n+1} \setminus l_n =: Y_n$ for some increasing sequence $\langle l_n \mid n < \omega \rangle$ in ω with $l_0 = 0$. Define a function ψ on $[\omega]^{<\omega}$ by

$$\psi(B) := \max\{\psi_n(B \cap Y_n) \mid n < \omega\} .$$

Then it is easy to see that ψ is a submeasure on ω and that $\lim_{n\to\omega} \hat{\psi}(\omega \setminus n) = 1 > 0$. Let $\mathcal{J} := \mathsf{Exh}(\psi) \in \mathbf{AP}$. Below we prove that $\mathcal{J}$ is the largest element in $(\mathbf{AP}, \leq_{\mathrm{KB}})$.

Take an arbitrary $\mathcal{I} \in \mathbf{AP}$. We show that $\mathcal{I} \leq_{\mathrm{KB}} \mathcal{J}$. By Lemma 2.3 take a nice submeasure φ on ω with $\mathcal{I} = \mathsf{Exh}(\varphi)$. By Lemma 2.4 it suffices to find a finite to one $\tau : \omega \to \omega$ such that $\psi(\tau^{-1}[A]) \leq \varphi(A)$ for all $A \in [\omega]^{<\omega}$.

Because φ is nice, there is an increasing sequence $\langle k_m \mid m < \omega \rangle$ in ω such that $k_0 = 0$, $k_1 = 1$, and $\varphi(k_{m+1} \setminus k_m) = 1$ for all $m < \omega$. Let $X_m := k_{m+1} \setminus k_m$ and $\varphi_m := \varphi \restriction \mathcal{P}(X_m)$ for each $m < \omega$. Note that $\varphi_m \in \Phi$ for all $m < \omega$. Note also that φ_0 is the smallest in $(\Phi, \trianglelefteq)$.

For each $n < \omega$ let m_n be the largest $m \leq n$ such that $\varphi_m \trianglelefteq \psi_n$. Here note that the mapping $n \mapsto m_n$ is finite to one because $\langle \psi_n \mid n < \omega \rangle$ is an increasing cofinal sequence in $(\Phi, \trianglelefteq)$. For each $n < \omega$ take $\tau_n : Y_n \to X_{m_n}$ witnessing that $\varphi_{m_n} \trianglelefteq \psi_n$, and let $\tau := \bigcup_{n<\omega} \tau_n$. Then τ is a finite to one function from ω to ω. Moreover for each $A \in [\omega]^{<\omega}$,

$$\begin{aligned}
\psi(\tau^{-1}[A]) &= \max\{\psi_n(\tau^{-1}[A] \cap Y_n) \mid n < \omega\} \\
&= \max\{\psi_n(\tau_n^{-1}[A \cap X_{m_n}]) \mid n < \omega\} \\
&\leq \max\{\varphi_{m_n}(A \cap X_{m_n}) \mid n < \omega\} \\
&\leq \varphi(A) .
\end{aligned}$$

Thus τ is as desired. $\square$

Next we prove Corollary 1.2:

Corollary 1.2. *There is $\mathcal{J} \in \mathbf{AP}$ such that $\mathcal{I} \leq_{\mathrm{KB}} \mathcal{J}$ for all $\mathcal{I} \in \mathbf{Fs}$.*

This follows from Theorem 1.1 and the lemma below:

Lemma 2.6. *For any $\mathcal{I} \in \mathbf{Fs}$ there is $\mathcal{J} \in \mathbf{AP}$ with $\mathcal{I} \subseteq \mathcal{J}$.*

Proof. Take an arbitrary $\mathcal{I} \in \mathbf{Fs}$. By Fact 2.1 let φ be a submeasure on ω with $\hat{\varphi}(\omega) = \infty$ and $\mathsf{Fin}(\varphi) = \mathcal{I}$. Then by induction on $n < \omega$ we can easily take $k_n \in \omega$ so that $k_0 = 0$, and $\varphi(k_{n+1} \setminus k_n) \geq n + 1$ for each $n < \omega$. Let $X_n := k_{n+1} \setminus k_n$. For each $n < \omega$ let ψ_n be a submeasure on X_n defined by

$$\psi_n(A) := \frac{1}{\varphi(X_n)} \cdot \varphi(A) .$$

for each $A \subseteq X_n$. Then define a submeasure ψ on ω by

$$\psi(A) := \max\{\psi_n(A \cap X_n) \mid n < \omega\}$$

for each $A \in [\omega]^{<\omega}$. Note that $\lim_{n\to\omega} \hat{\psi}(\omega \setminus n) = 1 > 0$. So $\mathcal{J} := \mathsf{Exh}(\psi) \in \mathbf{AP}$.

We check that $\mathcal{I} \subseteq \mathcal{J}$. Take an arbitrary $A \in \mathcal{I}$. Then $\alpha := \hat{\varphi}(A) < \infty$. Note that $\varphi(A \cap X_n) \leq \alpha$ for each $n < \omega$. So $\psi_n(A \cap X_n) \leq \frac{\alpha}{n+1}$ for each $n < \omega$. Thus $\lim_{n\to\omega} \hat{\psi}(A \setminus n) = 0$, that is, $A \in \mathsf{Exh}(\psi) = \mathcal{J}$. $\square$

3 Directedness of $(\mathbf{Bo}, \leq_{\mathrm{KB}})$

In this section we prove Theorem 1.3. After that, we will also give a proof of the fact, due to Katětov [6], that $(\mathbf{Bo}, \leq_{\mathrm{K}})$ does not have the largest element.

Now let us start to prove Theorem 1.3:

Theorem 1.3. $(\mathbf{Bo}, \leq_{\mathrm{KB}})$ *is countably upward directed.*

This follows from the following:

Theorem 3.1. *For any ideal $\mathcal{I} \in \mathbf{An}$ there is $\mathcal{J} \in \mathbf{Bo}$ with $\mathcal{I} \subseteq \mathcal{J}$. In particular,* $\mathbf{Bo}$ *is cofinal in* $(\mathbf{An}, \leq_{\mathrm{KB}})$.

Proposition 3.2. $(\mathbf{An}, \leq_{\mathrm{KB}})$ *is countably upward directed.*

Note that Theorem 3.1 is a variant of Luzin's separation theorem, which states that every disjoint two analytic sets are separated by a Borel set. In fact Theorem 3.1 can be proved using Luzin's separation theorem:

Proof of Theorem 3.1. Take an arbitrary $\mathcal{I} \in \mathbf{An}$. We construct a Borel $\mathcal{J} \supseteq \mathcal{I}$ step-by-step. First we claim the following:

Claim. *For any Borel $\mathcal{P} \subseteq \mathcal{P}(\omega)$ with $\mathcal{I} \subseteq \mathcal{P}$ there is a Borel $\mathcal{Q} \subseteq \mathcal{P}(\omega)$ such that*

(i) $\mathcal{I} \subseteq \mathcal{Q} \subseteq \mathcal{P}$,

(ii) if $A \in \mathcal{Q}$, and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{Q}$.

Proof of Claim. Suppose that $\mathcal{P}$ is a Borel subset of $\mathcal{P}(\omega)$ with $\mathcal{I} \subseteq \mathcal{P}$. By induction on $n < \omega$ we will take a Borel $\mathcal{Q}_n$ with $\mathcal{I} \subseteq \mathcal{Q}_n \subseteq \mathcal{P}$. The desired $\mathcal{Q}$ will be $\bigcap_{n<\omega} \mathcal{Q}_n$.

Let $\mathcal{Q}_0 := \mathcal{P}$. Suppose that $\mathcal{Q}_n$ has been taken. Then let

$$\mathcal{Q}'_{n+1} := \{A \in \mathcal{Q}_n \mid (\forall B \in \mathcal{I})\ A \cup B \in \mathcal{Q}_n\}.$$

Note that $\mathcal{Q}'_{n+1}$ is co-analytic, and $\mathcal{I} \subseteq \mathcal{Q}'_{n+1}$. By Luzin's separation theorem let $\mathcal{Q}_{n+1}$ be a Borel set with $\mathcal{I} \subseteq \mathcal{Q}_{n+1} \subseteq \mathcal{Q}'_{n+1}$.

Let $\mathcal{Q} := \bigcap_{n<\omega} \mathcal{Q}_n$. Then $\mathcal{Q}$ is Borel and satisfies (i). Note that if $A \in \mathcal{Q}_{n+1}$, and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{Q}_n$. Hence $\mathcal{Q}$ also satisfies (ii). □(Claim)

Using Claim, by induction on $n < \omega$, we will take a Borel $\mathcal{P}_n \subset \mathcal{P}(\omega)$ with $\mathcal{I} \subseteq \mathcal{P}_n$. The desired $\mathcal{J}$ will be $\bigcap_{n<\omega} \mathcal{P}_n$.

Let $\mathcal{P}_0 := \mathcal{P}(\omega) \setminus \{\omega\}$. Suppose that $\mathcal{P}_n$ has been taken. By Claim take a Borel $\mathcal{P}'_{n+1} \subseteq \mathcal{P}(\omega)$ such that

(i) $\mathcal{I} \subseteq \mathcal{P}'_{n+1} \subseteq \mathcal{P}_n$,

(ii) if $A \in \mathcal{P}'_{n+1}$, and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{P}'_{n+1}$.

Then let

$$\mathcal{P}''_{n+1} := \{A \in \mathcal{P}'_{n+1} \mid [(\forall B \subseteq A)\ B \in \mathcal{P}'_{n+1}] \wedge [(\forall B \in \mathcal{P}'_{n+1})\ A \cup B \in \mathcal{P}'_{n+1}]\}.$$

Then $\mathcal{P}''_{n+1}$ is co-analytic. Moreover $\mathcal{I} \subseteq \mathcal{P}''_{n+1}$ by (i) and (ii). By Luzin's separation theorem let $\mathcal{P}_{n+1}$ be a Borel subset of $\mathcal{P}(\omega)$ with $\mathcal{I} \subseteq \mathcal{P}_{n+1} \subseteq \mathcal{P}''_{n+1}$. Here note that

(iii) if $B \subseteq A \in \mathcal{P}_{n+1}$, then $B \in \mathcal{P}_n$,

(iv) if $A, B \in \mathcal{P}_{n+1}$, then $A \cup B \in \mathcal{P}_n$.

Let $\mathcal{J} := \bigcap_{n<\omega} \mathcal{P}_n$. Then $\mathcal{J}$ is Borel, and $\mathcal{I} \subseteq \mathcal{J}$. Moreover $\mathcal{J}$ is an ideal by (iii), (iv) and the fact that $\omega \notin \mathcal{P}_0$. Thus $\mathcal{J}$ is as desired. □

Next we prove Proposition 3.2. We use the following well-known fact due to Mathias [7] and Talagrand [12]. Its proof can be also found in Bartoszyński–Judah [1] (Theorem 4.1.2):

Fact 3.3 (Mathias [7], Talagrand [12]). *Let $\mathcal{I}$ be an ideal over ω which has the Baire property in $\mathcal{P}(\omega)$. Then there is an increasing sequence $\langle k_n \mid n < \omega\rangle$ in ω with $k_0 = 0$ such that $\bigcup_{n\in Z} k_{n+1} \setminus k_n \notin \mathcal{I}$ for any infinite $Z \subseteq \omega$.*

Proof of Proposition 3.2. Suppose that $\{\mathcal{I}_m \mid m < \omega\} \subseteq \mathbf{An}$. We show that there is $\mathcal{J} \in \mathbf{An}$ with $\mathcal{I}_m \leq_{\mathrm{KB}} \mathcal{J}$ for all $m < \omega$.

First note that each $\mathcal{I}_m$ has the Baire property because it is analytic. Using Fact 3.3, we can easily take an increasing sequence $\langle k_n \mid n < \omega\rangle$ in ω with $k_0 = 0$ such that $\bigcup_{n\in Z} k_{n+1} \setminus k_n \notin \mathcal{I}_m$ for any infinite $Z \subseteq \omega$ and any $m < \omega$. Let $X_n := (k_{n+1} \setminus k_n)^{n+1}$ for each $n < \omega$, and let $X := \bigcup_{n<\omega} X_n$. It suffices to find an analytic ideal $\mathcal{J}$ over X such that $\mathcal{I}_m \leq_{\mathrm{KB}} \mathcal{J}$ for all $m < \omega$.

For each $m < \omega$ define $\tau_m : X \to \omega$ as follows: For $(l_0, l_1, \ldots, l_n) \in X_n$, let $\tau_m(l_0, l_1, \ldots, l_n) := l_m$ if $m \leq n$, and let $\tau_m(l_0, l_1, \ldots, l_n) := 0$ otherwise. Let $\mathcal{J}$ be the set of all $B \in \mathcal{P}(X)$ such that $B \subseteq \bigcup_{m<\bar{m}} {\tau_m}^{-1}[A_m]$ for some $\bar{m} < \omega$ and some $\langle A_m \mid m < \bar{m}\rangle$ with $A_m \in \mathcal{I}_m$. We claim that $\mathcal{J}$ is as desired.

First it is easy to see that $\mathcal{J}$ is analytic in $\mathcal{P}(X)$. Next note that τ_m is a finite to one function from X to ω. Then, by the construction of $\mathcal{J}$, each τ_m witnesses that $\mathcal{I}_m \leq_{\mathrm{KB}} \mathcal{J}$. So all we have to prove is that $\mathcal{J}$ is an ideal over X. It follows from the construction of $\mathcal{J}$ that $\mathcal{J}$ is closed under taking finite unions and subsets, and $[X]^{<\omega} \subseteq \mathcal{J}$. Thus it suffices to show that $X \notin \mathcal{J}$.

Suppose that $\bar{m} < \omega$ and that $A_m \in \mathcal{I}_m$ for each $m < \bar{m}$. It is enough to show that $\bigcup_{m<\bar{m}} {\tau_m}^{-1}[A_m] \neq X$. By the choice of $\langle k_n \mid n < \omega\rangle$, for each $m < \bar{m}$ there are at most finitely many $n < \omega$ with $k_{n+1} \setminus k_n \subseteq A_m$. So we can take $n \geq \bar{m}$ such that $k_{n+1} \setminus k_n \not\subseteq A_m$ for any $m < \bar{m}$. For each $m < \bar{m}$ take $l_m \in k_{n+1} \setminus k_n$ with $l_m \notin A_m$, and let $l_m := k_n$ for each m with $\bar{m} \leq m \leq n$. Then $(l_0, l_1, \ldots, l_n) \in X_n \subseteq X$. But $(l_0, l_1, \ldots, l_n) \notin \bigcup_{m<\bar{m}} {\tau_m}^{-1}[A_m]$. □

Next we turn our attention to the fact that $(\mathbf{Bo}, \leq_{\mathrm{K}})$ does not have the largest element, which follows from the results in [6]. In fact, we have the following theorem among the results in [6], where an ideal $\mathcal{I}$ over ω is called an $\mathbf{\Sigma}^0_\xi$*-ideal* if it is $\mathbf{\Sigma}^0_\xi$ in $\mathcal{P}(\omega)$:

Theorem 3.4 (Katětov [6]). *Suppose that $1 \leq \xi < \omega_1$. Then there is $\mathcal{I} \in$ **Bo** such that $\mathcal{I} \not\leq_{\mathrm{K}} \mathcal{J}$ for any $\boldsymbol{\Sigma}^0_\xi$-ideal $\mathcal{J}$ over ω.*

Here we give a proof of this theorem, which is slightly simpler than the arguments in [6].

First we make some preliminaries. Let $\mathcal{C}$ denote the Cantor space. Let $\mathcal{I}$ be an ideal over ω, let $\langle \mathcal{P}_n \mid n < \omega \rangle$ be a sequence of subsets of $\mathcal{C}$, and let $\mathcal{Q}$ be a subset of $\mathcal{C}$. We write $\lim_{\mathcal{I}} \langle \mathcal{P}_n \mid n < \omega \rangle = \mathcal{Q}$ if

(i) $\{n < \omega \mid Q \notin \mathcal{P}_n\} \in \mathcal{I}$ for all $Q \in \mathcal{Q}$,

(ii) $\{n < \omega \mid Q \in \mathcal{P}_n\} \in \mathcal{I}$ for all $Q \in \mathcal{C} \setminus \mathcal{Q}$.

We will use the following:

Lemma 3.5. *Let $\mathcal{I}$ be an ideal over ω, let $\langle \mathcal{P}_n \mid n < \omega \rangle$ be a sequence of subsets of $\mathcal{C}$, and let $\mathcal{Q}$ be a subset of $\mathcal{C}$. Assume that $\lim_{\mathcal{I}} \langle \mathcal{P}_n \mid n < \omega \rangle = \mathcal{Q}$.*

(1) Let $\mathcal{J}$ be an ideal over ω with $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$, which is witnessed by $\tau : \omega \to \omega$. Then $\lim_{\mathcal{J}} \langle \mathcal{P}_{\tau(n)} \mid n < \omega \rangle = \mathcal{Q}$.

(2) Suppose that $1 \leq \xi < \omega_1$ and that $\mathcal{I}$ is a $\boldsymbol{\Sigma}^0_\xi$-ideal. Suppose also that each $\mathcal{P}_n$ is clopen in $\mathcal{C}$. Then $\mathcal{Q}$ is $\boldsymbol{\Sigma}^0_\xi$ in $\mathcal{C}$.

Proof. (1) If $Q \in \mathcal{Q}$, then $\{n < \omega \mid Q \notin \mathcal{P}_{\tau(n)}\} = \tau^{-1}[\{n < \omega \mid Q \notin \mathcal{P}_n\}] \in \mathcal{J}$. Also, if $Q \in \mathcal{C} \setminus \mathcal{Q}$, then $\{n < \omega \mid Q \in \mathcal{P}_{\tau(n)}\} = \tau^{-1}[\{n < \omega \mid Q \in \mathcal{P}_n\}] \in \mathcal{J}$. Thus $\lim_{\mathcal{J}} \langle \mathcal{P}_{\tau(n)} \mid n < \omega \rangle = \mathcal{Q}$.
(2) Define $f : \mathcal{C} \to \mathcal{P}(\omega)$ by $f(Q) := \{n < \omega \mid Q \notin \mathcal{P}_n\}$. Then it is easy to see that f is continuous and that $\mathcal{Q} = f^{-1}[\mathcal{I}]$. Hence $\mathcal{Q}$ is $\boldsymbol{\Sigma}^0_\xi$. □

Now we give a proof of Theorem 3.4:

Proof of Theorem 3.4. By Theorem 3.1 it suffices to find $\mathcal{I} \in$ **An** such that $\mathcal{I} \not\leq_{\mathrm{K}} \mathcal{J}$ for any $\boldsymbol{\Sigma}^0_\xi$ ideal $\mathcal{J}$.

Take a Borel $\mathcal{Q} \subseteq \mathcal{C}$ which is not $\boldsymbol{\Sigma}^0_\xi$ and an enumeration $\langle \mathcal{P}_n \mid n < \omega \rangle$ of all clopen subsets of $\mathcal{C}$. For each $Q \in \mathcal{C}$ let $A_Q := \{n < \omega \mid Q \in \mathcal{P}_n\}$. Then let $\mathcal{I}$ be the set of all $B \subseteq \omega$ such that

$$B \subseteq \left(\bigcup_{Q \in \mathcal{R}} \omega \setminus A_Q \right) \cup \left(\bigcup_{Q \in \mathcal{S}} A_Q \right)$$

for some $\mathcal{R} \in [\mathcal{Q}]^{<\omega}$ and some $\mathcal{S} \in [\mathcal{C} \setminus \mathcal{Q}]^{<\omega}$. We claim that $\mathcal{I}$ is as desired.

First we check that $\mathcal{I} \in$ **An**. It is easy to see that $\mathcal{I}$ is analytic. We show that $\mathcal{I}$ is an ideal. Note that $\mathcal{I}$ is closed under taking finite unions and subsets by its construction. Note also that for any $\mathcal{R} \in [\mathcal{Q}]^{<\omega}$ and any $\mathcal{S} \in [\mathcal{C} \setminus \mathcal{Q}]^{<\omega}$ there is $n < \omega$ with $\mathcal{P}_n \supseteq \mathcal{R}$ and $\mathcal{P}_n \cap \mathcal{S} = \emptyset$, i.e. $n \in A_Q$ for any $Q \in \mathcal{R}$ and $n \notin A_Q$ for any $Q \in \mathcal{S}$. Hence $\omega \notin \mathcal{I}$. To see that $[\omega]^{<\omega} \subseteq \mathcal{J}$, it is enough to prove that $\{n\} \in \mathcal{I}$ for all $n < \omega$. Suppose that $n < \omega$. Then $\mathcal{P}_n \neq \mathcal{Q}$ because $\mathcal{Q}$

is not $\mathbf{\Sigma}^0_\xi$. If there is $Q \in \mathcal{Q} \setminus \mathcal{P}_n$, then $Q \in \mathcal{Q}$, $n \in \omega \setminus A_Q$, and hence $\{n\} \in \mathcal{I}$. Otherwise, there is $Q \in \mathcal{P}_n \setminus \mathcal{Q}$. Then $Q \notin \mathcal{Q}$, $n \in A_Q$, and so $\{n\} \in \mathcal{I}$.

Next note that $\lim_{\mathcal{I}}\langle \mathcal{P}_n \mid n < \omega\rangle = \mathcal{Q}$ by the construction of $\mathcal{I}$. Recall also that $\mathcal{Q}$ is not $\mathbf{\Sigma}^0_\xi$ and that each $\mathcal{P}_n$ is clopen. Then it follows from Lemma 3.5 that $\mathcal{I} \not\leq_{\mathrm{K}} \mathcal{J}$ for any $\mathbf{\Sigma}^0_\xi$-ideal $\mathcal{J}$ over ω. Thus $\mathcal{I}$ is as desired. □

4 Questions

We end this paper with several questions.

We do not know the cofinal types of $(\mathbf{Bo}, \leq_{\mathrm{K}})$ and $(\mathbf{Bo}, \leq_{\mathrm{KB}})$. If the following question is answered affirmatively (this would strengthen Theorem 3.4), then $(\mathbf{Bo}, \leq_{\mathrm{K}})$ and $(\mathbf{Bo}, \leq_{\mathrm{KB}})$ are Tukey equivalent to $(\omega_1, <)$ by Theorem 1.3:

Question 4.1. *For any ξ with $1 \leq \xi < \omega_1$, is there $\mathcal{J} \in \mathbf{Bo}$ with $\mathcal{I} \leq_{\mathrm{KB}} \mathcal{J}$ (or $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$) for every $\mathbf{\Sigma}^0_\xi$-ideal $\mathcal{I}$?*

Note that there is such $\mathcal{J}$ for $\xi = 2$ by Corollary 1.2.

Next question is on the Rudin–Keisler order $\leq_{\mathrm{RK}}$ and the Rudin–Blass order $\leq_{\mathrm{RB}}$ on ideals, which are defined as follows: For an ideal $\mathcal{I}$ over X and an ideal $\mathcal{J}$ over Y,

$\mathcal{I} \leq_{\mathrm{RK}} \mathcal{J} \overset{\text{def}}{\Leftrightarrow}$ there is a function $\tau : Y \to X$ such that $A \in \mathcal{I}$ if and only if $\tau^{-1}[A] \in \mathcal{J}$ for any $A \subseteq X$,

$\mathcal{I} \leq_{\mathrm{RB}} \mathcal{J} \overset{\text{def}}{\Leftrightarrow}$ there is a finite to one function $\tau : Y \to X$ such that $A \in \mathcal{I}$ if and only if $\tau^{-1}[A] \in \mathcal{J}$ for any $A \subseteq X$.

Note that $\leq_{\mathrm{RK}} \subseteq \leq_{\mathrm{K}}$ and $\leq_{\mathrm{RB}} \subseteq \leq_{\mathrm{KB}}$. For details of these orders, see Hrušák [2] for example.

We do not know whether $\mathbf{AP}$ has the largest element with respect to these orders:

Question 4.2. *Does $(\mathbf{AP}, \leq_{\mathrm{RK}})$ or $(\mathbf{AP}, \leq_{\mathrm{RB}})$ have the largest element?*

As for the upward directedness of $\mathbf{Bo}$, it is easy to see that $(\mathbf{Bo}, \leq_{\mathrm{RK}})$ is upward directed: For any $\mathcal{I}_0, \mathcal{I}_1 \in \mathbf{Bo}$, their Fubini product $\mathcal{I}_0 \times \mathcal{I}_1$ is a Borel ideal on $\omega \times \omega$, and the projections from $\omega \times \omega$ to ω witness that $\mathcal{I}_0, \mathcal{I}_1 \leq_{\mathrm{RK}} \mathcal{I}_0 \times \mathcal{I}_1$. But we do not know whether $(\mathbf{Bo}, \leq_{\mathrm{RB}})$ is upward directed:

Question 4.3. *Is $(\mathbf{Bo}, \leq_{\mathrm{RB}})$ upward directed?*

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